AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **60(1)** (2014), Pages 79–108

On the perfect 1-factorisation problem for circulant graphs of degree 4

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Abstract

A 1-factorisation of a graph G is a partition of the edge set of G into 1-factors (perfect matchings); a perfect 1-factorisation of G is a 1-factorisation of G in which the union of any two of the 1-factors is a Hamilton cycle in G. It is known that for bipartite 4-regular circulant graphs, having order $2 \pmod 4$ is a necessary (but not sufficient) condition for the existence of a perfect 1-factorisation. The only known non-bipartite 4-regular circulant graphs that admit a perfect 1-factorisation are trivial (on 6 vertices). We prove several construction results for perfect 1-factorisations of a large class of bipartite 4-regular circulant graphs. In addition, we show that no member of an infinite family of non-bipartite 4-regular circulant graphs admits a perfect 1-factorisation. This supports the conjecture that there are no perfect 1-factorisations of any connected non-bipartite 4-regular circulant graphs of order at least 8.

1 Introduction and Notation

We consider graphs that are simple and undirected. The vertex set and edge set of a graph G are denoted $V(G)$ and $E(G)$, respectively. A graph G is *r-regular* if each vertex $v \in V(G)$ has degree r. A path with vertices $v_1, v_2, \dots, v_n$ and edges $\{\{v_i, v_{i+1}\} \mid i = 1, 2, \dots, n-1\}$ will be denoted by $[v_1, v_2, \dots, v_n]$ and a cycle with vertices $v_1, v_2, \dots, v_n$ and edges $\{\{v_i, v_{i+1}\} \mid i = 1, 2, \dots, n-1\} \cup \{v_n, v_1\}$ will be denoted by $(v_1, v_2, \dots, v_n)$. In any graph with vertex set $\mathbb{Z}_n$, the edge $\{x, y\}$ is said

* This research was carried out while the author was a PhD student in the School of Mathematics and Physics at The University of Queensland, supported by a UQ International Scholarship and UQ Centennial Scholarship.

to have *length* k where k is the length of the shortest path from x to y in the cycle $(0, 1, 2, \ldots, n-1)$.

A 1-*factor* (or *perfect matching*) of a graph G is a spanning 1-regular subgraph of G. A 1-*factorisation* of a graph G is a decomposition of G into edge-disjoint 1-factors. A *perfect 1-factorisation* (abbreviated P1F) is a 1-factorisation with the property that the union of every pair of distinct 1-factors of the 1-factorisation forms a Hamilton cycle. The terms *Hamilton graph* [14] and *strongly Hamilton graph* [16, 17, 24] have also been used in the literature to describe a graph that admits a P1F.

A well-known conjecture of Kotzig [15] states that the complete graph K_{2n} admits a P1F for all $n \geq 2$. This problem has received a lot of attention, however the conjecture remains open. It is known that K_{2n} admits a P1F when n is an odd prime [2] and when $2n-1$ is an odd prime [2, 14]. Seah [21] provides a survey of the P1F problem for complete graphs and Wanless [23] provides an up-to-date list of computational results. P1Fs of other classes of graphs have also been studied; for example, P1Fs of complete bipartite graphs have been studied in [7, 17] and P1Fs of cubic graphs have been studied in [5, 6, 11, 13, 14, 16, 19].

A *circulant graph* is a Cayley graph on a cyclic group. In particular, for n even and $S \subseteq \{1, 2, \ldots, \frac{n}{2}\}$, the circulant graph $\mathrm{Circ}(n, S)$ is defined to have vertex set $V = \{0, 1, \ldots, n-1\}$ and edge set $\{\{x, x+s \pmod{n}\} \mid x \in V, s \in S\}$. The set S is called the *connection set* of the graph. The 3-regular circulant graphs that admit a P1F were characterised in [11]. In this paper, we consider 4-regular circulant graphs, that is, $\mathrm{Circ}(n, \{a, b\})$ where $1 \leq a < b < \frac{n}{2}$. Observe that the graph $\mathrm{Circ}(n, \{a, b\})$ is connected if and only if $\gcd(a, b, n) = 1$ and every connected graph $\mathrm{Circ}(n, \{a, b\})$ is isomorphic to a graph $\mathrm{Circ}(n, \{a', b'\})$ for which $\gcd(a', b') = 1$ (see for example [3]). If a graph admits a 1-factorisation, then the graph is of even order and r-regular for some postive integer r. Hence we consider only circulant graphs of even order.

Two (undirected) circulant graphs $\mathrm{Circ}(n, \{d_1, d_2, \ldots, d_k\})$ and $\mathrm{Circ}(n, \{d'_1, d'_2, \ldots, d'_k\})$ are said to be *conjugate by a multiplier* [20] if there exists an $m \in \mathbb{Z}_n^*$ such that

$$\{md'_1, md'_2, \ldots, md'_k\} = \{\pm d_1, \pm d_2, \ldots, \pm d_k\} \text{ modulo } n,$$

where $\mathbb{Z}_n^*$ denotes the set of residues modulo n that are relatively prime to n. In particular, the permutation $x \mapsto mx$ for $x \in \mathbb{Z}_n$ is an isomorphism between $\mathrm{Circ}(n, \{d'_1, d'_2, \ldots, d'_k\})$ and $\mathrm{Circ}(n, \{d_1, d_2, \ldots, d_k\})$. In 1967 Adam [1] conjectured that two circulant graphs are isomorphic if and only if they are conjugate by a multiplier. While it is known that the conjecture is false in general (see for example [20]), the conjecture holds in the case $k = 2$.

Theorem 1 [9, 10] *Two* 3- *or* 4-*regular circulant graphs* $\mathrm{Circ}(n, \{d_1, d_2\})$ *and* $\mathrm{Circ}(n, \{d'_1, d'_2\})$ *are isomorphic if and only if they are conjugate by a multiplier.*

It is known that every connected, even order circulant graph has a 1-factorisation [18, 22]. A long-standing conjecture states that every Cayley graph contains a Hamilton cycle, and the conjecture is known to be true for Cayley graphs on abelian groups

[8, 25]. In fact, a connected 4-regular Cayley graph on a finite abelian group can be decomposed into two Hamilton cycles [4]. However, these results do not guarantee the existence of a P1F, and thus the problem of characterising the circulant graphs (of degree at least 4) that admit a P1F remains open.

