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Minimum degree and the existence of semiregular factors in planar graphs

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Abstract

Let G be a simple planar graph and let α be a positive integer such that $1 \leq \alpha \leq 3$ and $\delta(G) \geq \alpha + 2$. For every pair of edges e_1, e_2 of G, the graph $G - \{e_1, e_2\}$ contains an $[\alpha, \alpha+1]$-factor.

1 Introduction and preliminaries

All graphs considered are simple and finite. We refer the reader to [1] for standard graph theoretic terms not defined in this paper.

Let G be a graph. The degree $d_G(u)$ of a vertex u in G is the number of edges of G incident with u. The minimum degree of G is denoted by $\delta(G)$. If X and Y are subsets of $V(G)$, the set and the number of the edges of G joining X to Y are denoted by $E_G(X,Y)$ and $e_G(X,Y)$, respectively. For any set X of vertices in G, the subgraph induced by X is denoted by $G[X]$ and the neighbour set of X in G by $N_G(X)$. The number of connected components of G is denoted by $\omega(G)$. A cut edge of G is an edge such that $\omega(G - \{e\}) > \omega(G)$.

A bipartite graph is one whose vertex set can be partitioned into two subsets X and Y, so that each edge has one end in X and one end in Y; such a partition (X,Y) is called a bipartition of the graph. The following result is a well-known characterization of bipartite graphs.

Theorem 1 (Bondy [1]). *A graph is bipartite if and only if it contains no odd cycle.*

Let G be a graph and let g and f be two integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for all $x \in V(G)$. Then a $[g,f]$-factor of G is a spanning subgraph F satisfying $g(x) \leq d_F(x) \leq f(x)$ for all $x \in V(G)$. If $g(x) = \alpha$ and $f(x) = b$ for all $x \in V(G)$, then we will call such a $[g,f]$-factor, an $[\alpha,b]$-factor. If f, g are both constant functions taking the same value k then we will call such an $[\alpha, b]$-factor, a k-factor. Thus a k-factor of G is a k-regular spanning subgraph of G. A k-factor is also called regular factor and a $[k, k+1]$-factor is often called semiregular factor.

The following theorem due to Lovász [4] is a necessary and sufficient condition for a graph to have a $[g, f]$-factor.

Lovász's Theorem. *Let G be a graph and g and f be integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for all $x \in V(G)$. Then G has a $[g, f]$-factor if and only if*

$$q_G(D, S) + \sum_{x \in S}(g(x) - d_{G-D}(x)) \leq \sum_{x \in D} f(x)$$

for all disjoint sets $D, S \subseteq V(G)$, where $q_G(D, S)$ denotes the number of components H of $(G - D) - S$ such that $g(x) = f(x)$ for all $x \in V(H)$ and

$$e_G(S, V(H)) + \sum_{x \in V(H)} f(x) \equiv 1 \pmod 2.$$

A graph is said to be planar or embeddable in the plane, if it can be drawn in the plane so that its edges intersect only at their ends. Such a drawing of a planar graph G is called a planar embedding of G. It can be regarded as a graph isomorphic to G and we sometimes refer to it as a plane graph. A planar embedding of a planar graph divides the plane into a number of connected regions, called faces, each bounded by edges of the graph. We shall denote by $F(G)$ and $\Phi(G)$ the set and the number respectively of faces of a plane graph G.

Each plane graph has exactly one unbounded face called the exterior face. For every plane graph G, we denote the boundary of a face f of G by $b(f)$. If G is connected, $b(f)$ can be regarded as a closed walk in which each cut edge of G in $b(f)$ is traversed twice. A face f is said to be incident with the vertices, and edges in its boundary. If e is a cut edge in G, just one face is incident with e, otherwise there are two faces incident with e. The degree $d_G(f)$, of a face f of G is the number of edges with which it is incident (cut edges are counted twice).

The following proposition and theorems related to planar graphs are well-known results.

