AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **60(1)** (2014), Pages 198–207

Dominant Shi regions with a fixed separating wall: bijective enumeration*

Alessandro Conflitti   Ricardo Mamede

CMUC, Centre for Mathematics
University of Coimbra
Apartado 3008, 3001–454 Coimbra
Portugal
conflitt@mat.uc.pt   mamede@mat.uc.pt

Eleni Tzanaki

Department of Applied Mathematics
University of Crete, P. O. Box 2208
Heraklion, Crete 71409
Greece etzanaki@tem.uoc.gr

Abstract

We present a purely combinatorial proof by means of an explicit bijection, of the exact number of dominant regions having as a separating wall the hyperplane associated to the longest root in the m-extended Shi hyperplane arrangement of type A and dimension $n-1$.

1 Overview

Currently, a very active research area is the investigation of the combinatorics of hyperplane arrangements, see e.g. [18, 25] for an introduction on the topic. In particular, it is well know, see [1, 2, 6, 20, 21, 22, 23, 24], that there is a useful and efficient way to encode dominant regions and dominant alcoves of the m-extended Shi arrangement in the affine space of type A with dimension $n-1$, based on some Young tableaux of a staircase partition whose filling must obey several arithmetic conditions, which are called Shi tableaux.

* This work was partially supported by the Centro de Matemática da Universidade de Coimbra (CMUC), funded by the European Regional Development Fund through the program COMPETE and by the Portuguese Government through the FCT - Fundação para a Ciência e a Tecnologia under the project PEst-C/MAT/UI0324/2011.

Shi tableaux together with another combinatorial object called *abacus*, see e.g. [12, 26], proved to be a very powerful tool to investigate hyperplane arrangements, but since they are quite not very easy to handle, because of the arithmetic constraints on the filling, recently some bijections were designed between some Shi tableaux, for instance the one associated to some specific regions, and other families of tableaux which are much easier to work on, for instance cores, see [4, 9, 10, 14, 15, 19], and m-staggered staircases, see [7, 8].

Nevertheless, at the moment it is often impossible to prove enumerative results on these hyperplane arrangements in a purely bijective way. Usually in order to count selected families of regions of the m-extended Shi arrangement, one uses the bijection between these regions and an associated family of Young tableaux, but then in order to enumerate the corresponding class of Young tableaux some algebraic non-constructive method is frequently used, namely abstract results connected with representation theory.

In particular, in [8, Corollary 4.5] it is proved that there are exactly m^{n-2} dominant regions having as a separating wall the hyperplane associated to the longest root, but the enumeration of the Young tableaux associated to such regions is made in a very abstract non-constructive way combining the use of generating functions with some geometric results which are quite complicated, and the whole is very arduous and laborious to follow.

In this brief note we present a purely combinatorial proof of the aforementioned result designing an explicit bijection between the involved Young tableaux and the class of words of length $n-2$ over the finite alphabet $\{0, \ldots, m-1\}$.

We remark that the only bijective proofs known so far, viz. [3, 5, 16], deal with the whole set of dominant regions without further constraints, therefore as far as we are aware of, this is the first bijective result involving dominant regions with additional interesting properties, namely having a specific fixed separating wall.

2 Notation

We follow closely the notation used in [7, 8]. For more comprehensive references we refer to [1, 2, 25].

Let $\{\varepsilon_1, \ldots, \varepsilon_n\}$ be the standard basis of $\mathbb{R}^n$ and $\langle , \rangle$ be the bilinear form for which this is an orthonormal basis. Let $\alpha_j = \varepsilon_j - \varepsilon_{j+1}$. Then $\Pi = \{\alpha_1, \ldots, \alpha_{n-1}\}$ is a basis of

$$V = \{(a_1, \ldots, a_n) \in \mathbb{R}^n \mid \sum_{j=1}^{n} a_j = 0\}.$$

For $i \leq j$, we define $\alpha_{i,j} = \alpha_i + \ldots + \alpha_j$, so in this notation we have that $\alpha_i = \alpha_{i,i} = \varepsilon_i - \varepsilon_{i+1}$.