For even n, a connected circulant graph $G = \mathrm{Circ}(n, \{a, b\})$ with $1 \leq a < b \leq \frac{n}{2}$ is bipartite if and only if a and b are both odd. The following necessary condition for the existence of P1Fs of bipartite graphs is attributed to Kotzig [14] (see for example [17]).

Theorem 2 [14] *If G is a bipartite r-regular graph with $r > 2$ and G admits a P1F, then $|V(G)| \equiv 2 \pmod 4$.*

Corollary 3 *Suppose n is even and $G = \mathrm{Circ}(n, \{a, b\})$, where a and b are both odd. If G admits a P1F, then $n \equiv 2 \pmod 4$.*

The first question that arises is whether or not this condition is sufficient for the existence of a P1F of a bipartite circulant graph. For 3-regular circulant graphs it is sufficient, and furthermore, the 3-regular bipartite circulant graphs of order 2 (mod 4) are the only 3-regular circulant graphs (of order at least 8) that admit a P1F [11].

Theorem 4 [11] *If $n > 6$, then a connected 3-regular circulant graph G of order n admits a P1F if and only if $n \equiv 2 \pmod 4$ and G is bipartite.*

In [11], P1Fs of 4-regular circulant graphs were also briefly studied. A computer search showed that for $6 < n < 30$, a connected 4-regular circulant graph G admits a P1F if and only if G is bipartite of order $2 \pmod 4$. Thus, for small values of n, the necessary condition given by Corollary 3 is sufficient; it was also shown that the condition is sufficient if the circulant graph is isomorphic to $\mathrm{Circ}(n, \{1, 3\})$ [11]. However, the condition is not sufficient in general since $\mathrm{Circ}(30, \{1, 11\})$ was shown by computer search to not admit a P1F. In [12] the P1F problem was considered for a class of Cayley graphs, which led to the result that $\mathrm{Circ}(30, \{1, 11\})$ is the smallest member of an infinite family of bipartite 4-regular circulant graphs of order $2 \pmod 4$ for which a P1F does not exist.

Theorem 5 [12] *Suppose $k \equiv 2 \pmod 4$ and $k > 6$. If $k \equiv 10 \pmod{12}$ then $\mathrm{Circ}(3k, \{1, k + 1\})$ does not admit a P1F.*

Further study of graphs with a similar structure to the family defined in Theorem 5 led to results showing that several infinite families of bipartite 4-regular circulant graphs of order $2 \pmod 4$ do admit a P1F. It remains an open problem to characterise the bipartite 4-regular circulant graphs of order $2 \pmod 4$ that admit a P1F.

The only known non-bipartite 4-regular circulant graphs that admit a P1F are of order 6. In [11] it was shown that $\mathrm{Circ}(n, \{1, 2\})$ does not admit a P1F for $n > 6$

and in [12] another infinite family of non-bipartite 4-regular circulant graphs, namely $\mathrm{Circ}(n, \{1, \frac{n}{2} - 1\})$ for $n \equiv 2 \pmod 4$ ($n > 6$), was shown to not admit a P1F. These results support the following conjecture.

Conjecture 6 [11] *Suppose n is even and $n > 6$. If $G = \mathrm{Circ}(n, \{a, b\})$ is a connected non-bipartite 4-regular circulant graph, then G does not admit a P1F.*

In this paper we continue the study of P1Fs of 4-regular circulant graphs. In Section 2, we present construction results that demonstrate the existence of P1Fs for many classes of bipartite 4-regular circulant graphs of order $2 \pmod 4$. In Section 3 we prove that Conjecture 6 holds for yet another infinite class of non-bipartite 4-regular circulant graphs.

2 P1Fs of Bipartite Circulant Graphs

The following notation is used in some of the proofs in this section. Suppose $G = \mathrm{Circ}(n, S)$ and $P = [v_0, v_1, \ldots, v_k]$ is a path of length k in G. For any $\ell \in \mathbb{Z}_n$, we define the path $P + \ell = [v_0 + \ell, v_1 + \ell, \ldots, v_k + \ell]$, where $v_i + \ell$ is calculated modulo n.

We begin with a result that proves the existence of P1Fs for infinitely many classes of bipartite 4-regular circulant graphs.

Theorem 7 *Let $n \geq 14$ such that $n \equiv 2 \pmod 4$. If $5 \leq b < \frac{n}{2}$ is an odd integer such that $\gcd(n, b) = 1$ and $\gcd(n, b-1) = \gcd(n, b+1) = 2$, then any 4-regular circulant graph isomorphic to $\mathrm{Circ}(n, \{1, b\})$ admits a P1F.*

Proof. We construct four disjoint 1-factors A, B, C, D as follows, where $x + i$ is calculated modulo n.

$$\begin{aligned} A &= \{\{x, x+1\} \mid x \text{ is even}\}; \\ B &= \{\{x, x+1\} \mid x \text{ is odd}\}; \\ C &= \{\{x, x+b\} \mid x \text{ is even}\}; \\ D &= \{\{x, x+b\} \mid x \text{ is odd}\}. \end{aligned}$$

These 1-factors clearly form a 1-factorisation of $G = \mathrm{Circ}(n, \{1, b\})$. Observe that the graph with edges given by $A \cup B$ is the subgraph consisting of all edges of length 1 in G, which is a Hamilton cycle. Also, the graph with edges given by $C \cup D$ is the subgraph consisting of all edges of length b in G which is a Hamilton cycle (since n and b are relatively prime).

Let H_1 be the graph with edges given by $A \cup C$. Then H_1 is the union of the following paths of length two

$$P_i = [i(b-1), i(b-1) + b, (i+1)(b-1)] \text{ for } i = 0, 1, \ldots, \tfrac{n-2}{2},$$

where vertex labels are calculated modulo n. It is clear that the start and end vertices of consecutive paths coincide and that the end vertex of $P_{\frac{n-2}{2}}$, which is $\frac{n}{2}(b-1) \equiv 0 \pmod{n}$, is the start vertex of P_0. Furthermore, vertices at distance two in the union of the paths P_i have vertex labels that differ by $b-1$. Suppose that H_1 is not a Hamilton cycle. Then there exists a positive integer $i < \frac{n}{2}$ such that $i(b-1) = mn$ for some integer m (since b is odd, it cannot be that $i(b-1)+b = mn$). Since $b-1$ is even, let $k = \frac{b-1}{2}$. Then $ik = m\frac{n}{2}$, so k divides $m\frac{n}{2}$. Since $\gcd(n, b-1) = 2$, it follows that $\gcd(\frac{n}{2}, k) = 1$ and hence that k divides m. Thus $m \geq k$, so $ik \geq k\frac{n}{2}$ and hence $i \geq \frac{n}{2}$, which is a contradiction. Therefore, H_1 is a Hamilton cycle.

