AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **60(2)** (2014), Pages 109–127

Prüfer codes for acyclic hypertrees

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Abstract

We describe a new Prüfer code based on star-reductions which works for infinite acyclic hypertrees.

1 Main results

A famous theorem attributed to Cayley states that there are n^{n-2} finite trees with vertices $\{1, \dots, n\}$. Prüfer gave in [3] a beautiful proof by constructing a one-to-one correspondence between such trees and elements in the set $\{1, \dots, n\}^{n-2}$ of all n^{n-2} words of length $n-2$ in the alphabet $\{1, \dots, n\}$. More precisely, we obtain the Prüfer code of a tree with $n \geq 2$ vertices $\{1, \dots, n\}$ by successively pruning smallest leaves and writing down their neighbours until reaching a tree reduced to a unique edge. Selivanov in [5] generalized Prüfer's construction to acyclic hypertrees. A different Prüfer code for arbitrary (not necessarily acyclic) finite hypertrees was described by C. Rényi and A. Rényi in [4].

Prüfer's construction and its subsequent generalizations can be succinctly described as "pruning trees". The aim of this paper is to describe a Prüfer code based on a different kind of simplification, star-reduction, which merges hyperedges until reaching the trivial hypertree consisting of a unique hyperedge containing all vertices. The resulting Prüfer code respects degrees (a vertex of degree a occurs with multiplicity $a-1$ in the corresponding Prüfer word) a property shared with Prüfer's original construction. Its definition is perhaps slightly less straightforward but it has the additional feature that it works for trees and acyclic hypertrees which are infinite, a property which fails to hold in the general case for the classical construction of Prüfer.

The rest of this paper is organized as follows: Section 2 recalls briefly the definition of acyclic hypertrees. Section 3 describes the Prüfer partition. The Prüfer

* This work has been partially supported by the LabEx PERSYVAL-Lab (ANR–11-LABX-0025). The author is a member of the project-team GALOIS supported by this LabEx.

partition of an ordinary rooted tree is trivial and carries no information. It is however a necessary ingredient for the Prüfer code of an acyclic hypertree with hyperedges of size larger than 2. Section 4 contains an intuitive definition of Prüfer codes and reviews a few enumerative results. Sections 5 and 6 recall the definition of the classical Prüfer code and its generalization to finite rooted acyclic hypertrees. Sections 2-6 are expository and contain nothing new.

Sections 7-8 describe the Prüfer code $T \longmapsto (\mathcal{P}, W^*)$ based on star-reductions (mergings of hyperedges). As far as I am aware, this construction has not appeared elsewhere in the litterature.

Section 9 generalizes the construction of the Prüfer code based on star-reductions to infinite acyclic hypertrees.

The final Section 10 illustrates the extension to infinite trees by an example which gives rise to bijections in the set S_n of all elements in the symmetric group of $\{1, \ldots, n\}$.

2 Hypergraphs and hypertrees

A *hypergraph* is a pair $(\mathcal{V}, \mathcal{E})$ consisting of a set $\mathcal{V}$ of *vertices* and of a set $\mathcal{E}$ of *hyperedges* given by subsets of $\mathcal{V}$ containing at least two elements. In order to avoid complications we require moreover that no hyperedge is a subset of another hyperedge in a hypergraph. A hypergraph is *finite* if it contains only finitely many vertices and finitely many hyperedges.

Except in Sections 9 and 10 we consider henceforth mainly only finite hypergraphs (and hypertrees) consisting of a finite number of vertices and hyperedges. We denote by $\text{size}(e)$ the cardinality of a hyperedge e, defined as the number of vertices contained in e, and by $\deg(v)$ the degree of a vertex v given by the number of hyperedges containing v. A vertex of degree 1 is a *leaf*. Both numbers $\text{size}(e)$ and $\deg(v)$ can be arbitrary (perhaps infinite) cardinal numbers. A hypergraph is *locally finite* if it has only edges of finite size and vertices of finite degree.

Two distinct vertices $v, w \in \mathcal{V}$ of a common hyperedge are *adjacent* or *neighbours*. A *path* of *length* l joining two vertices $v, w \in \mathcal{V}$ is a sequence $v = v_0, v_1, \ldots, v_l = w$ involving only consecutively adjacent (and distinct) vertices. A *cycle* is a closed path involving only distinct vertices. A hypergraph is *connected* if two vertices can always be joined by a path. The *distance* between two vertices v, w of a connected hypergraph is the length of a shortest path joining v and w. *Geodesics* (or *shortest paths*) are paths $\ldots, v_i, \ldots, v_j, \ldots$ with v_i, v_j at distance $|i-j|$ for all indices i, j. We set $d(v, w) = \infty$ if v and w belong to different connected components.

A connected hypergraph is an *acyclic hypertree* if every cycle with more than one vertex is contained in a unique hyperedge. In particular, two distinct hyperedges of an acyclic hypertree have at most one vertex in common (otherwise we get a cycle consisting of two common vertices contained in the intersection of two different hyperedges which is contained in two different hyperedges) and geodesics between two given vertices are unique, a property which we will use in a crucial way in the sequel. The definition of a general hypertree (a connected hypergraph admitting a tree as a host graph where a host graph is a graph with the same vertex-set inducing connected subgraphs in every hyperedge) will not be needed in the sequel. Observe that two hyperedges of a general hypertree can intersect in more than one vertex. This destroys in general uniqueness of geodesics.

Remark 2.1. The terminology "acyclic hypertree" can be motivated as follows: Associate to a hypergraph the cell-complex with simplices defined by hyperedges, a hyperedge of size s corresponding to an $(s-1)-$dimensional simplex, with simplices glued together along common vertices. Two simplices intersect thus in general in a finite set of vertices and not in common simplex as usual in simplicial complexes. The cell-complex associated in this way to a hypergraph $(\mathcal{V}, \mathcal{E})$ is connected and acyclic (and contractible) if and only if $(\mathcal{V}, \mathcal{E})$ is an acyclic hypertree.

Proposition 2.2. *(i) We have*

$$\sum_{e \in \mathcal{E}} \text{size}(e) = \sum_{v \in \mathcal{V}} \deg(v) \geq n + k - 1 \tag{1}$$

for a connected finite hypergraph with n vertices and k hyperedges.

(ii) A connected finite hypergraph is an acyclic hypertree if and only if equality holds in (1).

Corollary 2.3. *We have*

$$\sum_{e \in \mathcal{E}} (\text{size}(e) - 1)) = n - 1 \tag{2}$$

and

$$\sum_{v \in \mathcal{V}} (\deg(v) - 1)) = k - 1 \tag{3}$$

for a finite acyclic hypertree with n vertices and k hyperedges.

Proof of Proposition 2.2 Inequality (1) holds obviously for a finite (hyper)graph consisting of $n \geq 1$ isolated vertices with equality only in the case $n = 1$. Removing a hyperedge of size a from a given hypergraph reduces $\sum_{e \in \mathcal{E}} \text{size}(e)$ and $\sum_{v \in \mathcal{V}} \deg(v)$ by $a \geq 2$ and $n + k - 1$ by 1. This implies inequality (1) by induction.

In order to prove (ii), we consider a leaf v in a finite acyclic hypertree. If v belongs to a hyperedge e' of size > 2, we remove v from e'. This reduces $\sum_{e \in \mathcal{E}} \text{size}(e)$, $\sum_{v \in \mathcal{V}} \deg(v)$ and n by 1 and does not change the number k of hyperedges. If v belongs to an ordinary edge e' (of size 2), we remove v and e'. This reduces $\sum_{e \in \mathcal{E}} \text{size}(e)$ and $\sum_{v \in \mathcal{V}} \deg(v)$ by 2 and both numbers n, k by 1. Equality in (1) holds thus for acyclic hypertrees by induction on the number of vertices.