The elements of $\Delta = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}$ are called *roots*, and a root α is defined to be *positive*, written $\alpha > 0$, if $\alpha \in \Delta^+ = \{\varepsilon_i - \varepsilon_j \mid i < j\}$. We let $\Delta^- = -\Delta^+$ and write $\alpha < 0$ if $\alpha \in \Delta^-$. Then Π is the set of simple roots, and the highest root is

$\theta = \alpha_{1,n-1} = \sum_{j=1}^{n-1} \alpha_j$.

For each $\alpha \in \Delta$, $k \in \mathbb{Z}$ we define an affine reflecting hyperplane

$$H_{\alpha,k} = \{v \in V \mid \langle v, \alpha \rangle = k\}.$$

Note that $H_{-\alpha,-k} = H_{\alpha,k}$, so without loss of generality we might consider $k \in \mathbb{N}$.

The *m-extended Shi hyperplane arrangement* is defined as the collection of hyperplanes

$$\mathcal{H}_m^{n-1} = \{H_{\alpha,k} \mid \alpha \in \Delta^+, -m < k \leq m\}.$$

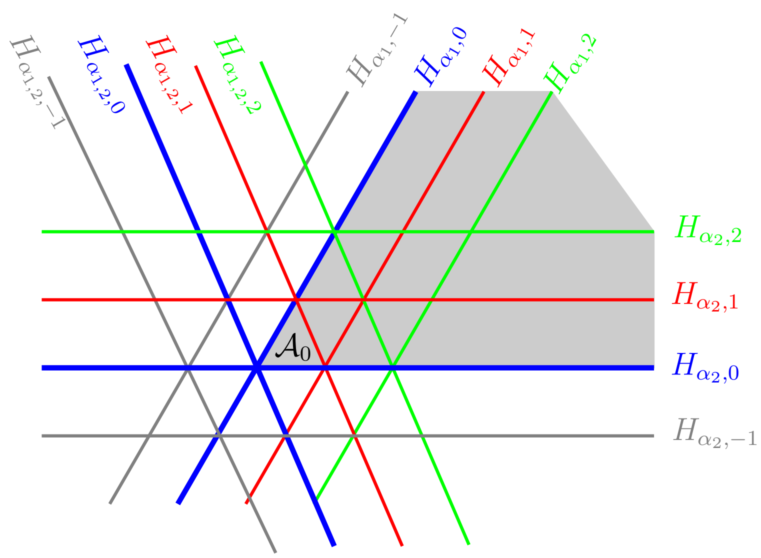
Note that this arrangement can actually be defined for all crystallographic root systems of finite type, but here we are concerned only with type A; very little is known about the combinatorics of m-extended Shi arrangements with root system of other types, see [11, 13, 17].

The connected components of the hyperplane arrangement complement $V \setminus \bigcup_{H \in \mathcal{H}_m^{n-1}} H$ are called *regions* of the m-extended Shi hyperplane arrangement.

Denote the closed half-spaces $\{v \in V \mid \langle v, \alpha \rangle \geq k\}$ and $\{v \in V \mid \langle v, \alpha \rangle \leq k\}$ by $H_{\alpha,k}^+$ and $H_{\alpha,k}^-$, respectively; then the *dominant* (or *fundamental*) chamber of V is $\bigcap_{i=1}^{n-1} H_{\alpha_i,0}^+$. A *dominant region* of the m-extended Shi hyperplane arrangement is a region contained in the dominant chamber.

Each connected component of $V \setminus \bigcup_{\substack{\alpha \in \Delta^+ \\ k \in \mathbb{Z}}} H_{\alpha,k}$ is called an *alcove*, and the *fundamental alcove* is $\mathcal{A}_0$, the interior of $H_{\theta,1}^- \cap \bigcap_{i=1}^{n-1} H_{\alpha_i,0}^+$. A *dominant alcove* is an alcove contained in the dominant chamber.