Consider the graph H_2 with edges given by $A \cup D$. Then H_2 is the union of the following paths of length two

$$P_i = [i(b+1), i(b+1)+b, (i+1)(b+1)] \text{ for } i = 0, 1, \ldots, \tfrac{n-2}{2},$$

where vertex labels are calculated modulo n. Again it is clear that the start and end vertices of consecutive paths coincide and that the end vertex of $P_{\frac{n-2}{2}}$, which is $\frac{n}{2}(b+1) \equiv 0 \pmod{n}$, is the start vertex of P_0. Furthermore, vertices at distance two in the union of the paths P_i have vertex labels that differ by $b+1$. By the same argument as above, with $b-1$ replaced with $b+1$, it follows that H_2 is a Hamilton cycle.

By symmetry, the same holds for graphs with edge sets $B \cup D$ and $B \cup C$. Therefore, the 1-factorisation A, B, C, D is a P1F. $\square$

Theorem 7 establishes the existence of P1Fs of 4-regular circulant graphs for many connection sets. We next consider connection sets $\{1, 5\}$ and $\{1, 9\}$ in more detail.

2.1 Connection Set $\{1, 5\}$

Theorem 7 provides us with the following result on P1Fs of $\mathrm{Circ}(n, \{1, 5\})$.

Corollary 8 *Suppose n is an integer such that $n \geq 14$, $n \equiv 2 \pmod{4}$ and n is not divisible by 3 or 5. Then $\mathrm{Circ}(n, \{1, 5\})$ admits a P1F.*

The cases not covered by Corollary 8 are $n \equiv 6 \pmod{12}$ and $n \equiv 10 \pmod{20}$. The computer results provided in [11] show that $\mathrm{Circ}(30, \{1, 5\})$ admits a P1F. Theorem 9 provides a P1F construction when $n \equiv 6 \pmod{12}$; however, the question of whether or not there is a general construction for a P1F of $\mathrm{Circ}(n, \{1, 5\})$ where $n \equiv 10 \pmod{20}$ (and $n > 30$) remains open.

Theorem 9 *If $n \geq 18$ is an integer such that $n \equiv 6 \pmod{12}$, then $\mathrm{Circ}(n, \{1, 5\})$ admits a P1F.*

Proof. Suppose $n \geq 18$ and $n \equiv 6 \pmod{12}$. We construct four disjoint 1-factors A, B, C, D as follows, where $x + i$ is calculated modulo n.

$$\begin{aligned}
A &= \{\{x, x+1\} \mid x = 0, 5 \text{ or } 10 \leq x \leq n-4 \text{ such that } x \equiv 2, 6, 10 \pmod{12}\} \\
&\quad \cup \{\{x, x+5\} \mid x = 2 \text{ or } 4 \leq x \leq n-2 \text{ such that } x \equiv 0, 4, 8 \pmod{12}\}; \\
B &= \{\{x, x+1\} \mid x = 4 \text{ or } 9 \leq x \leq n-1 \text{ such that } x \equiv 1, 5, 9 \pmod{12}\} \\
&\quad \cup \{\{x, x+5\} \mid x = 1 \text{ or } 3 \leq x \leq n-3 \text{ such that } x \equiv 3, 7, 11 \pmod{12}\}; \\
C &= \{\{x, x+1\} \mid x = 1, 3 \text{ or } 7 \leq x \leq n-2 \text{ such that } x \equiv 0, 4, 7 \pmod{12}\} \\
&\quad \cup \{\{x, x+5\} \mid x = 0 \text{ or } 6 \leq x \leq n-8 \text{ such that } x \equiv 6, 9, 10 \pmod{12}\}; \\
D &= \{\{x, x+1\} \mid x = 2, 6, 8 \text{ or } 11 \leq x \leq n-3 \text{ such that } x \equiv 3, 8, 11 \pmod{12}\} \\
&\quad \cup \{\{x, x+5\} \mid x = 5 \text{ or } 13 \leq x \leq n-1 \text{ such that } x \equiv 1, 2, 5 \pmod{12}\}.
\end{aligned}$$

It is easy to see that A, B, C, D is a 1-factorisation when $n \equiv 6 \pmod{12}$. To show that it is a P1F, we consider each pair of 1-factors in turn. When $n \geq 30$, let $g = \frac{n-30}{12}$.

Consider the graph with edge set $A \cup B$. Define the paths P, Q, R and, when $n \geq 30$, S_k, T_k, U_k for $k = 0, 1, \ldots, g$, where all vertex labels are computed modulo n:

$$\begin{aligned}
P &= [0, 1, 6, 5, 4, 9, 10, 11, 16] \\
Q &= [n-2, 3, 8, 13, 14, 15] \\
R &= [n-3, 2, 7, 12, 17, 18] \\
S_k &= [16, 21, 22, 23, 28] + 12k \\
T_k &= [15, 20, 25, 26, 27] + 12k \\
U_k &= [18, 19, 24, 29, 30] + 12k.
\end{aligned}$$

When $n = 18$, $A \cup B$ is $P \cup Q \cup R$, which is a Hamilton cycle. When $n \geq 30$, $A \cup B$ is $P \cup S_0 \cup \cdots \cup S_g \cup Q \cup T_0 \cup \cdots \cup T_g \cup R \cup U_0 \cup \cdots \cup U_g$. It is easy to see that the start and end vertices of consecutive paths coincide. Since the end vertex of one path is also the start vertex of the next path, we say that a path *includes* a vertex if it lies on the path but is not the end vertex. Paths S_k, T_k and U_k each cover four distinct residue classes modulo 12 and include the vertices $\{15, 16, 18\} \cup \{19, \ldots, n-4\} \cup \{n-1\}$. Paths P, Q and R together include the vertices $\{0, 1, \ldots, 14\} \cup \{17\} \cup \{n-3, n-2\}$. Thus, the graph with edge set $A \cup B$ is a Hamilton cycle.