A finite hypergraph which is not an acyclic hypertree contains either a non-leaf in a hyperedge of size ≥ 3 or it contains a cycle consisting of at least 2 ordinary edges.

Removing such a non-leaf in the first case reduces $\sum_{e\in\mathcal{E}}\mathrm{size}(e)$ and $\sum_{v\in\mathcal{V}}\deg(v)$ but keeps $n+k-1$ unchanged. Removing an ordinary leaf in a nontrivial cycle reduces $\sum_{e\in\mathcal{E}}\mathrm{size}(e)$ and $\sum_{v\in\mathcal{V}}\deg(v)$ by 2 and $n+k-1$ by 1. Inequality (1) is thus strict for such a hypergraph. □

A hypertree is *trivial* if it is reduced to a unique hyperedge. Equivalently, a connected graph is a trivial hypertree if all its vertices are leaves.

Proposition 2.4. *(i) We have $k < n$ for a finite acyclic hypertree with n vertices and k hyperedges.*

(ii) Every finite acyclic hypertree with $n \geq 2$ vertices contains at least two leaves.

Proof Since every hyperedge e contains at least $\mathrm{size}(e) \geq 2$ vertices we have $n-1 = \sum_{e\in\mathcal{E}}(\mathrm{size}(e)-1) \geq k$. This shows (i). Since $n-1 > k-1 = \sum_{v\in\mathcal{V}}(\deg(v)-1)$ there exists at least two vertices contributing nothing to the sum $\sum_{v\in\mathcal{V}}(\deg(v)-1)$. □

A connected infinite hypergraph is an *acyclic hypertree* if every connected subgraph induced by a finite number of vertices (with hyperedges given by intersections containing at least two vertices of original hyperedges with the finite subset of vertices under consideration) is a finite acyclic hypertree. Equivalently, an infinite hypergraph is an acyclic hypergraph if all its connected finite subhypergraphs are acyclic hypertrees.

Proposition 2.5. *Two vertices v, w at distance l in an acyclic hypertree T are joined by a unique shortest path $v = v_0, v_1, \ldots, v_l = w$ defining a unique sequence $e_1, \ldots, e_l$ of hyperedges such that $\{v_{i-1}, v_i\} \subset e_i$.*

We leave the proof to the reader. □

3 The Prüfer partition of a rooted acyclic hypertree

A *rooted acyclic hypertree* has a marked root vertex r among its vertices. The root vertex r induces a *marked vertex* e_* closest (at minimal distance) to the root in every hyperedge e of a rooted acyclic hypertree. In particular, the root-vertex r is the marked vertex of every hyperedge containing r. Removal of the marked vertex e_* from e yields the *reduced hyperedge* $e' = e \setminus \{e_*\}$ consisting of all *unmarked vertices* of e. No reduced hyperedge contains the root. We use the notation $\{v_1, \ldots, v_{k-1}\}_w$ for a hyperedge $e = e' \cup \{e_*\}$ of size k with marked vertex $e_* = w$ and reduced hyperedge $e' = \{v_1, \ldots, v_{k-1}\}$, see Section 3.2 for an example.

Proposition 3.1. *A non-root vertex v of a rooted acyclic hypertree is the marked vertex of $\deg(v)-1$ hyperedges. The root r is the marked vertex of $\deg(r)$ hyperedges.*

The proof is left to the reader. □

Corollary 3.2. *Reduced hyperedges of a rooted acyclic hypertree T with vertices $\mathcal{V}$, root r and k hyperedges partition the set $\mathcal{V} \setminus \{r\}$ into k non-empty subsets.*

The *Prüfer partition* $\mathcal{P}(T)$ of an $r-$rooted acyclic hypertree T with vertices $\mathcal{V}$ is the partition of $\mathcal{V}' = \mathcal{V} \setminus \{r\}$ into reduced hyperedges given by Corollary 3.2.

The definition of Prüfer partitions can easily be generalized to arbitrary (not necessarily locally finite) infinite rooted acyclic hypertrees.

In the sequel, we will mainly consider acyclic hypertrees with non-root vertices $\mathcal{V}'$ given by (perhaps infinite) subsets of $\{1, 2, \dots\}$. The Prüfer partition of such an acyclic hypertree is thus given by a partition of $\mathcal{V}'$.

3.1 Partition maps

A map $p : E \longrightarrow E$ of a set E is *idempotent* if $p = p \circ p$. Equivalently, a map $p : E \longrightarrow E$ is idempotent if its image $p(E)$ is its set of fixed points. Idempotent maps of a set E are in one-to-one correspondence with partitions of E decorated with a marked element in each part. Indeed, such a decorated partition gives rise to an idempotent map by sending each element to the marked element of its part. In the opposite direction, an idempotent map p gives rise to a partition with marked elements given by fixed points and parts given by preimages of fixed points.

A set E is well-ordered if E is endowed with an order relation which yields a least element in every non-empty subset of E. A map $p : E \longrightarrow E$ of a well-ordered set is *lowering* if $p(x) \leq x$ for all x.

Well-ordering a set E selects least elements as the canonical marked elements in parts of a partition. Partitions of a well-ordered set E are thus in one-to-one correspondence with maps $p : E \longrightarrow E$ which are idempotent and lowering. We call such a map a *partition map*.

3.2 An example of a Prüfer partition

We consider the finite rooted acyclic hypertree T with vertices $1, \dots, 14$, root-vertex 14 and 8 hyperedges given by

$$\{1, 10, 12\}_8, \{2\}_1, \{3, 9\}_4, \{4, 7\}_8, \{5\}_{14}, \{6\}_4, \{8, 13\}_{14}, \{11\}_7$$

where $\{3, 9\}_4$ for example represents a hyperedge of size 3 with marked vertex 4 and reduced hyperedge $\{3, 9\}$.

Figure 1 shows T with hyperedges represented by shaded polygons or ordinary edges.

The Prüfer partition $\mathcal{P}(T)$ of the acyclic hypertree T with root vertex 14 is thus the partition

$$\{1, 10, 12\} \cup \{2\} \cup \{3, 9\} \cup \{4, 7\} \cup \{5\} \cup \{6\} \cup \{8, 13\} \cup \{11\}$$

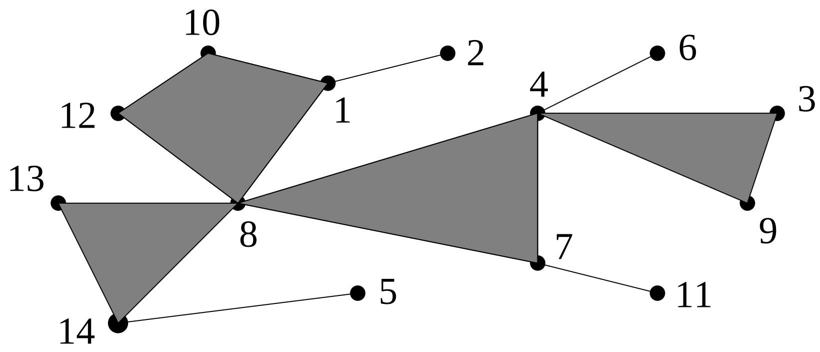

Figure 1: An acyclic hypertree

defined by the union of all reduced hyperedges of T. The corresponding partition map is given by

$$p(1, 10, 12) = 1,\ p(2) = 2,\ p(3, 9) = 3,\ p(4, 7) = 4,$$
$$p(5) = 5,\ p(6) = 6,\ p(8, 13) = 8,\ p(11) = 11\ .$$

3.3 The spine of a rooted acyclic hypertree

We associate in this short digressional section an ordinary rooted tree (the spine) to every rooted acyclic hypertree.