The following example shows $\mathcal{H}_2^2$; the dominant chamber is the part with a gray background.



It is known, see [1, Corollary 1.3], that in $\mathcal{H}_m^{n-1}$ the number of dominant regions is

$$\frac{1}{n}\binom{(m+1)\,n}{n-1},$$

and the number of bounded dominant regions is

$$\frac{1}{n}\binom{(m+1)\,n-2}{n-1}.$$

Given n, m positive integers, there is a known way, see e.g. [2, 6, 8], to represent a dominant region of m-Shi hyperplane arrangement in $\mathbb{R}^n$ as a Shi tableau, i.e. the Young diagram (using the English notation, but with the labeling of the columns from right to left) of a staircase partition $\{(n-j)_{j=1}^{n-1}\}$, with a special filling, i.e. the value $k_{i,j}$ associated to the cell in the row i and column $n-j$ must satisfy some special requirements. Namely, the entries $k_{i,j}$ are nonnegative integers not greater than m, they are non-increasing along rows and columns, and for any $i \leq t < j$ the following conditions must hold: if $k_{i,j} = m$ then $k_{i,t} + k_{t+1,j} \geq m-1$, whereas if $k_{i,j} \leq m-1$ then either

$$k_{i,j} = k_{i,t} + k_{t+1,j} \tag{1}$$

or

$$k_{i,j} = k_{i,t} + k_{t+1,j} + 1. \tag{2}$$

We present as an example the Shi tableaux for $n = 5$, where the filling must satisfy the conditions described above.

$k_{1,4}$	$k_{1,3}$	$k_{1,2}$	$k_{1,1}$
$k_{2,4}$	$k_{2,3}$	$k_{2,2}$	
$k_{3,4}$	$k_{3,3}$		
$k_{4,4}$			

For instance, this means that given m if $k_{1,4} = m$ then $\min\{k_{1,1} + k_{2,4}, k_{1,2} + k_{3,4}, k_{1,3} + k_{4,4}\} \geq m-1$ whereas if $k_{1,4} \leq m-1$ then

$$k_{1,4} = k_{1,1} + k_{2,4} + b_1 = k_{1,2} + k_{3,4} + b_2 = k_{1,3} + k_{4,4} + b_3,$$

where $b_1, b_2, b_3 \in \{0, 1\}$.

So an explicit example of a m-Shi tableau for $n = 5$ and any $m \geq 4$ is

4	3	2	2
2	1	0	
1	1		
0			

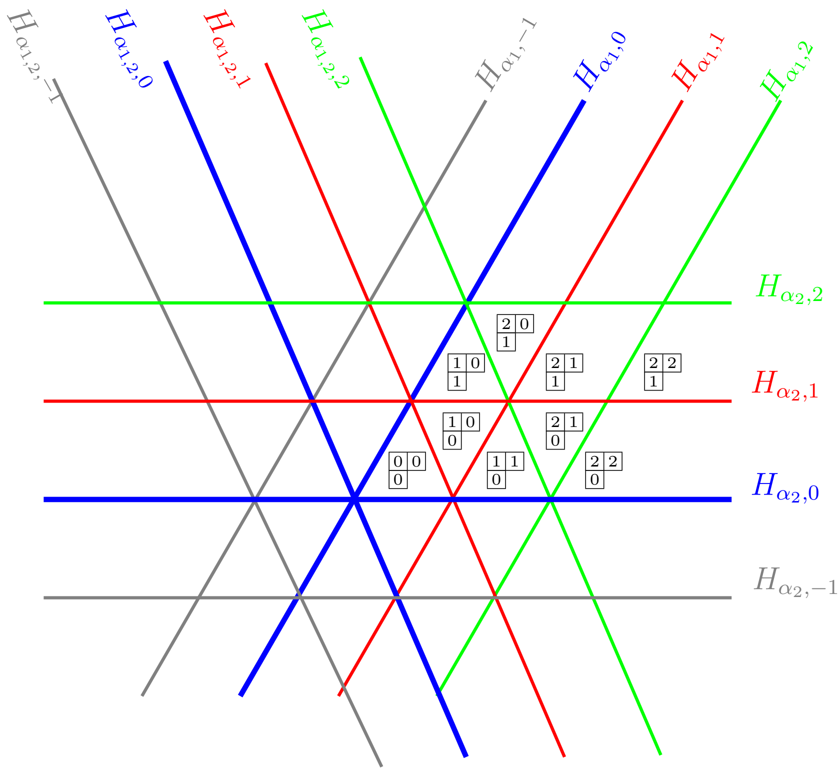
. (3)