Proposition 1. *If G is planar, then every subgraph of G is also planar.*

Theorem 2 (Euler's formula). *If G is a connected plane graph, then*

$$|V(G)| - |E(G)| + \Phi(G) = 2.$$

Theorem 3. *If G is a plane graph, then*

$$\sum_{f \in F(G)} d_G(f) = 2|E(G)|.$$

The following result can be derived from Theorems 2 and 3.

Corollary 1. *If G is a connected plane triangle-free graph with at least three vertices, then*

$$|E(G)| \leq 2|V(G)| - 4.$$

Proof. For every face f of a connected plane triangle-free graph G having at least three vertices, we have $d_G(f) \geq 4$. So

$$\sum_{f \in F(G)} d_G(f) \geq 4\Phi(G).$$

Thus by using Theorem 3,

$$|E(G)| \geq 2\Phi(G). \tag{1}$$

By Theorem 2 and (1), we obtain

$$|V(G)| - |E(G)| + \frac{|E(G)|}{2} \ \geq \ 2$$

or

$$|E(G)| \ \leq 2 \ |V(G)| - 4.$$

□

Theorem 4. *For every planar graph G, $\delta(G) \leq 5$.*

The discussion concerning the existence of a k-factor or an $[\alpha, b]$-factor in a planar graph is meaningful only for the cases when $k \leq 5$ and $\alpha \leq 5$ respectively, by using Theorem 4.

The existence of such k-factors in planar graphs were studied recently by the author [3] and related results for the existence of connected $[\alpha, b]$-factors can also be found in [2]. As was demonstrated in [3], high minimum degree of a planar graph cannot guarantee the existence of a regular factor. It appears that this does not apply for semiregular factors.

The main purpose of this paper is to present the following sufficient condition for the existence of semiregular factors in a planar graph based on its minimum degree.

Theorem 5. *Let G be a planar graph and let α be a positive integer such that $1 \leq \alpha \leq 3$ and $\delta(G) \geq \alpha + 2$. For every pair of edges e_1, e_2 of G, the graph $G - \{e_1, e_2\}$ contains an $[\alpha, \alpha + 1]$-factor.*

2 Proof of main result

To obtain the proof of Theorem 5, we shall need an auxiliary lemma.

Lemma 1. *Let G be a graph and α, b be two positive integers such that $b > \alpha$. Suppose that there exist $D, S \subseteq V(G)$, such that $D \cap S = \emptyset$ and*

$$\sum_{x\in S}(\alpha - d_{G-D}(x)) > b|D|. \tag{2}$$

If $S \cup D$ is minimal with respect to (2), then

(i) $d_{G-D}(x) \leq \alpha - 1$ *for every* $x \in S$, *and*

(ii) $|N_G(x) \cap S| \geq 3$ *for every* $x \in D$.

Proof. (i) Suppose that there exists $u \in S$ such that $d_{G-D}(u) \geq \alpha$.

Define $S^{'} = S - \{u\}$. Then

$$\begin{aligned}\sum_{x\in S'}(\alpha - d_{G-D}(x)) &= \sum_{x\in S}(\alpha - d_{G-D}(x)) - (\alpha - d_{G-D}(u)) \\ &\geq \sum_{x\in S}(\alpha - d_{G-D}(x)).\end{aligned}$$

Thus by using (2),

$$\sum_{x\in S'}(\alpha - d_{G-D}(x)) > b|D|,$$

contradicting the minimality of $D \cup S$ with respect to (2).

(ii) Suppose that there exists $u \in D$ such that $|N_G(u) \cap S| \leq 2$.