The *spine* of a rooted acyclic hypertree T with root r and vertices $\mathcal{V}$ is the ordinary tree $Sp(T)$ with root r, vertices $\overline{\mathcal{V}} = \mathcal{P} \cup \{r\}$ where elements of $\mathcal{P}$ are parts involved in the Prüfer partition $\mathcal{P}$ of T and with edges $\{A, B\}$ if an element of A is adjacent to an element of B. There is an obvious projection $\pi : \mathcal{V} \longrightarrow \overline{\mathcal{V}}$ defined by $\pi(r) = r$ and $\pi(v) = e'$ if $v \in \mathcal{V} \setminus \{r\}$ is contained in the reduced hyperedge e'. Edges of $Sp(T)$ are in one-to-one correspondence with hyperedges of T and are given by $\{e', \pi(e_*)\}$ for a hyperedge e of T. An ordinary rooted tree is its proper spine (up to isomorphism).

We state the following result without proof:

Proposition 3.3. *(i)* $Sp(T) = T$ *if and only if* T *is an ordinary rooted tree.*

(ii) $d(\pi(v), \pi(w)) \leq d(v, w)$ *for* $v, w \in \mathcal{V}$ *and* $d(\pi(v), \pi(r)) = d(v, r)$ *for every vertex* v *in* $\mathcal{V}$.

Remark 3.4. *There exists a second natural map which associates a rooted tree to every rooted acyclic hypertree and which does not modify the set of vertices: replace every hyperedge* e *of* T *with* $(\text{size}(e) - 1)$ *ordinary edges given by* $\{v, e_*\}$ *for* v *in* e'. *Proposition 3.3 holds also for this construction except for the inequality of assertion (ii) which has to be replaced by the opposite inequality.*

4 Glue maps, Prüfer words and enumeration of labelled trees

We consider a fixed rooted acyclic hypertree T with root r and non-root vertices $\mathcal{V}'$. We denote by $g : \mathcal{V}' \longrightarrow \mathcal{V}$ the map defined by $g(v) = w$ where w is parent of v, ie. w is the unique neighbour of v closer to the root-vertex r than v. The map g sends thus every vertex v of a reduced hyperedge e' to the marked vertex e_* of the associated hyperedge $e = e' \cup \{e_*\}$. We extend g to all vertices $\mathcal{V} = \mathcal{V}' \cup \{r\}$ of T by setting $g(r) = r$. We say that the map g defines the marked vertices of T and we call g the *glue-map* of the $r-$rooted tree T.

The sequence $v, g(v), g^2(v) = g(g(v)), \ldots$ of iterates of g is eventually constant and defines (up to repetitions of the root vertex) the unique geodesic joining a vertex v of T to the root vertex r.

Given a partition $\mathcal{P}$ of a set $\mathcal{V}' = \mathcal{V} \setminus \{r\}$, a map $g : \mathcal{V} \longrightarrow \mathcal{V}$ is $\mathcal{P}-$*admissible* if there exists an $r-$rooted acyclic hypertree with vertices $\mathcal{V}$, Prüfer partition $\mathcal{P}$ and glue-map g.

Proposition 4.1. *(i) Given a partition $\mathcal{P}$ of $\mathcal{V}' = \mathcal{V} \setminus \{r\}$, a map $g : \mathcal{V} \longrightarrow \mathcal{V}$ is $\mathcal{P}-$admissible if and only if the restriction of g to a part of $\mathcal{P}$ is constant and for every vertex v there exists an integer $k = k(v)$ such that $g^k(v) = g^{k+1}(v) = r$ where g^k denotes the $k-$fold iterate $g \circ g \circ \cdots \circ g$ of g.*

(ii) $\mathcal{P}-$admissible partitions are in one-to-one correspondance with $r-$rooted acyclic hypertrees having vertices $\mathcal{V}$ and Prüfer partition $\mathcal{P}$.

Proof Associate to a part e' of $\mathcal{P}$ the hyperedge with vertices $e' \cup \{g(e')\}$. The condition $g^k(v) = r$ shows that these hyperedges define a connected hypergraph. Since we have equality in inequality (2) for every finite connected subhypergraph containing r, the resulting connected hypergraph is an acyclic hypertree. This shows (i). Assertion (ii) is obvious. □

Remark 4.2. A glue map g of an acyclic hypertree T with partition map p is completely determined by $g(r) = r$ and by its restriction $g|_{p(\mathcal{V}')} : p(\mathcal{V}') \longrightarrow \mathcal{V}$ to the set of fixed points (smallest elements in reduced hyperedges) of p.

A *Prüfer word* is a one-to-one correspondence between the set of $\mathcal{P}-$admissible maps $\mathcal{V}' \longrightarrow \mathcal{V}$ and the set $\mathcal{V}^{k-1}$ of all words of length $k-1$ (with k denoting the number of non-empty parts in the partition $\mathcal{P}$ of $\mathcal{V}'$) in the alphabet $\mathcal{V}$ satisfying the following two conditions:

1. The degree of a non-root vertex v in the acyclic hypertree associated to a $\mathcal{P}-$admissible map g is one more than the number of occurences of the vertex v in the word $W \in \mathcal{V}^{k-1}$ corresponding to g.

2. The Prüfer word is given by a simple algorithm which is fast (polynomial in any reasonable sense) for finite acyclic hypertrees.

A Prüfer word is thus an elegant way to recover the loss of information of the map $T \longmapsto \mathcal{P}(T)$ induced by the Prüfer partition.

Formula (3) implies that Condition (1) in a Prüfer word is also fulfilled by the root-vertex. Exempting the root from Condition (1) is motivated by Section 9 dealing with infinite hypergraphs.

A *Prüfer code* is a map $T \longmapsto (\mathcal{P}, W)$ where $\mathcal{P}$ is the Prüfer partition of the non-root vertices V' of an acyclic hypertree T and where W is a Prüfer word.

Sections 5 and 6 contain a description of a Prüfer code due to Selivanov, see [5]. It uses removal of hyperedges of a certain type and decreases the number of vertices.

The main result of this paper is a construction of a new Prüfer code. It is based on merging hyperedges with a common intersection into one larger hyperedge and does not change the set of vertices. It has moreover the interesting feature that it works for suitably defined infinite acyclic hypertrees, as outlined in Section 9 and illustrated in Section 10.

We denote by $\mathcal{T}(\mathcal{V}, \mathcal{P})$ the set of all finite rooted acyclic hypertrees with vertices $\mathcal{V} = \mathcal{V}' \cup \{r\}$ and with a given fixed Prüfer partition $\mathcal{P}$ of $\mathcal{V}'$. The existence of Prüfer codes implies easily the following standard result in enumerative combinatorics:

Corollary 4.3. *Associating to an acyclic hypertree $T \in \mathcal{T}(\mathcal{V}, \mathcal{P})$ the monomial*

$$w(T) = \prod_{v \in \mathcal{V}} x_v^{\deg(v)-1}$$

we have

$$\sum_{T \in \mathcal{T}(\mathcal{V},\mathcal{P})} w(T) = \left(\sum_{v \in \mathcal{V}} x_v \right)^{k-1}$$

where k denotes the number of (non-empty) parts in $\mathcal{P}$.

In particular, there are

$$S_2(k, n-1) n^{k-1}$$

acyclic hypertrees with k hyperedges and vertices $\{1, \ldots, n\}$ where $S_2(k, n)$ denotes the Stirling number of the second kind enumerating the number of partitions of $\{1, \ldots, n\}$ into k non-empty subsets.