Very sketchily, given a dominant region R in the m-Shi arrangement, $k_{i,j}$ is the minimum number of integer translates of $H_{\alpha_{i,j},0}$ that separates R from the origin (or equivalently from $\mathcal{A}_0$), whereas given a m-Shi tableau, the corresponding region R consists of those points x such that

$$\begin{cases} k_{i,j} < \langle \alpha_{i,j}, x\rangle < k_{i,j}+1 & \text{if } 0 \leq k_{i,j} < m, \\ k_{i,j} < \langle \alpha_{i,j}, x\rangle & \text{if } k_{i,j} = m, \end{cases}$$

for all $1 \leq i \leq j \leq n$.

We show some examples in $\mathcal{H}_2^2$.



Two open regions are *separated* by a hyperplane H if they lie in different closed half-spaces relative to H, and a *wall* of a region is a hyperplane in $\mathcal{H}_m^{n-1}$ which supports a facet of that region. A *separating wall* for a region R is a wall of R which separates R from the fundamental alcove $\mathcal{A}_0$.

Hence, for instance, there are two dominant regions having $H_{\alpha_{1,2},2}$ as a separating wall, namely the regions associated to the Young tableaux

$$\begin{array}{|c|c|}\hline 2 & 0 \\ \hline 1 \\ \cline{1-1}\end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 0 \\ \cline{1-1}\end{array},$$

and three dominant regions having $H_{\alpha_2,1}$ as a separating wall, namely the regions associated to the Young tableaux

$$\begin{array}{|c|c|}\hline 1 & 0 \\ \hline 1 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}.$$

3 Main result

We give a purely combinatorial proof of the following result.

Theorem 3.1. *There are exactly m^{n-2} dominant regions having $H_{\theta,m}$ as a separating wall.*

Proof. It is known, see e.g. [7, 8], that a dominant region admits $H_{\theta,m}$ as a separating wall if and only if its corresponding Shi tableau has the following property:

$$\begin{cases} k_{1,n-1} = m, & \\ k_{1,t} + k_{t+1,n-1} = m-1 & \text{for all } 1 \leq t \leq n-2. \end{cases} \tag{4}$$

Note that obviously not all Shi tableaux satisfy this condition: for example the tableau (3) does not, whereas the following

7	6	5	3	1	1
5	5	4	1	0	
5	5	3	1		
3	3	2			
1	1				
0					

does.

We design a bijection between these Shi tableaux and the set of all words on an alphabet with m symbols having length $n-2$, i.e. $n-2$ letters.
Without loss of generality we consider the set $\{0, 1, \ldots, m-1\}$ to be the alphabet, so there is a natural total order on it.

We describe the map which associates to each such word a suitable Shi tableau, i.e. we give a procedure to univocally and bijectively determine the filling of the tableau from the given word.