Consider the graph with edge set $A \cup C$. Define the paths P, Q and, when $n \geq 30$, S_k for $k = 0, 1, \ldots, g$, where all vertex labels are computed modulo n:

$$\begin{aligned}
P &= [0, 1, 2, 7, 8, 13, 12, 17, 16] \\
Q &= [n-2, 3, 4, 9, 14, 15, 10, 11, 6, 5, 0] \\
S_k &= [16, 21, 26, 27, 22, 23, 18, 19, 20, 25, 24, 29, 28] + 12k.
\end{aligned}$$

When $n = 18$, $A \cup C$ is $P \cup Q$, which is a Hamilton cycle. When $n \geq 30$, $A \cup C$ is $P \cup S_0 \cup \cdots \cup S_g \cup Q$. It is straightforward to check that the union of these paths is a Hamilton cycle.

Consider the graph with edge set $A \cup D$. Define the paths P, Q and, when $n \geq 30$, S_k, T_k for $k = 0, 1, \ldots, g$, where all vertex labels are computed modulo n:

$$\begin{aligned} P &= [n-5, 0, 1, n-4, n-3, n-2, 3, 2, 7, 6, 5, 10, 11, 12, 17] \\ Q &= [n-1, 4, 9, 8, 13] \\ S_k &= [17, 22, 23, 24, 29] + 12k \\ T_k &= [13, 18, 19, 14, 15, 16, 21, 20, 25] + 12k. \end{aligned}$$

When $n = 18$, $A \cup D$ is $P \cup Q$, which is a Hamilton cycle. When $n \geq 30$, $A \cup D$ is $P \cup S_0 \cup \cdots \cup S_g \cup Q \cup T_0 \cup \cdots \cup T_g$. It is straightforward to check that the union of these paths is a Hamilton cycle.

Consider the graph with edge set $B \cup C$. Define the paths P, Q and, when $n \geq 30$, S_k, T_k for $k = 0, 1, \ldots, g$, where all vertex labels are computed modulo n:

$$\begin{aligned} P &= [0, 5, 4, 3, 8, 7, 12, 13, 14, 9, 10, 15] \\ Q &= [n-3, 2, 1, 6, 11, 16, 17, 18] \\ S_k &= [15, 20, 19, 24, 25, 26, 21, 22, 27] + 12k \\ T_k &= [18, 23, 28, 29, 30] + 12k \end{aligned}$$

When $n = 18$, $B \cup C$ is $P \cup Q$, which is a Hamilton cycle. When $n \geq 30$, $B \cup C$ is $P \cup S_0 \cup \cdots \cup S_g \cup Q \cup T_0 \cup \cdots \cup T_g$. It is straightforward to check that the union of these paths is a Hamilton cycle.

Consider the graph with edge set $B \cup D$. Define the paths P, Q and, when $n \geq 30$, S_k for $k = 0, 1, \ldots, g$, where all vertex labels are computed modulo n:

$$\begin{aligned} P &= [n-1, 0, n-5, n-4, 1, 6, 7, 12, 11, 16, 15] \\ Q &= [n-3, 2, 3, 8, 9, 10, 5, 4, n-1] \\ S_k &= [15, 20, 21, 22, 17, 18, 13, 14, 19, 24, 23, 28, 27] + 12k. \end{aligned}$$

When $n = 18$, $B \cup D$ is $P \cup Q$, which is a Hamilton cycle. When $n \geq 30$, $B \cup D$ is $P \cup S_0 \cup \cdots \cup S_g \cup Q$. It is straightforward to check that the union of these paths is a Hamilton cycle.

Consider the graph with edge set $C \cup D$. Define the paths P, Q, R and, when $n \geq 30$, S_k, T_k, U_k for $k = 0, 1, \ldots, g$, where all vertex labels are computed modulo n:

$$\begin{aligned} P &= [n-5, 0, 5, 10, 15, 16, 17] \\ Q &= [n-1, 4, 3, 2, 1] \\ R &= [19, 14, 9, 8, 7, 6, 11, 12, 13] \\ S_k &= [17, 22, 27, 28, 29] + 12k \\ T_k &= [1, n-4, n-9, n-10, n-11] - 12k \\ U_k &= [13, 18, 23, 24, 25] + 12k. \end{aligned}$$

When $n = 18$, $C \cup D$ is $P \cup Q \cup R$, which is a Hamilton cycle. When $n \geq 30$, $C \cup D$ is $P \cup S_0 \cup \cdots \cup S_g \cup Q \cup T_0 \cup \cdots \cup T_g \cup R \cup U_0 \cup \cdots \cup U_g$. It is easy to see that S_k, T_k and

U_k each cover four distinct residue classes modulo 12 and the start and end vertices of consecutive paths coincide. Thus the union of these paths is a Hamilton cycle.

Therefore, the 1-factors A, B, C, D form a perfect 1-factorisation. □

2.2 Connection Set $\{1, 9\}$

Theorem 7 provides us with the following result on P1Fs of $\mathrm{Circ}(n, \{1, 9\})$.

Corollary 10 *Suppose n is an integer such that $n \geq 22$, $n \equiv 2 \pmod 4$ and n is not divisible by 3 or 5. Then $\mathrm{Circ}(n, \{1, 9\})$ admits a P1F.*

The cases not covered by Corollary 10 are $n \equiv 6 \pmod{12}$ and $n \equiv 10 \pmod{20}$. The computer results provided in [11] show that $\mathrm{Circ}(30, \{1, 9\})$ admits a P1F. Theorem 11 provides a P1F construction when $n \equiv 10 \pmod{20}$; however, the question of whether or not there is a general construction for a P1F of $\mathrm{Circ}(n, \{1, 9\})$ where $n \equiv 6 \pmod{12}$ (and $n > 30$) remains open.