5 Hyperedges of leaf-type and the map $T \longmapsto W(T)$

The next two sections describe Selivanov's generalization of Prüfer codes to acyclic finite hypertrees, see [5].

A hyperedge e of a rooted acyclic hypertree T is of *leaf-type* if all vertices of the associated reduced hyperedge e' are leaves.

Proposition 5.1. *Every finite rooted acyclic hypertree not reduced to its root has a hyperedge of leaf-type.*

Proof A hyperedge containing a vertex at maximal distance from the root-vertex is of leaf-type. □

Proposition 5.2. *Given a hyperedge e of a non-trivial acyclic hypertree T with vertices $\mathcal{V}$, root r and hyperedges $\mathcal{E}$, the set $\mathcal{E} \setminus \{e\}$ is the set of hyperedges of an r-rooted acyclic hypertree with vertices $\mathcal{V} \setminus e'$ if and only if e is of leaf-type.*

We leave the proof to the reader. □

We consider henceforth acyclic hypertrees with vertices given by a finite subset of $\mathbb{N}$, rooted at the largest vertex and with hyperedges totally ordered according to their smallest unmarked vertex. We construct the *Prüfer word* $W(T)$ of such an acyclic hypertree T by successively removing the smallest hyperedge of leaf-type until reaching a trivial acyclic hypertree reduced to a unique hyperedge and by writing down the sequence of marked vertices of the removed hyperedges.

The following result is useful for the computation of the Prüfer word of a tree given as a list of hyperedges:

Proposition 5.3. *A hyperedge e of a rooted acyclic hypertree T is of leaf-type if and only if no element of the associated reduced hyperedge $e' = e \setminus \{e_*\}$ occurs as a marked vertex among the other hyperedges of T.*

We leave the easy proof to the reader. □

5.1 An example of a Prüfer word

The Prüfer word $w_1 \dots w_7$ of the acyclic hypertree T represented by Figure 1 of Section 3.2 can be computed as follows: We start with the increasing sequence of all hyperedges, ordered according to their smallest non-marked vertex. We iterate then the following loop: We search the first hyperedge e of leaf-type using for example Proposition 5.3. We remove e and we write down the marked vertex e_* of the removed hyperedge e. We stop if only a unique hyperedge remains.

For our example represented in Figure 1, we get the increasing sequences of hyperedges

$$\{1,10,12\}_8, \{\mathit{2}\}_{\mathit{1}}, \{\mathit{3},\mathit{9}\}_{\mathit{4}}, \{4,7\}_8, \{\mathit{5}\}_{\mathit{14}}, \{\mathit{6}\}_{\mathit{4}}, \{8,13\}_{14}, \{\mathit{11}\}_{\mathit{7}}$$
$$\{\mathit{1},\mathit{10},\mathit{12}\}_{\mathit{8}}, \{\mathit{3},\mathit{9}\}_{\mathit{4}}, \{4,7\}_8, \{\mathit{5}\}_{\mathit{14}}, \{\mathit{6}\}_{\mathit{4}}, \{8,13\}_{14}, \{\mathit{11}\}_{\mathit{7}}$$
$$\{\mathit{3},\mathit{9}\}_{\mathit{4}}, \{4,7\}_8, \{\mathit{5}\}_{\mathit{14}}, \{\mathit{6}\}_{\mathit{4}}, \{8,13\}_{14}, \{\mathit{11}\}_{\mathit{7}}$$
$$\{4,7\}_8, \{\mathit{5}\}_{\mathit{14}}, \{\mathit{6}\}_{\mathit{4}}, \{8,13\}_{14}, \{\mathit{11}\}_{\mathit{7}}$$
$$\{4,7\}_8, \{\mathit{6}\}_{\mathit{4}}, \{8,13\}_{14}, \{\mathit{11}\}_{\mathit{7}}$$
$$\{4,7\}_8, \{8,13\}_{14}, \{\mathit{11}\}_{\mathit{7}}$$
$$\{\mathit{4},\mathit{7}\}_{\mathit{8}}, \{8,13\}_{14}$$
$$\{\mathit{8},\mathit{13}\}_{\mathit{14}}$$

with hyperedges of leaf-type in italics. The acyclic hypertree T corresponds thus to the Prüfer word 1 8 4 14 4 7 8.

6 The inverse map $(\mathcal{P}, W) \longmapsto T$

A part e' in a partition $\mathcal{P}$ with k non-empty parts of a subset $\mathcal{S}$ of $\{1, \ldots, n-1\}$ is of *leaf-type with respect to a word* $W \in \mathbb{N}^*$ over the alphabet $\mathbb{N}$ if W involves no elements of e'.

Lemma 6.1. *A partition $\mathcal{P}$ into k non-empty parts of a subset of $\{1, \ldots, n-1\}$ contains at least one part of leaf-type with respect to a word W in the set $\{1, \ldots, n\}^{k-1}n$.*

Proof The last letter n of W does not occur in any part of $\mathcal{P}$ and the number of remaining letters in W is one less than the number of parts in $\mathcal{P}$. □

We consider a pair $(\mathcal{P}, W)$ consisting of a Prüfer partition $\mathcal{P}$ of $\mathcal{S} \subset \{1, \ldots, n-1\}$ into k non-empty parts and of a Prüfer word $W \in (\mathcal{S} \cup \{n\})^{k-1}$. In order to construct the associated acyclic hypertree T rooted at n, it is enough to determine the glue map g defining the marked vertex $e_* = g(e')$ of every reduced hyperedge e' appearing in $\mathcal{P}$. This can be achieved as follows: We order the elements of $\mathcal{P}$ totally according to their smallest element and we augment $W = w_1 \ldots w_{k-1}$ by adding a last letter $w_k = n$. We have thus $W = w_1 \ldots w_{k-1} n$. We iterate now the following loop: By Lemma 6.1 there exists a smallest part e' of $\mathcal{P}$ which is of leaf-type with respect to $W = w_1 \ldots w_{k-1}$. We get in this way the hyperedge e'_{w_1} (given by all elements in e' and by the marked vertex w_1) of T. We remove now e' from $\mathcal{P}$, we erase w_1 in W and we iterate until $\mathcal{P}$ is empty.

6.1 An example for the inverse map $(\mathcal{P}, W) \longmapsto T$

We reconstruct the acyclic hypertree T of Figure 1 from its Prüfer code consisting of the Prüfer partition

$$\{1, 10, 12\} \cup \{2\} \cup \{3, 9\} \cup \{4, 7\} \cup \{5\} \cup \{6\} \cup \{8, 13\} \cup \{11\}$$

and of the Prüfer word $W = 1\ 8\ 4\ 14\ 4\ 7\ 8$. The computation for T is as follows

$$\begin{array}{l|l}
\{1,10,12\},\{\mathit{2}\},\{\mathit{3},\mathit{9}\},\{4,7\},\{\mathit{5}\},\{\mathit{6}\},\{8,13\},\{\mathit{11}\} & 1 \\
\{\mathit{1},\mathit{10},\mathit{12}\},\{\mathit{3},\mathit{9}\},\{4,7\},\{\mathit{5}\},\{\mathit{6}\},\{8,13\},\{\mathit{11}\} & 8 \\
\{\mathit{3},\mathit{9}\},\{4,7\},\{\mathit{5}\},\{\mathit{6}\},\{8,13\},\{\mathit{11}\} & 4 \\
\{4,7\},\{\mathit{5}\},\{\mathit{6}\},\{8,13\},\{\mathit{11}\} & 14 \\
\{4,7\},\{\mathit{6}\},\{8,13\},\{\mathit{11}\} & 4 \\
\{4,7\},\{8,13\},\{\mathit{11}\} & 7 \\
\{\mathit{4},\mathit{7}\},\{8,13\} & 8 \\
\{\mathit{8},\mathit{13}\} & 14
\end{array}$$

where the first columns displays relevant sets of reduced hyperedges with hyperedges of leaf-type in italics and where the last column contains the letters of the Prüfer word augmented with an additional letter $w_8 = 14$ representing the root-vertex. We get the hyperedges

$$\{1, 10, 12\}_8, \{2\}_1, \{3, 9\}_4, \{4, 7\}_8, \{5\}_{14}, \{6\}_4, \{8, 13\}_{14}, \{11\}_7$$

defining the acyclic hypertree T of Figure 1 by marking (indexing) the first reduced hyperedge of leaf type (written in italics) of every row with the corresponding letter of W.