Given a word $w = w_1 \cdots w_{n-2}$ let $\widetilde{w} = \widetilde{w}_1 \cdots \widetilde{w}_{n-2}$ be its sorted word, i.e. a permutation of the letters such that $\widetilde{w}_j \leq \widetilde{w}_{j+1}$ for all $j = 1, \ldots, n-3$. Moreover, for $j = 1, \ldots, n-2$, we define $\mathrm{Pos}\,(\widetilde{w}_j)$ in the following way:

if $j = 1$ or $\widetilde{w}_j > \widetilde{w}_{j-1}$

$$\mathrm{Pos}\,(\widetilde{w}_j) = \min\{1 \leq l \leq n-2 \text{ such that } w_l = \widetilde{w}_j\},$$

whereas if $j > 1$ and $\widetilde{w}_j = \widetilde{w}_{j-1}$

$$\mathrm{Pos}\,(\widetilde{w}_j) = \min\{\mathrm{Pos}(\widetilde{w}_{j-1}) + 1 \leq l \leq n-2 \text{ such that } w_l = \widetilde{w}_j\}.$$

For example, if $w = 61513$ then $\widetilde{w} = 11356$ and
$(\mathrm{Pos}\,(\widetilde{w}_1), \mathrm{Pos}\,(\widetilde{w}_2), \mathrm{Pos}\,(\widetilde{w}_3), \mathrm{Pos}\,(\widetilde{w}_4), \mathrm{Pos}\,(\widetilde{w}_5)) = (2, 4, 5, 3, 1)$.

We set $k_{1,n-1} = m$, and for all $j = 1, \ldots, n-2$ we set

$$k_{1,j} = \widetilde{w}_j$$

and

$$k_{j+1,n-1} = m - 1 - k_{1,j} = m - 1 - \widetilde{w}_j.$$

In other words, we are filling the tableau in the following way: according to the property (4) we set $k_{1,n-1} = m$, and the remaining cells of the first row are filled with the decreasing reordering of w. Then we fill the first column of the tableau according to the property (4).
Therefore, since now property (4) is completely fulfilled, in order to prove the Theorem it is enough to fill the remaining cells of the tableau in such a way that the entries are nonnegative integers strictly less than m and non-increasing along rows and columns, each of them satisfies either condition (1) or condition (2), and such that the whole operation is invertible.

For all $t = 2, \ldots, n-2$ and all $j = t, \ldots, n-2$ we set

$$k_{t,j} = \begin{cases} k_{1,j} - k_{1,t-1} & \text{if } \operatorname{Pos}(\widetilde{w}_{t-1}) < \operatorname{Pos}(\widetilde{w}_j) \\ k_{1,j} - k_{1,t-1} - 1 & \text{if } \operatorname{Pos}(\widetilde{w}_{t-1}) > \operatorname{Pos}(\widetilde{w}_j)\,; \end{cases}$$

in other words, we define

$$k_{t,j} = k_{1,j} - k_{1,t-1} - b_{t,j},$$

where

$$b_{t,j} = \begin{cases} 0 & \text{if } \operatorname{Pos}(\widetilde{w}_{t-1}) < \operatorname{Pos}(\widetilde{w}_j) \\ 1 & \text{if } \operatorname{Pos}(\widetilde{w}_{t-1}) > \operatorname{Pos}(\widetilde{w}_j)\,, \end{cases} \tag{5}$$

and therefore the numbers $b_{t,j}$ encode the inversion table of the word $w = w_1 \cdots w_{n-2}$. We remark that each entry $k_{t,j}$ of this filling is a nonnegative integer strictly less than m and satisfies either condition (1) or condition (2) with $i = 1$ (and with a rescaled t, i.e. shifted by 1).

This filling produces a Shi tableau: all entries are nonnegative integers which are non-increasing along rows and columns, and each entry, save $k_{1,n-1} = m$, is strictly less than m and satisfies either condition (1) or condition (2) for all $i = 1, \ldots, n-1$; checking that is quite a long, dull and tedious procedure, but it is completely straightforward from the definition, so we skip the details.