Theorem 11 *If $n \geq 30$ is an integer such that $n \equiv 10 \pmod{20}$, then $\mathrm{Circ}(n, \{1, 9\})$ admits a P1F.*

Proof. First observe that $n \equiv 10 \pmod{20}$. We construct four disjoint 1-factors A, B, C, D as follows, where all vertex labels are calculated modulo n.

$$\begin{aligned}
A &= \{\{x, x+1\} \mid x = 0, 3, 5, 9 \text{ or} \\
&\qquad\qquad 13 \leq x \leq n-4 \text{ such that } x \equiv 2, 6, 10, 13, 18 \pmod{20}\} \\
&\quad \cup \{\{x, x+9\} \mid x = 2, n-2 \text{ or} \\
&\qquad\qquad 8 \leq x \leq n-10 \text{ such that } x \equiv 0, 8, 12, 15, 16 \pmod{20}\}; \\
B &= \{\{x, x+1\} \mid x = 2, 7, 11 \text{ or} \\
&\qquad\qquad 16 \leq x \leq n-2 \text{ such that } x \equiv 0, 4, 8, 12, 16 \pmod{20}\} \\
&\quad \cup \{\{x, x+9\} \mid x = 0, 4 \text{ or} \\
&\qquad\qquad 6 \leq x \leq n-4 \text{ such that } x \equiv 2, 6, 10, 14, 18 \pmod{20}\}; \\
C &= \{\{x, x+1\} \mid x = 1, 6, 10 \text{ or} \\
&\qquad\qquad 15 \leq x \leq n-3 \text{ such that } x \equiv 3, 7, 11, 15, 19 \pmod{20}\} \\
&\quad \cup \{\{x, x+9\} \mid x = 3 \text{ or} \\
&\qquad\qquad 5 \leq x \leq n-1 \text{ such that } x \equiv 1, 5, 9, 13, 17 \pmod{20}\}; \\
D &= \{\{x, x+1\} \mid x = 4, 8, 12 \text{ or} \\
&\qquad\qquad 14 \leq x \leq n-1 \text{ such that } x \equiv 1, 5, 9, 14, 17 \pmod{20}\} \\
&\quad \cup \{\{x, x+9\} \mid x = 1, 7 \text{ or} \\
&\qquad\qquad 11 \leq x \leq n-3 \text{ such that } x \equiv 3, 4, 7, 11, 19 \pmod{20}\}.
\end{aligned}$$

It is easy to see that A, B, C, D is a 1-factorisation when $n \equiv 10 \pmod{20}$. To show that it is a P1F, we consider each pair of 1-factors in turn. When $n \geq 50$, define $h = \frac{n-50}{20}$.

Consider the graph with edge set $A \cup B$. Define the paths P, Q, R and, when $n \geq 50$, S_k, T_k, U_k for $k = 0, 1, \ldots, h$:

$$\begin{aligned} P &= [n-8, 1, 0, 9, 10, 19, 18, 27, 26] \\ Q &= [28, 29, 20, 21, 12, 11, 2, 3, 4, 13, 14, 23, 22] \\ R &= [n-4, 5, 6, 15, 24, 25, 16, 17, 8, 7, n-2] \\ S_k &= [22, 31, 30, 39, 38, 47, 46] + 20k \\ T_k &= [26, 35, 44, 45, 36, 37, 28] + 20k \\ U_k &= [48, 49, 40, 41, 32, 33, 34, 43, 42] + 20k. \end{aligned}$$

Observe that when $n = 30$, the graph with edge set $A \cup B$ is $P \cup R \cup Q$, which is a Hamilton cycle. When $n \geq 50$, then the graph with edge set $A \cup B$ is the following union of paths, where all vertex labels are computed modulo n:

$$A \cup B = P \cup T_0 \cup Q \cup (\bigcup_{i=0}^{h-1} S_i \cup T_{i+1} \cup U_i) \cup S_h \cup R \cup U_h$$

It is easy to see that the start and end vertices of consecutive paths coincide. In particular, observe that $n-4$ is where S_h and R coincide, and $n-2$ is where R and U_h coincide and $n-8$ is where U_h and P coincide. Again, we say that a path includes a vertex if it lies on the path but is not the end vertex. Paths S_k, T_k each cover six distinct residue classes modulo 20 and the remaining 8 residue classes modulo 20 are covered by U_k. Paths P, Q and R include the vertices $n-8$, $n-4$ and all vertices from 0 to 29 except 22 and 26; the paths S_k, T_k and U_k, for $k = 0, 1, \ldots, h$, include the vertices 22,26 and all vertices from 30 to $n-1$ except $n-8$ and $n-4$. Thus, the graph with edge set $A \cup B$ is a Hamilton cycle.

Consider the graph with edge set $C \cup D$. Define the paths P, Q, R and, when $n \geq 50$, S_k, T_k, U_k for $k = 0, 1, \ldots, h$:

$$\begin{aligned} P &= [13, 12, 3, n-6, n-7, 2, 1, 10, 11, 20, 19, 28, 27] \\ Q &= [n-3, 6, 7, 16, 15, 14, 5, 4, n-5] \\ R &= [25, 26, 17, 18, 9, 8, n-1, 0, n-9, n-8, n-17] \\ S_k &= [33, 24, 23, 32, 31, 40, 39, 48, 47] + 20k \\ T_k &= [45, 46, 37, 38, 29, 30, 21, 22, 13] + 20k \\ U_k &= [27, 36, 35, 34, 25] + 20k. \end{aligned}$$

When $n = 30$, it is clear that $C \cup D = P \cup Q \cup R$, which is a Hamilton cycle. When $n \geq 50$, then the graph with edge set $C \cup D$ is the Hamilton cycle formed by the following union of paths, where all vertex labels are computed modulo n:

$$C \cup D = S_h \cup Q \cup T_h \cup (\bigcup_{i=0}^{h-1} S_{h-1-i} \cup U_{h-i} \cup T_{h-1-i}) \cup P \cup U_0 \cup R.$$

It is straightforward to check that the union of these paths is a Hamilton cycle.