7 Star-reduction and the map $T \longmapsto W^*(T)$

A *hyperstar* is an acyclic hypertree of diameter at most 2. The *center* of a hyperstar of diameter 2 is the unique vertex adjacent to all other vertices. It is given by the intersection of at least two hyperedges. Every vertex is a center of the trivial hyperstar reduced to a unique hyperedge.

The hyperstar $St(v)$ of an acyclic hypertree T at a vertex v of T is the subtree of T formed by v and by all its neighbours. Its hyperedges are all hyperedges of T which contain v.

The *star-reduction* of T at a vertex v is the acyclic hypertree $*_v(T)$ obtained by replacing all hyperedges of T involved in the hyperstar $St(v)$ by a unique hyperedge consisting of all vertices in $St(v)$.

Proposition 7.1. *(i) We have $*_v(T) = T$ if and only if v is leaf of T.*

(ii) v is a leaf of $*_v(T)$.

(iii) We have $*_v(*_w(T)) = *_w(*_v(T))$ for any pair of vertices v, w in T.

Proofs are easy and left to the reader.

Assertion (iii) of Proposition 7.1 allows to define $*_S(T)$ for a subset S of vertices. Given an acyclic hypertree T with vertices $\{1, \dots, n\}$, we use the shorthand $*_{\leq v}(T)$ for the star-reduction $*_{\{1,\dots,v\}}(T)$ at the subset $\{1, \dots, v\}$ of all vertices not exceeding v. All vertices $1, \dots, v$ of $*_{\leq v}(T)$ are leaves. Similarly, we use $*_{<v}(T)$ for $*_{\{1,\dots,v-1\}}(T)$ using the convention $*_{<1}(T) = T$.

The Prüfer word $W^*(T)$ of an acyclic hypertree T with vertices $\{1, \dots, n\}$ and k hyperedges is defined as follows:

We set $W^*(T) = n^{k-1}$ if T is a hyperstar with k hyperedges centered at its root-vertex n.

Otherwise, there exists a smallest non-leaf $v < n$ in T and we can define the increasing sequence (ordered by smallest unmarked elements) $\mathcal{E}_v = (e_1, \dots, e_{k-1})$ of all $k - 1$ hyperedges not containing the smallest non-leaf v as an unmarked vertex. In other words, the sequence $\mathcal{E}_v$ is obtained by removing the unique hyperedge e containing v in its reduced hyperedge e' from the increasing sequence (ordered by smallest unmarked elements) of all k hyperedges of T. Since v is a non-leaf there exist an increasing sequence $\mathcal{S}_v = 1 \leq i_1 < \cdots < i_a \leq k - 1$ consisting of all $a = \deg(v) - 1 > 0$ indices $i_1, \dots, i_a$ such that the hyperedges $e_{i_1}, \dots, e_{i_a}$ of $\mathcal{E}_v$ have marked vertex v. We set $w_{i_1} = \cdots = w_{i_a} = v$ for the a letters of $W^*(T) = w_1 w_2 \dots$ with indices $i_1, \dots, i_a$ in $\mathcal{S}_v$. The subword formed by the $k - 1 - a$ remaining letters

$$w_1 \dots w_{i_1-1} \widehat{w_{i_1}} w_{i_1+1} \dots \widehat{w_{i_2}} \dots w_{i_a-1} \widehat{w_{i_a}} w_{i_a+1} \dots w_{k-1}$$

of $W^*(T)$ corresponds to the Prüfer word $W^*(*_v(T))$ of the star reduction $*_v(T)$ of T at the vertex v.

Remark 7.2. The set $\mathcal{T}(n)$ of all acyclic hypertrees with vertices $1, \ldots, n$ rooted at n carries two interesting additional structures:

1. It is a ranked poset for the order-relation given by $T \geq *_S T$ for any subset $S \subset \{1, \ldots, n\}$ of vertices. This poset has a unique minimal element given by the trivial acyclic hypertree consisting of a unique hyperedge. Its rank function $rk(T)$ is given by the number $nl(T)$ of vertices which are non-leaves and its Möbius function is $(-1)^{nl(T)}$.

2. The elements of $\mathcal{T}(n)$ are the vertices of a rooted acyclic hypertree. The root vertex is again the trivial acyclic hypertree reduced to a unique hyperedge with vertices $1, \ldots, n$. The ancestor of a non-trivial tree T is the star-reduction $s_a(T)$ with respect to the smallest vertex a which is not a leaf.

7.1 An example for the construction of the Prüfer word $W^*(T)$

We illustrate the computation of the Prüfer word $W^*(T)$ by using once more our favourite tree with vertices $1, \ldots, 14$, root-vertex 14 and hyperedges

$$\{1, 10, 12\}_8, \{2\}_1, \{3, 9\}_4, \{4, 7\}_8, \{5\}_{14}, \{6\}_4, \{8, 13\}_{14}, \{11\}_7$$

depicted in Figure 1 of Section 3.2.

Non-leaves of T in increasing order are $1, 4, 7, 8, 14$ and increasing sequences of all hyperedges involved in non-trivial star-reductions $*_{\leq v}(T)$ of T are given by

v	hyperedges of $*_{<v}(T)$
1	$\{\mathit{1}, \mathit{10}, \mathit{12}\}_{\mathit{8}}, \{2\}_1, \{3, 9\}_4, \{4, 7\}_8, \{5\}_{14}, \{6\}_4, \{8, 13\}_{14}, \{11\}_7,$
4	$\{1, 2, 10, 12\}_8, \{3, 9\}_4, \{\mathit{4}, \mathit{7}\}_{\mathit{8}}, \{5\}_{14}, \{6\}_4, \{8, 13\}_{14}, \{11\}_7$
7	$\{1, 2, 10, 12\}_8, \{\mathit{3}, \mathit{4}, \mathit{6}, \mathit{7}, \mathit{8}, \mathit{9}\}_{\mathit{8}}, \{5\}_{14}, \{8, 13\}_{14}, \{11\}_7$
8	$\{1, 2, 10, 12\}_8, \{3, 4, 6, 7, 9, 11\}_8, \{5\}_{14}, \{\mathit{8}, \mathit{13}\}_{\mathit{14}}$
	$\{1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13\}_{14}, \{5\}_{14}.$

The first column indicates the smallest non-leaf of $*_{<v}(T)$. The second column consists of the increasing list of all hyperedges of $*_{<v}(T)$ with the hyperedge containing the smallest leaf as an unmarked vertex in italics.