Define $D^{'} = D - \{u\}$. Then

$$\begin{aligned}\sum_{x\in S} d_{G-D'}(x) &= \sum_{x\in S} d_{G-D}(x) + |N_G(u) \cap S| \\ &\leq \sum_{x\in S} d_{G-D}(x) + 2.\end{aligned}$$

Thus

$$\begin{aligned}\alpha|S| - \sum_{x\in S} d_{G-D'}(x) &\geq \alpha|S| - (\sum_{x\in S} d_{G-D}(x) + 2) \\ &> b|D| - 2 \qquad \text{by (2)} \\ &= b(|D| - 1) + b - 2 \\ &= b|D^{'}| + b - 2\end{aligned}$$

contradicting the minimality of $D \cup S$ with respect to (2), since $b \geq 2$. □

Proof of Theorem 5. Suppose that there exists a pair of edges e_1, e_2 of G such that the graph $G - \{e_1, e_2\}$ does not contain an $[\alpha, \alpha+1]$-factor.

Let $X = \{e_1, e_2\}$ and $G^* = G - X$. Then by Lovász's Theorem there exist $D, S \subseteq V(G^*)$, such that $D \cap S = \emptyset$ and

$$\alpha|S| - \sum_{x \in S} d_{G^*-D}(x) > (\alpha+1)|D|. \tag{3}$$

If we assume that $D \cup S$ is minimal with respect to (3), then by Lemma 1

$$d_{G^*-D}(x) \leq \alpha - 1 \text{ for every } x \in S \tag{4}$$

and

$$|N_{G^*}(x) \cup S| \geq 3 \text{ for every } x \in D. \tag{5}$$

First we note that $S \neq \emptyset$, by using (3). Furthermore for every $x \in S$,

$$\begin{aligned} \alpha + 2 \leq \delta(G) \leq d_G(x) &\leq d_{G-D}(x) + |D| \\ &\leq d_{G^*-D}(x) + |X| + |D| \\ &\leq \alpha + 1 + |D| \qquad \text{by (4) and } |X| = 2. \end{aligned}$$

Thus $D \neq \emptyset$.

For every $x \in S$, we also have

$$d_{G^*-D}(x) + |N_{G^*}(x) \cap D| = d_{G^*}(x) \geq d_G(x) - |X|. \tag{6}$$

But $d_G(x) \geq \delta(G) \geq \alpha + 2$ and $|X| = 2$. So by using (4), (6) yields

$$|N_{G^*}(x) \cap D| \geq 1. \tag{7}$$

Define a new graph H such that $V(H) = D \cup S$ and $E(H) = E_{G^*}(D, S)$. Clearly H is a bipartite subgraph of G with bipartition (D, S) and so by Proposition 1, H is also a planar graph. Moreover we can derive from (5) and (7) that for every component C of H, $|V(C)| \geq 4$. Hence Corollary 1 yields

$$|E(H)| \leq 2(|D| + |S|) - 4 \tag{8}$$

since the bipartite graph H is triangle-free by Theorem 1. But

$$\begin{aligned} |E(H)| &= \sum_{x \in S} d_{G^*}(x) - \sum_{x \in S} d_{G^*-D}(x) \\ &\geq \sum_{x \in S} d_G(x) - 2|X| - \sum_{x \in S} d_{G^*-D}(x) \\ &\geq \delta(G)|S| - 2|X| - \sum_{x \in S} d_{G^*-D}(x) \\ &\geq (\alpha+2)|S| - 4 - \sum_{x \in S} d_{G^*-D}(x) \end{aligned}$$

and thus by using (8),

$$2|D| \geq \alpha|S| - \sum_{x \in S} d_{G^*-D}(x). \tag{9}$$

Now (3) can be written as

$$\alpha|S| - \sum_{x \in S} d_{G^*-D}(x) > (\alpha - 1)|D| + 2|D|$$

and so by using (9), $0 > (\alpha - 1)|D|$ contradicting the fact that α is a positive integer.

This completes the proof of Theorem 5. □

3 Sharpness

We next show that the conditions of Theorem 5 are, in some sense, best possible. We first notice that the number of deleted edges cannot be increased. Let G be a planar graph such that $\delta(G) = \alpha + 2$ and let $u \in V(G)$ be a vertex such that $d_G(u) = \alpha + 2$. If we delete from G more than two edges having u as an end vertex, then the resulting graph G^* will have minimum degree less than α and clearly G^* cannot possess an $[\alpha, \alpha + 1]$-factor.