We next define five types of path structures which will be used for the next four pairs of 1-factors, where $k = 0, 1, \ldots, g$ with $g = \frac{n-30}{20}$.

$$\begin{aligned}
M_k(x) &= [x, x+9, x+10, x+11, x+20] + 20k \\
V_k(x) &= [x, x-1, x+8, x+9, x+10, x+1, x+2, x+11, x+20] + 20k \\
W_k(x) &= [x, x+9, x+18, x+19, x+20] + 20k \\
Y_k(x) &= [x, x-1, x-10, x-19, x-20] - 20k \\
Z_k(x) &= [x, x-1, x+8, x+17, x+18, x+9, x+10, x+11, x+20] + 20k.
\end{aligned}$$

It is straightforward to see that, for example, $M_0(9) \in A \cup C$ and therefore $M_i(9) \in A \cup C$ for $i = 1, 2 \ldots, g$. The graphs with edge sets $A \cup C$, $A \cup D$, $B \cup C$ and $B \cup D$ are given below, where all vertex labels are computed modulo n.

$$\begin{aligned}
A \cup C &= [2, 11, 10, 9] \cup \bigcup_{i=0}^{g} M_i(9) \cup [n-1, 8, 17] \cup \bigcup_{i=0}^{g} M_i(17) \cup [7, 6, 5, 14] \cup \\
&\quad \bigcup_{i=0}^{g} V_i(14) \cup [4, 3, 12] \cup \bigcup_{i=0}^{g} W_i(12),
\end{aligned}$$

$$\begin{aligned}
A \cup D &= [10, 9, 8] \cup \bigcup_{i=0}^{g} M_i(8) \cup [n-2, 7, 16] \cup \bigcup_{i=0}^{g} M_i(16) \cup [6, 5, 4] \cup \bigcup_{i=0}^{g} Y_i(4) \cup \\
&\quad [14, 13, 12] \cup \bigcup_{i=0}^{g} M_i(12) \cup [2, 11, 20] \cup \bigcup_{i=0}^{g} M_i(20),
\end{aligned}$$

$$\begin{aligned}
B \cup C &= [0, 9, 18] \cup \bigcup_{i=0}^{g} M_i(18) \cup [8, 7, 6] \cup \bigcup_{i=0}^{g} M_i(6) \cup [n-4, 5, 14] \cup \bigcup_{i=0}^{g} M_i(14) \cup \\
&\quad [4, 13, 22] \cup \bigcup_{i=0}^{g} M_i(22) \cup [12, 11, 10] \cup \bigcup_{i=0}^{g} M_i(10),
\end{aligned}$$

$$\begin{aligned}
B \cup D &= [9, 8, 7] \cup \bigcup_{i=0}^{g} M_i(7) \cup [n-3, 6, 15] \cup \bigcup_{i=0}^{g} Z_i(15) \cup [5, 4, 13, 12, 11] \cup \\
&\quad \bigcup_{i=0}^{g} M_i(11) \cup [1, 10, 19] \cup \bigcup_{i=0}^{g} \cup M_i(19).
\end{aligned}$$

It is straightforward to check that each of these graphs is a Hamilton cycle. Therefore, the 1-factors A, B, C, D form a perfect 1-factorisation. $\square$

2.3 Connection Set $\{1, \frac{n}{2} - 2\}$

We consider the family of bipartite 4-regular circulant graphs $\mathrm{Circ}(n, \{1, b\})$ where b is as large as possible. Since b must be odd for the graph to be bipartite, it follows that b is either $\frac{n}{2}-1$ or $\frac{n}{2}-2$, depending on whether $n \equiv 0, 2 \pmod 4$, respectively. When $n \equiv 0 \pmod 4$, $\mathrm{Circ}(n, \{1, \frac{n}{2}-1\})$ does not admit a P1F by Corollary 3. When $n \equiv 2 \pmod 4$, $\mathrm{Circ}(n, \{1, \frac{n}{2}-2\})$ admits a P1F provided that the gcd conditions in Theorem 7 are satisfied. However, the next result shows that a P1F of $\mathrm{Circ}(n, \{1, \frac{n}{2}-2\})$ exists when $n \equiv 2 \pmod 4$ irrespective of the gcd conditions from Theorem 7.

Theorem 12 *If $n \geq 14$ and $n \equiv 2 \pmod 4$, then $\mathrm{Circ}(n, \{1, \frac{n}{2}-2\})$ admits a P1F.*

Proof. We construct four disjoint 1-factors A, B, C, D as follows, where all vertex labels are calculated modulo n. Let $k = \frac{n}{2}$.

$$\begin{aligned}
A &= \{\{x, x+1\} \mid x \in \{0, 2\} \cup \{8, 10, 12, \ldots, k+1\} \cup \{k+7, k+9, \ldots, n-2\}\} \\
&\quad \cup \{\{x, x+k-2\} \mid x = 5, 6, 7, k+6\}; \\
B &= \{\{x, x+1\} \mid x \in \{5, 7, 9, \ldots, k-2\} \cup \{k+4, k+6, k+8, \ldots, n-1\}\} \\
&\quad \cup \{\{x, x+k-2\} \mid x = 2, 3, 4, k+3\}; \\
C &= \{\{x, x+1\} \mid x = 3, k, k+2, k+5\} \\
&\quad \cup \{\{x, x+k-2\} \mid x \in \{0, 1, k+4\} \cup \{k+7, k+8, k+9, \ldots, n-1\}\}; \\
D &= \{\{x, x+1\} \mid x = 1, 4, 6, k+3\} \\
&\quad \cup \{\{x, x+k-2\} \mid x = k+5 \text{ or } x \in \{8, 9, 10, \ldots, k+2\}\};
\end{aligned}$$

We describe the graphs formed by the six pairs of 1-factors, and show that each of these is a Hamilton cycle. First consider $A \cup B$. It is straightforward to check that for $n \geq 14$, $A \cup B$ is the following Hamilton cycle:

$$\begin{aligned}
A \cup B = \; &(0, 1, k+3, 5, 6, k+4, k+5, \\
&7, 8, 9, \ldots, k, \\
&2, 3, k+1, k+2, 4, \\
&k+6, k+7, k+8, \ldots, n-1).
\end{aligned}$$

For the next pairs of 1-factors, we define the following types of paths in $\mathrm{Circ}(n, \{1, \frac{n}{2}-2\})$:

$$\begin{aligned}
P_i &= [i, i+1, k+i+3, k+i+4, i+2] \\
Q_i &= [i, i+1, k+i-1, k+i, i+2]
\end{aligned}$$

Consider the graph induced by $A \cup C$. If $n = 14$, then $A \cup C = (0, 1, 6, 11, 2, 3, 4, 13, 12, 7, 8, 9, 10, 5)$ and if $n = 18$, then $A \cup C = (0, 1, 8, 9, 10, 11, 12, 5, 16, 17, 6, 13, 2, 3, 4, 15, 14, 7)$, both of which are Hamilton cycles in the respective circulant graphs. For $n \geq 22$, define the following path:

$$\begin{aligned}
X = \; &[k-3, k-2, 0, 1, k-1, k, k+1, k+2, k+3, 5, k+7, \\
&k+8, 6, k+4, 2, 3, 4, k+6, k+5, 7, k+9, k+10, 8]
\end{aligned}$$

For $n = 22$, $A \cup C = X$ is a Hamilton cycle. For $n \geq 26$, $A \cup C = X \cup P_8 \cup P_{10} \cup \cdots \cup P_{k-5}$, which is a Hamilton cycle.