Removing the italicized hyperedges from the sequences in the second column, we get the sequence $\mathcal{E}_v$ and the sequence $\mathcal{S}_v$ which determines the positions of the letters $1, 4, 7, 8$ and 14 in the Prüfer word $W^*(T) = w_1 \ldots w_7$ of T:

v	$\mathcal{E}_v$	$\mathcal{S}_v$	$\mathcal{E} \setminus \mathcal{E}_v$
1	$\{2\}_1, \{3, 9\}_4, \{4, 7\}_8, \{5\}_{14}, \{6\}_4, \{8, 13\}_{14}, \{11\}_7$	1	$\{1, 10, 12\}_8$
4	$\{1, 2, 10, 12\}_8, \{3, 9\}_4, \{5\}_{14}, \{6\}_4, \{8, 13\}_{14}, \{11\}_7$	2, 4	$\{4, 7\}_8$
7	$\{1, 2, 10, 12\}_8, \{5\}_{14}, \{8, 13\}_{14}, \{11\}_7$	4	$\{3, \ldots\}_8$
8	$\{1, 2, 10, 12\}_8, \{3, 4, 6, 7, 9, 11\}_8, \{5\}_{14}$	1, 2	$\{8, 13\}_{14}$
15	$\{1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13\}_{14}, \{5\}_{14}.$		

The necessary data for the final computation of $W^*(T)$ are summarized by

v	$\mathcal{S}_v$	$*_1$	$*_2$	$*_3$	$*_4$	$*_5$	$*_6$	$*_7$
1	1	1	$*_1$	$*_2$	$*_3$	$*_4$	$*_5$	$*_6$
4	$2,4$	1	$*_1$	4	$*_2$	4	$*_3$	$*_4$
7	4	1	$*_1$	4	$*_2$	4	$*_3$	7
8	$1,2$	1	8	4	8	4	$*_3$	7
14		1	8	4	8	4	14	7

and give 1 8 4 8 4 14 7 for the Prüfer word $W^*(T)$ of T.

From an algorithmic point of view it is perhaps more straightforward to work with the partition map and with the glue map of T. Writing $p(w)_{g(w)}$ for the image of the vertex $w \in \{1,\ldots,13\}$, the partition map and the glue map of $*_{\leq v}(T)$ (with the convention $*_0(T) = T$) are given by

v	1	2	3	4	5	6	7	8	9	10	11	12	13
0	1_8	2_1	3_4	4_8	5_{14}	6_4	4_8	8_{14}	3_4	1_8	11_7	1_8	8_{14}
1	1_8	1_8								1_8		1_8	
4			3_8	3_8		3_8	3_8		3_8				
7											3_8		
8	1_{14}	1_{14}	1_{14}	1_{14}		1_{14}	1_{14}	1_{14}	1_{14}	1_{14}	1_{14}	1_{14}	1_{14}

(unchanged values are omitted).

Computing the partition map p_i and the glue map g_i of $*_{\leq i}(T)$ is straightforward by induction on i: For v a non-root vertex define $p_0(v)$ as the minimal element of the unique reduced hyperedge e' containing v and define $g_0(v)$ as the marked vertex $e_* = e \setminus e'$ of the unique hyperedge e whose associated reduced hyperedge $e' = E \setminus \{e_*\} \subset e$ contains v.

We suppose now p_{i-1} and g_{i-1} constructed. We consider

$$a = \min\left(p_{i-1}(i), \min_{v \in \mathcal{V}', g_{i-1}(v)=i} p_{i-1}(v)\right)$$

and we set $p_i(v) = a$ if either $p_{i-1}(v) = p_{i-1}(i)$ or $g_{i-1}(v) = i$. We leave $p_i(v) = p_{i-1}(v)$ unchanged otherwise, ie. if $p_{i-1}(v) \neq p_{i-1}(i)$ and $g_{i-1}(v) \neq i$. The integer a is of course the minimal element in the unique reduced hyperedge of $*_{\leq i}(T)$ which contains i.

We set $g_i(v) = g_{i-1}(v)$ if $g_{i-1}(v) \neq i$ and we set $g_i(v) = g_{i-1}(i)$ if $g_{i-1}(v) = i$. Otherwise stated, the marked vertex $g_{i-1}(i)$ of the unique reduced hyperedge e' in $*_{<i}(T)$ which contains i is not affected by star-reduction at i except if it is equal to i. In this case it is replaced by the marked vertex $g_{i-1}(i)$ of the unique reduced hyperedge in $*_{<i}(T)$ which contains i.

Using Remark 4.2 we can condense the computations for $\mathcal{S}_v$ to

v		$\mathcal{S}_v$	
1	$2_1, 3_4, 4_8, 5_{14}, 6_4, 8_{14}, 11_7$	1	1_8
4	$1_8, 3_4, 5_{14}, 6_4, 8_{14}, 11_7$	2, 4	4_8
7	$1_8, 5_{14}, 11_7$	3	3_8
8	$1_8, 3_8, 5_{14}$	1, 2	8_{14}

by choosing smallest unmarked representatives in hyperedges.

8 The inverse map $(\mathcal{P}, W^*) \longmapsto T$

Given a Prüfer code $(\mathcal{P}, W^*)$ where $\mathcal{P}$ is a partition of $\{1, \ldots, n-1\}$ into k non-empty parts and where $W^* \in \{1, \ldots, n\}^{k-1}$ is a word of length $k-1$ with letters in $\{1, \ldots, n\}$, there exists a unique acyclic hypertree T such that $\mathcal{P} = \mathcal{P}(T)$ is the Prüfer partition of T and $W^* = W^*(T)$, defined by the construction of Section 7, is the Prüfer word of T.

If $W^* = n^{k-1}$, the pair $(\mathcal{P}, W^*)$ is Prüfer code of the hyperstar centered at the root-vertex n with reduced hyperedges given by the parts of $\mathcal{P}$. Otherwise there exists a smallest letter $v < n$ occuring with strictly positive multiplicity $a > 0$ in W^*. We denote by $\mathcal{P}_v = (s_1, \ldots, s_{k-1})$ the increasing sequence (ordered with respect to smallest elements) of all $k-1$ parts not containing the vertex v of $\mathcal{P}$. If $i_1, \ldots, i_a$ are the a indices of all letters equal to v in the word $W^* = w_1 \ldots w_{k-1}$ then the parts $s_{i_1}, \ldots, s_{i_a}$ correspond to all reduced hyperedges of a (not yet constructed) acyclic hypertree T with marked vertex v. Denoting by e'_v the unique part of $\mathcal{P}$ containing v, we consider the partition $\tilde{\mathcal{P}}$ obtained by merging the $a+1$ parts $s_{i_1}, \ldots, s_{i_a}$ and e'_v into a larger part $\tilde{e}'$. We denote by $\widetilde{W^*}$ the word of length $k-1-a$ obtained by removing all a letters equal to v from W^*. The Prüfer word $\widetilde{W^*}$ of the pair $(\widetilde{\mathcal{P}}, \widetilde{W^*})$ contains no letter $\leq a$. It is thus by descending induction on a the Prüfer code of a unique acyclic hypertree $\widetilde{T}$ with $\{1, \ldots, a\}$ contained in the set of leaves. More precisely, the recursively defined acyclic hypertree $\widetilde{T}$ is the star-reduction $*_v(T)$ at v of the acyclic hypertree T corresponding to $(\mathcal{P}, W^*)$. The hyperedge $\tilde{e}$ associated to the part $\tilde{e}'$ of $\widetilde{T}$ splits into $a+1$ hyperedges of T in the obvious way: a hyperedges with marked vertex v have reduced hyperedges $s_{i_1}, \ldots, s_{i_a}$. The marked vertex of the hyperedge corresponding to the last part e'_v involved in $\tilde{e}'$ is given by the marked vertex of the hyperedge associated to $\tilde{e}'$ in $\widetilde{T} = *_v(T)$. This defines the acyclic hypertree T uniquely.