We next show that the minimum degree condition is also best possible by describing a family of planar graphs G having slightly lower minimum degree and not having the properties implied by Theorem 5. In fact as we will see, the conclusions of Theorem 5 will not hold even if we do not delete edges from G and even if we are looking for any $[\alpha, n]$-factor where $n \geq \alpha$, instead of an $[\alpha, \alpha + 1]$-factor.

We construct such graphs G as follows. We start from two vertices u, v which are joined to (i) all vertices of $2n + 1$ copies of K_α, when $1 \leq \alpha \leq 2$, (ii) all vertices of a cycle of length $2n + 1$ when $\alpha = 3$; where in both the above cases n is a positive integer such that $n \geq \alpha$. Clearly the resulting graph G is planar, $\delta(G) = \alpha + 1$ and, as we will show, G does not have an $[\alpha, n]$-factor.

Let $D = \{u, v\}$ and $S = V(G) - D$. Then

$$\sum_{x \in S} (\alpha - d_{G-D}(x)) > n|D|$$

since $d_{G-D}(x) = \alpha - 1$ for every $x \in S$, $|S| \geq 2n + 1$ and $|D| = 2$.

Hence G does not have an $[\alpha, n]$-factor by Lovász's Theorem.

Theorem 5 is a sufficient condition for a planar graph to have [1, 2], [2, 3], [3, 4]-factors in terms of its minimum degree. A natural question that may arise is whether high minimum degree of a planar graph can guarantee the existence of a $[4, n]$-factor or of a $[5, n]$-factor for any $n \geq 4$ or $n \geq 5$, respectively. We will show that this is not possible by describing again a family of planar graphs. We construct such graphs G as follows. We start by taking $2n + 1$ copies $H_1, H_2, \ldots, H_{2n+1}$ of the plane graph illustrated in Fig. 1.

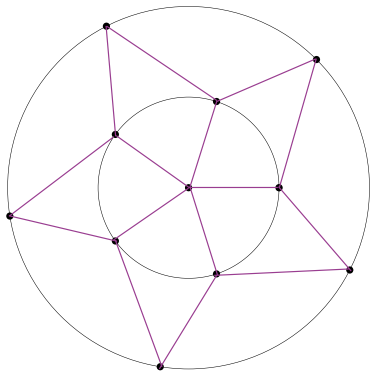

Figure 1:

In this plane graph, as we can see, the exterior face is incident with 5 vertices. Let $\{u_{i,1}, u_{i,2}, u_{i,3}, u_{i,4}, u_{i,5}\}$ be the set of vertices of the exterior face of every such copy H_i for $i = 1, 2, \ldots, 2n+1$. We place $H_1, H_2, \ldots, H_{2n+1}$ in a circular order and between H_i and H_{i+1} we add vertex v_i for $i = 1, 2, \ldots, 2n+1$ (where addition is taken modulo $2n+1$). We also add vertex w_1 inside the circular ordering of H_i's, v_i's and vertex w_2 outside. We join v_i to $u_{i,4}, u_{i,5}, u_{i+1,1}$, vertex w_1 to $u_{i,2}, u_{i,3}, v_i$ and vertex w_2 to v_i, for $i = 1, 2, \ldots, 2n+1$. The resulting graph G is clearly planar, $\delta(G) = 5$ and G does not have a $[4, n]$-factor or a $[5, n]$-factor for every $n \geq 4$ or $n \geq 5$, respectively. For the proof of our last statement, we define $D = \{w_1, w_2\}$ and $S = \{v_1, v_2, \ldots, v_{2n+1}\}$. Then

$$\sum_{x \in S} (4 - d_{G-D}(x)) > n|D|$$

since $\sum_{x \in S} (4 - d_{G-D}(x)) = 2n+1$ and $|D| = 2$. Thus by Lovász's Theorem, G does not have a $[4, n]$-factor and so does not also possess a $[5, n]$-factor.

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