Consider the graph induced by $A \cup D$. If $n = 14$, then $A \cup D = (0, 1, 2, 3, 12, 7, 6, 11, 10, 5, 4, 13, 8, 9)$, which is a Hamilton cycle. For $n \geq 18$, define the following path:

$$Y = [k+1, k+2, 0, 1, 2, 3, k+5, 7, 6, k+4, k+3, 5, 4, k+6, 8, 9, k+7, k+8, 10]$$

For $n = 18$, $A \cup D = Y$ is a Hamilton cycle. For $n \geq 22$, $A \cup D = Y \cup Q_{10} \cup Q_{12} \cup \cdots \cup Q_{k-1}$, which is again a Hamilton cycle.

Consider the graph induced by $B \cup C$. If $n = 14$, then $B \cup C = (0, 5, 6, 1, 10, 9, 4, 3, 8, 7, 2, 11, 12, 13)$, which is a Hamilton cycle. Define the following path:

$$Z = [k-2, k-1, 1, k+3, k+2, 4, 3, k+1, k, 2, k+4, k+5, k+6, k+7, 5]$$

If $n \geq 18$, then $B \cup C = Z \cup P_5 \cup P_7 \cup \cdots \cup P_{k-4}$, which is a Hamilton cycle.

Consider the graph induced by $B \cup D$. If $n = 14$ then $B \cup D = (7, 2, 1, 10, 11, 12, 3, 8, 13, 0, 9, 4, 5, 6)$, which is a Hamilton cycle. For $n \geq 18$, define the following path:

$$W = [k, 2, 1, k+3, k+4, k+5, 3, k+1, n-1, 0, k+2, 4, 5, 6, 7, 8, k+6, k+7, 9]$$

If $n = 18$ then $B \cup D = W$, which is a Hamilton cycle. If $n \geq 22$, then $B \cup D = W \cup Q_9 \cup Q_{11} \cup \cdots \cup Q_{k-2}$, which is a Hamilton cycle.

For the last pair of 1-factors, $C \cup D$, we define the following types of paths in $\mathrm{Circ}(n, \{1, \frac{n}{2} - 2\})$:

$$\begin{aligned} S_i &= [i, k+i+2, i+4] \\ T_i &= [i, k+i-2, i-4] \end{aligned}$$

We first describe the Hamilton cycle formed by the graph with edge set $C \cup D$ for small values of n:

$$C \cup D = \begin{cases} (8, 13, 12, 3, 4, 5, 0, 9, 10, 11, 2, 1, 6, 7) & \text{if } n = 14 \\ (8, 15, 14, 3, 4, 5, 16, 9, 10, 17, 6, 7, 0, 11, 12, 13, 2, 1) & \text{if } n = 18 \\ (8, 17, 16, 3, 4, 5, 18, 9, 0, 13, 14, 15, 2, 1, 10, 19, 6, 7, 20, 11, 12, 21) & \text{if } n = 22. \end{cases}$$

The general description of $C \cup D$, for $n \geq 26$, depends on the congruence class of n modulo 8. If $n \equiv 2 \pmod 8$, let $g = \frac{k-13}{4}$. Then the graph with edge set $C \cup D$ is the following Hamilton cycle

$$[8, k+6, k+5, 3, 4, 5, k+7, 9] \cup \bigcup_{i=0}^{g} S_{9+4i} \cup [k, k+1, n-1, k-3] \cup \bigcup_{i=0}^{g} T_{k-3-4i} \cup$$
$$[6, 7, k+9, 11] \cup \bigcup_{i=0}^{g} S_{11+4i} \cup [k+2, k+3, k+4, 2, 1, k-1] \cup \bigcup_{i=0}^{g} T_{k-1-4i}.$$

If $n \equiv 6 \pmod 8$ let $g = \frac{k-11}{4}$. Then the graph with edge set $C \cup D$ is the following Hamilton cycle

$$[8, k+6, k+5, 3, 4, 5, k+7, 9] \cup \bigcup_{i=0}^{g} S_{9+4i} \cup [k+2, k+3, k+4, 2, 1, k-1] \cup$$

$$\bigcup_{i=0}^{g} T_{k-1-4i} \cup [6, 7, k+9, 11] \cup \bigcup_{i=0}^{g-1} S_{11+4i} \cup [k, k+1, n-1, k-3] \cup \bigcup_{i=0}^{g-1} T_{k-3-4i}.$$

Therefore A, B, C, D is a P1F of $\mathrm{Circ}(n, \{1, \frac{n}{2} - 2\})$ when $n \equiv 2 \pmod 4$. $\square$

Lemma 13 *Suppose $n \equiv 2 \pmod 4$ so that $n = 4s + 2$ for some positive integer $s \geq 2$. Then*

$$\mathrm{Circ}(n, \{1, \tfrac{n}{2} - 2\}) \cong \begin{cases} \mathrm{Circ}(n, \{1, s\}) & \text{if } s \text{ is odd} \\ \mathrm{Circ}(n, \{1, s+1\}) & \text{if } s \text{ is even} \end{cases}$$

Proof. Suppose s is odd. It is straightforward to show that $\gcd(s, n) = 1$, so $s \in \mathbb{Z}_n^*$. We show the desired isomorphism is achieved by the multiplier s. Observe that $s\{1, \frac{n}{2} - 2\} = s\{1, 2s - 1\} = \{s, s(2s - 1)\}$. Since s is odd, let $s = 2\ell + 1$. Now $s(2s-1) = \ell n + 1$, and hence $s(2s-1) \equiv 1 \pmod n$. Therefore, $\mathrm{Circ}(n, \{1, \frac{n}{2} - 2\}) \cong \mathrm{Circ}(n, \{1, s\})$, by Theorem 1.