8.1 An example for the inverse map

We reconstruct the acyclic hypertree T of Figure 1 from its Prüfer code $(\mathcal{P}, W^*)$ consisting of the Prüfer partition

$$\mathcal{P} = \{1, 10, 12\}_8, \{2\}_1, \{3, 9\}_4, \{4, 7\}_8, \{5\}_{14}, \{6\}_4, \{8, 13\}_{14}, \{11\}_7$$

(with parts totally ordered by minimal elements) and of the Prüfer word $W^* = 1\ 8\ 4\ 8\ 4\ 14\ 7$.

We have

$$\begin{array}{r|l} 1 & \{\mathit{1}, \mathit{10}, \mathit{12}\}, \{2\}_1, \{3,9\}, \{4,13\}, \{5\}, \{6\}_4, \{8,13\}, \{11\} \\ 4 & \{1,2,10,12\}, \{3,9\}_4, \{\mathit{4}, \mathit{7}\}, \{5\}, \{6\}_4, \{8,13\}, \{11\} \\ 7 & \{1,2,10,12\}, \{\mathit{3}, \mathit{4}, \mathit{6}, \mathit{7}, \mathit{9}\}, \{5\}, \{8,13\}, \{11\}_7 \\ 8 & \{1,2,10,12\}_8, \{3,4,6,7,9,11\}_8, \{5\}, \{\mathit{8}, \mathit{13}\} \\ 15 & \{1,2,3,4,6,7,8,9,10,11,12,13\}_{14}, \{5\}_{14} \end{array}$$

The indices $1, 4, 7, 8$ are added according to the positions of the letters $1, 4, 7, 8$ and in the words

$$\begin{array}{rcl} W^* & = & 1\ 8\ 4\ 8\ 4\ 14\ 7 \\ W^* \setminus \{1\} & = & 8\ 4\ 8\ 4\ 14\ 7 \\ W^* \setminus \{1,4\} & = & 8\ 8\ 14\ 7 \\ W^* \setminus \{1,4,7\} & = & 8\ 8\ 14 \end{array}$$

after removal of the italicized part containing the index under consideration. Reduced hyperedges of the last row are all marked by the root 14.

Parts in every row are completely ordered according to smallest elements and are obtained from the parts of the previous row by merging all parts involving the vertex considered in the previous row (and by copying the remaining parts).

The marked vertex of a reduced hyperedge e' is now given by the index of the first indexed superset $\tilde{e}' \supset e'$ encountered when moving down the rows. We get thus the hyperedges

$$\{1,10,12\}_8, \{2\}_1, \{3,9\}_4, \{4,7\}_8, \{5\}_{14}, \{6\}_4, \{8,13\}_{13}, \{11\}_7$$

of our favourite acyclic hypertree T depicted in Figure 1.

The following table illustrates the algorithm by giving partition maps for $*_{\leq v}(T)$ and by giving partial glue-maps (denoted by $p(v)_{g(v)}$, see Remark 4.2)

v	1	2	3	4	5	6	7	8	9	10	11	12	13	
p_0	1	2	3	4	5	6	4	8	3	1	11	1	8	
1	$2_1, 3, 4, 5, 6, 8, 11$													1
p_1	1	1	3	4	5	6	4	8	3	1	11	1	8	
4	$1, 3_4, 5, 6_4, 8, 11$													4
p_4	1	1	3	3	5	3	3	8	3	1	11	1	8	
7	$1, 5, 8, 11_7$													3
p_7	1	1	3	3	5	3	3	8	3	1	3	1	8	
8	$1_8, 3_8, 5$													8
p_8	1	1	1	1	5	1	1	1	1	1	1	1	1	
	$1_{14}, 5_{14}$													

The table encodes the information for the glue map g as follows: Suppose we want to determine the marked vertex $g(7)$ of the reduced hyperedge containing vertex 7. The second row (denoted by p_0) contains the information for the partition map of $*_{\leq 0}(T) = T$. It shows that the smallest element in the reduced hyperedge (of T) containing 7 is 3. This implies $g(7) = g(4)$ and we are reduced to compute $g(4)$. Nothing interesting happens to vertex 4 during the star-reduction at vertex 1. After that, vertex 4 is involved in the star-reduction at vertex 4 and becomes an element of the reduced hyperedge of $*_{\leq 4}(T)$ with smallest element $p_4(4) = 3$. We switch thus our attention to the vertex 3. The next row indicates that the hyperedge containing 3 of $*_{\leq 4}(T)$ has marked vertex 8. We have thus $g(7) = 8$ for the value $g(7)$ of the glue map g at the vertex 8.

Proceeding similarly we get the complete information

v	1	2	3	4	5	6	7	8	9	10	11	12	13
g	8	1	4	8	14	4	4	14	4	8	7	8	14

for the glue map g of the tree T depicted in Figure 1.

9 Infinite acyclic hypertrees

The construction of the Prüfer code $(\mathcal{P}, W^*)$ based on mergings of hyperedges works perfectly well for infinite acyclic hypertrees with vertices $\{1,2,3,\dots\} \cup \{\infty\}$ rooted at ∞. It encodes such an acyclic hypertree T with infinitely many hyperedges by a Prüfer partition of $\{1,2,\dots\}$ into infinitely many non-empty parts and an infinite Prüfer word $W^* = w_1w_2\cdots \in (\mathbb{N} \cup \{\infty\})^{\mathbb{N}}$ with an arbitrary vertex v (which can be the root vertex) of T occuring $\deg(v) - 1$ times in W^* where $\deg(v)$ can be infinite.

The Prüfer map is however not onto: A pair $(\mathcal{P}, W^*)$ consisting of a partition of $\{1,2,\dots\}$ into infinitely many parts and an infinite word $W^* \in (\mathbb{N} \cup \{\infty\})^{\mathbb{N}}$ corresponds in general to no infinite rooted acyclic hypertree. In order to have a one-to-one correspondence, we introduce in this section ideally rooted acyclic hypertrees with vertices $\{1,2,\dots\} \cup \{\infty\}$. Such objects are hyperforests having at most one component which is an ordinary (finite or infinite) rooted acyclic hypertree together with an arbitrary large (and perhaps infinite) number of infinite trees with marked ends playing the role of the root vertex ∞.

Observe that infinite acyclic hypertrees with vertices $\{1,2,\dots\}$ rooted at ∞ which have only finitely many hyperedges are essentially the same as finite acyclic hypertrees from the point of view of the Prüfer word W^*. We leave the easy discussion for this class of rooted acyclic hypertrees to the reader.

9.1 Ends of acyclic hypertrees and ideally rooted acyclic hypertrees

Two infinite geodesics $\gamma, \gamma' : \mathbb{N} \longrightarrow \mathcal{V}$ of an infinite hypergraph G are *equivalent* if $d(\gamma(n), \gamma'(n))$ is ultimately constant. Equivalence classes of such infinite geodesics are called *ends* of G.

An *ideally rooted acyclic hypertree* is a hyperforest with a choice of an end in every connected component not containing the root vertex. We call the connected component containing the root of an ideally rooted acyclic hypertree the *root component*. The root component can be finite (and perhaps reduced to its root) or infinite. All other components are *ideal components*. They contain always infinitely many hyperedges.

An ideally rooted acyclic hypertree has a marked vertex e_* in every hyperedge. The marked vertex e_* of a hyperedge e in the root component is defined in the usual way as the unique vertex of e which is closest to the root. The marked vertex e_* of a hyperedge e in an ideal component C is defined as the unique vertex closest to $\gamma(n)$ for n huge enough where $\gamma : \mathbb{N} \longrightarrow \mathcal{V}$ is a fixed geodesic defining the equivalence class of the marked end of C. We leave it to the reader to show that e_* is well defined and depends only on the equivalence class of γ.