Moreover this operation is invertible, viz. that there is a unique and bijective way to recover the word given the Shi tableau, since the Shi tableau encodes both a sorted word and its inversion table.

□

For the sake of clarity, we present an example of such bijection: given the word $w = 61513$ over the alphabet $\{0, 1, 2, 3, 4, 5, 6\}$, its corresponding 7-Shi tableau is

7	6	5	3	1	1
5	**4**	4	2	0	
5	**4**	**3**	2		
3	**2**	**1**			
1	**1**				
0					

,

and an entry $k_{t,j}$ is written in bold if and only if the corresponding $b_{t,j}$ in (5) equals 1.

We now present an example of the inverse construction: consider the tableaux

$$\begin{array}{|c|c|c|c|c|c|}\hline 6&4&3&2&2&1\\\hline 4&2&2&1&1\\\cline{1-5} 3&1&1&0\\\cline{1-4} 3&1&0\\\cline{1-3} 2&1\\\cline{1-2} 1\\\cline{1-1}\end{array} \quad ;$$

very easily we get $\widetilde{w} = 12234$ and the inversion table

$$I = \begin{array}{|c|c|c|c|}\hline 1&1&0&0\\\hline 1&1&0\\\cline{1-3} 0&0\\\cline{1-2} 1\\\cline{1-1}\end{array}$$

given by (5). Then by the definition of the bijection and the first line of I we get that $\widetilde{w}_4 = 3$ and $\widetilde{w}_5 = 4$ are the only elements whose Pos is greater that $\mathrm{Pos}(\widetilde{w}_1)$, therefore we have $w = \{3,4\}1\{2,2\} = \{3,4\}122$. The second and the third line add no information (since they encode the positions of the two 2s, which are already determined), but by the forth line we get $\mathrm{Pos}\,(\widetilde{w}_5) < \mathrm{Pos}\,(\widetilde{w}_4)$, and therefore $w = 43122$.

The following Corollary illustrates a nice enumerative result which follows very easily from our bijection.

Corollary 3.2. *The number of Shi tableaux which are associated to dominant regions having $H_{\theta,m}$ as a separating wall, and such that they have a fixed first row $(k_{1,n-1},\dots,k_{1,1})$ equals the number of words having the letters $\{k_{1,1},\dots,k_{1,n-1}\}$ with the right multiplicity , i.e. the number of permutations of the set $\{k_{1,1},\dots,k_{1,n-1}\}$.*

Proof. Obvious by the definition of the bijection presented in the proof of Theorem 3.1. □

Acknowledgements

The authors would like to heartily thank the anonymous referees for a patient and careful reading of the manuscript and very helpful advice.

References

[1] C. A. Athanasiadis, Generalized Catalan numbers, Weyl groups and arrangements of hyperplanes, *Bull. Lond. Math. Soc.* **36** (2004), 294–302.

[2] C. A. Athanasiadis, On a refinement of the generalized Catalan numbers for Weyl groups, *Trans. Amer. Math. Soc.* **357** (2005), 179–196.

[3] C. A. Athanasiadis and S. Linusson, A simple bijection for the regions of the Shi arrangement of hyperplanes, *Discrete Math.* **204** (1999), 27-39.

[4] C. Berg, B. Jones and M. Vazirani, A bijection on core partitions and a parabolic quotient of the affine symmetric group, *J. Combin. Theory Ser. A* **116** (2009), 1344–1360.

[5] P. Cellini and P. Papi, Ad-nilpotent ideals of a Borel subalgebra. II, *J. Algebra* **258** (2002), 112–121.

[6] S. Fishel, M. Kallipoliti and E. Tzanaki, Facets of the Generalized Cluster Complex and Regions in the Extended Catalan Arrangement of Type A, *Electron. J. Combin.* Volume **20**, Issue 4 (2013), Paper 7, 21 pp.

[7] S. Fishel, E. Tzanaki and M. Vazirani, Counting Shi regions with a fixed separating wall, *Discrete Math. Theor. Comput. Sci. Proceedings*, 23rd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2011), (2011), 351–362.

[8] S. Fishel, E. Tzanaki and M. Vazirani, Counting Shi regions with a fixed separating wall, *Ann. Comb.* **17** (2013), 671–693, available at http://dx.doi.org/10.1007/s00026-013-0201-x

[9] S. Fishel and M. Vazirani, A bijection between (bounded) dominant Shi regions and core partitions, *Discrete Math. Theor. Comput. Sci. Proceedings*, 22nd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2010), (2010), 283–294.

[10] S. Fishel and M. Vazirani, A bijection between dominant Shi regions and core partitions, *European J. Combin.* **31** (2010), 2087–2101.

[11] M. D. Haiman, Conjectures on the quotient ring by diagonal invariants, *J. Algebraic Combin.* **3** (1994) 17–76.

[12] G. D. James, Some combinatorial results involving Young diagrams, *Math. Proc. Cambridge Philos. Soc.* **83** (1978), 1–10.

[13] C. Krattenthaler, L. Orsina and P. Papi, Enumeration of ad-Nilpotent $\mathfrak{b}$-Ideals for Simple Lie Algebras, *Adv. in Appl. Math.* **28** (2002), 478–522.

[14] L. Lapointe and J. Morse, Tableaux on $k+1$-cores, reduced words for affine permutations, and k-Schur expansions, *J. Combin. Theory Ser. A* **112** (2005), 44–81.

[15] A. Lascoux, Ordering the affine symmetric group, *Algebraic combinatorics and applications*, Proceedings of the Euroconference, ALCOMA, Gößweinstein, Germany, September 12-19, 1999, A. Betten et al. (ed.), Springer, Berlin (2001), 219–231.

[16] K. Mészáros, Labeling the regions of the type C_n Shi arrangement, *Electron. J. Combin.* Volume **20**, Issue 2 (2013), Paper 31, 12 pp.

[17] A. O. Morris and A. K. Yaseen, Some combinatorial results involving shifted Young diagrams, *Math. Proc. Cambridge Philos. Soc.* **99** (1986), 23–31.

[18] P. Orlik and H. Terao, *Arrangements of hyperplanes*, Springer-Verlag, Berlin 1992.

[19] M. J. Richards, Some decomposition numbers for Hecke algebras of general linear groups, *Math. Proc. Cambridge Philos. Soc.* **119** (1996), 383–402.

[20] J.-Y. Shi, The Kazhdan-Lusztig cells in certain affine Weyl groups, *Lecture Notes in Mathematics* **1179**, Springer-Verlag, Berlin (1986).

[21] J.-Y. Shi, Alcoves corresponding to an affine Weyl group, *J. Lond. Math. Soc. (2)* **35** (1987), 42–55 (1987).

[22] J.-Y. Shi, Sign types corresponding to an affine Weyl group, *J. Lond. Math. Soc. (2)* **35** (1987), 56–74.

[23] J.-Y. Shi, The number of $\oplus$-sign types, *Quart. J. Math.* **48** (1997), 93–105.

[24] R. P. Stanley, Hyperplane arrangements, parking functions and tree inversions, in "Mathematical essays in honor of Gian-Carlo Rota's 65th birthday" B. E. Sagan, et al. (ed.), *Prog. Math.* **161**, Birkhäuser, Boston, MA, (1998) 359–375.

[25] R. P. Stanley, An Introduction to Hyperplane Arrangements, *Geometric Combinatorics* (E. Miller, V. Reiner, and B. Sturmfels, eds.), IAS/Park City Mathematics Series, vol. 13, American Mathematical Society, Providence, RI, 2007, pp. 389–496.

[26] M. Wildon, Counting partitions on the abacus, *Ramanujan J.* **17** (2008), 355–367.

(Received 28 Dec 2013; revised 26 June 2014)