9.2 Partition maps and glue maps of ideally rooted acyclic hypertrees

Partition maps of ideally rooted acyclic hypertrees are idempotent lowering maps of the set $\mathbb{N}* = \mathbb{N} \setminus \{0\}$ into itself. Glue maps are maps of the set $\mathbb{N} * \cup \{\infty\}$ admitting the fixpoint $g(\infty) = \infty$ as their only recurrent element. Extending partition maps by $p(\infty) = \infty$, pairs of maps $p, g : \mathbb{N}^* \cup \{\infty\} \longrightarrow \mathbb{N}^* \cup \{\infty\}$ fixing $\infty = p(\infty) = g(\infty)$ formed by a lowering idempotent map p and a map g with $\infty = g(\infty)$ as its unique recurrent element correspond to partition maps and the glue maps of ideally rooted trees if and only if $g = g \circ p$.

9.3 The Prüfer code of an ideally rooted acyclic hypertree

Proofs are straightforward and omitted in this informal Section.

The Prüfer partition $\mathcal{P} = \mathcal{P}(T)$ of an ideally rooted acyclic hypertree T is defined in the obvious way as the partition of the set $\mathcal{V} \setminus \{r\}$ of non-root vertices with parts $e \setminus \{e_*\}$ given by all reduced hyperedges.

The *glue-map* of an ideally rooted acyclic hypertree T is the map $g : \mathcal{V} \longrightarrow \mathcal{V}$ having the root $r = g(r)$ as its unique fixpoint and given by $g(v) = e_*$ for a non-root vertex v arising as an unmarked element of the hyperedge e.

The Prüfer word $W^*(T)$ of an ideally rooted acyclic hypertree T with vertices $\{1, 2, \dots\} \cup \{\infty\}$ rooted at ∞ is well-defined and given by an infinite word $w_1 w_2 \dots$ with a finite letter $n \in \mathbb{N}$ occuring exactly $\deg(n) - 1$ times. The degree $\deg(n)$ of a vertex n can be finite or infinite. The letter ∞ corresponding to the root vertex occurs at most $\deg(\infty) - 1$ times in $W^*(T)$ where $\deg(\infty) \in \{0, 1, 2, \dots\} \cup \{\infty\}$ is defined as the degree of the vertex ∞ in the root-component.

The exact number of occurences of ∞ in $W^*(T)$ can be strictly smaller than $\deg(\infty) - 1$. More precisely, we order the connected components of $T \setminus \{\infty\}$ according to their smallest vertex. Denoting by $\widetilde{C}$ the smallest ideal component, we erase all connected components $\geq \widetilde{C}$ from T and we denote by $\widetilde{\deg}(\infty)$ the degree of ∞ in the resulting ordinary rooted acyclic $\widetilde{\text{hypertree}}$. The number of occurences of the letter ∞ in $W^*(T)$ is then given by $\max(0, \widetilde{\deg}(\infty) - 1)$.

We have now a one-to-one correspondence between the set of ideally rooted acyclic hypertrees having vertices $\{1, 2, \dots\} \cup \{\infty\}$ rooted at ∞ and having infinitely many hyperedges and the set of Prüfer codes consisting of a Prüfer partition of $\mathbb{N}$ into infinitely many non-empty parts and an infinite Prüfer word which can be an arbitrary element of $\{\{1, 2, \dots,\} \cup \{\infty\}\}^{\mathbb{N}}$.

Ideally rooted acyclic hypertrees (with infinitely many hyperedges) which are locally finite correspond to Prüfer codes with Prüfer partitions involving only finite parts and with Prüfer words involving all letters $\{\{1, 2, \dots,\} \cup \{\infty\}\}$ with finite multiplicity.

10 An example giving rise to bijections of S_n

The simplest infinite tree with vertices $\mathbb{N} \cup \{\infty\}$, rooted at ∞, is given by a halfline originating at the root ∞ with vertices $v_1, v_2, \dots \in \mathbb{N}$ at distance $1, 2, \dots$ of the root-vertex ∞. Setting $v_0 = \infty$, each vertex v_i other than the root-vertex v_0 has thus exactly two neighbours v_{i-1} and v_{i+1}. The root vertex ∞ has a unique neighbour v_1. Such a tree is completely described by the permutation $i \longmapsto v_i$ of the set $\{1, 2, 3, \dots\}$ and every permutation σ of $\{1, 2, \dots\}$ describes a unique such tree. The Prüfer word W^* of such a tree yields a permutation ψ of $\{1, 2, \dots\}$. (Caution: not every permutation of $\{1, 2, \dots\}$ corresponds to such a tree: most permutations give rise to trees with ideal components.) The Prüfer partition of such a tree is of course the trivial partition of $\{1, 2, \dots\}$ into singletons and thus carries no information.

A particularly nice subset of permutations is given by so-called "finitely-supported" permutations moving only finitely many elements of the infinite set $\{1, 2, \dots\}$. Such a permutation σ satisfies $\sigma(m) = m$ for every integer m larger than some natural integer n and thus can be considered as an element of the finite permutation group S_n acting in the usual way on $\{1, \dots, n\}$. It is easy to see that the Prüfer word W^* of such a tree has this property again. Thus the Prüfer word defines a bijection of S_n which respects the the obvious inclusion of S_{n-1} in S_n as the subset of all permutations fixing n.

We describe now this map for $n \leq 4$. We write $\begin{pmatrix} \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{pmatrix}$ for a permutation $i \longmapsto \sigma(i)$ of $\{1, 2, \dots, n\}$.

In the case $n = 1$ there is a unique permutation. It fixes every element of $\{1, 2, \dots\}$ and the associated Prüfer word W^* is again the identity permutation.

The unique non-trivial permutation σ in S_2 (extendend to a permutation of $\{1, 2, 3, \dots\}$ by setting $\sigma(i) = i$ for all $i > 2$) gives again rise to $W^*(\sigma) = \sigma$.

In the case $n = 3$, the image $W^*(\sigma)$ of σ is already known for the two permutations of $S_2 \subset S_3$. The remaining four permutations form two orbits defined by the image $\sigma(3)$ of the largest integer 3. Note that we have always $\psi(i) = \sigma(i) = i$ for the partition ψ encoded by the Prüfer word $\psi = W^*(\sigma)$ of a partition σ such that

$\sigma(i) = i$ for all $i > n$.

In the case $n = 4$, the map $\sigma \longmapsto W^*(\sigma)$ gives rise to two orbits

$$\begin{pmatrix} 2 & 3 & 4 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 2 & 4 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 3 & 2 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 2 & 4 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 4 & 2 & 1 \end{pmatrix}$$

associated to permutations such that $\sigma(4) = 1$. We have finally one orbit

$$\begin{matrix} \begin{pmatrix} 1 & 3 & 4 & 2 \end{pmatrix}, \begin{pmatrix} 4 & 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 3 & 1 & 4 & 2 \end{pmatrix}, \\ \begin{pmatrix} 3 & 4 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 4 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 4 & 3 & 1 & 2 \end{pmatrix} \end{matrix}$$

associated to all permutations such that $\sigma(4) = 2$ and one orbit

$$\begin{matrix} \begin{pmatrix} 1 & 2 & 4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 4 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 2 & 1 & 3 \end{pmatrix}, \\ \begin{pmatrix} 2 & 1 & 4 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 4 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 1 & 2 & 3 \end{pmatrix} \end{matrix}$$

consisting of all permutations with $\sigma(4) = 3$.

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(Received 9 Apr 2013; revised 29 Oct 2013)