Next suppose s is even. It is straightforward to show that $\gcd(s + 1, n) = 1$, so $s + 1 \in \mathbb{Z}_n^*$. We show the desired isomorphism is achieved by the multiplier $s + 1$. Observe that $(s+1)\{1, \frac{n}{2} - 2\} = (s+1)\{1, 2s-1\} = \{s+1, (s+1)(2s-1)\}$. Since s is even, let $s = 2\ell$. Now $(s+1)(2s-1) = \ell n - 1$, and hence $(s+1)(2s-1) \equiv -1 \pmod n$. Therefore, $\mathrm{Circ}(n, \{1, \frac{n}{2} - 2\}) \cong \mathrm{Circ}(n, \{1, s+1\})$, by Theorem 1. $\square$

Combining Lemma 13 and Theorem 12 we obtain the following corollary on the existence of P1Fs for classes of bipartite 4-regular circulant graphs, many of which are not covered by Theorem 7.

Corollary 14 *Suppose $n \geq 14$ and $n \equiv 2 \pmod 4$ so that $n = 4s + 2$ for some positive integer $s \geq 3$. If s is odd, then $\mathrm{Circ}(n, \{1, s\})$ admits a P1F. If s is even, then $\mathrm{Circ}(n, \{1, s+1\})$ admits a P1F.*

3 A Nonexistence Result

For n an even integer, $6 < n \leq 30$, it has been shown by computer search that $\mathrm{Circ}(n, \{1, 4\})$ does not admit a P1F [11]. The following lemma will be useful to prove that this result holds for all even integers $n > 6$.

Lemma 15 *Suppose n is an even integer, $n \geq 16$. If $\mathrm{Circ}(n, \{1, 4\})$ admits a P1F then for every $x \in \mathbb{Z}_n$, the edges $\{x, x+4\}$ and $\{x+2, x+6\}$ (where vertex labels are calculated modulo n) belong to distinct 1-factors of the P1F.*

Proof. Suppose that there is a P1F of Circ$(n, \{1, 4\})$ and the four 1-factors are coloured red, blue, green and yellow. Suppose that for some $x \in \mathbb{Z}_n$, the edges $\{x, x+4\}$ and $\{x+2, x+6\}$ belong to the same 1-factor; without loss of generality, assume $x = 0$ and the red 1-factor contains both the edges $\{0, 4\}$ and $\{2, 6\}$. Based on the possible colours for edges $\{1, 5\}$ and $\{3, 7\}$, there are five cases to consider. We derive a contradiction in each case.

For any $x \in \mathbb{Z}_n$, consider the edges $\{x, x+4\}, \{x+2, x+6\}, \{x+1, x+5\}$ and $\{x+3, x+7\}$ (where vertex labels are calculated modulo n). We say that these edges have a *Case 1 configuration* if $\{x, x+4\}, \{x+2, x+6\}$ are one colour and $\{x+1, x+5\}, \{x+3, x+7\}$ are a second colour. They have a *Case 2 configuration* if they are coloured with exactly three distinct colours such that the pair of edges with the same colour is either $\{x, x+4\}, \{x+2, x+6\}$ or $\{x+1, x+5\}, \{x+3, x+7\}$. They have a *Case 3 configuration* if they are all the same colour.

Case 1: Suppose $\{1, 5\}$ and $\{3, 7\}$ are the same colour, but not red; say without loss of generality they are green. Then $\{n-1, 3\}$, $\{n-3, 1\}$ and $\{5, 9\}$ are red; similarly $\{n-2, 2\}$, $\{4, 8\}$ and $\{6, 10\}$ are green. Without loss of generality, assume $\{0, 1\}$ is blue and $\{1, 2\}$ is yellow. Then $\{2, 3\}$, $\{4, 5\}$ and $\{6, 7\}$ are blue and $\{3, 4\}$ and $\{5, 6\}$ are yellow as in Figure 1. If $\{n-1, 0\}$ is yellow then $(n-1, 0, 4, 3)$ is a red-yellow 4-cycle, which is a contradiction; hence $\{n-1, 0\}$ is green. If $\{7, 8\}$ is yellow then $(3, 4, 8, 7)$ is a green-yellow 4-cycle, a contradiction; thus $\{7, 8\}$ is red (see Figure 1). However, now $(n-1, 0, 4, 8, 7, 3)$ is a red-green 6-cycle, which is a contradiction. Thus, Case 1 is impossible.

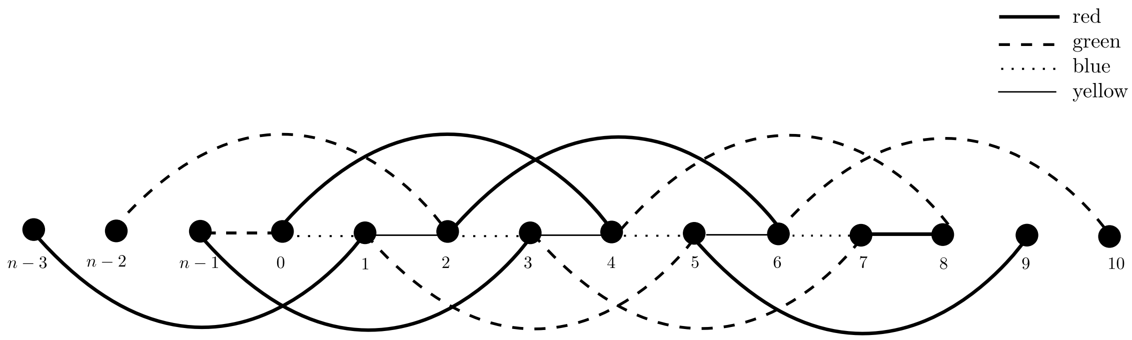

Figure 1: Case 1 is impossible.

Case 2: Suppose $\{1, 5\}$ and $\{3, 7\}$ are different colours and both not red, say without loss of generality that $\{1, 5\}$ is green and $\{3, 7\}$ is blue. Then $\{n-3, 1\}$, $\{n-1, 3\}$ and $\{5, 9\}$ are red.

Suppose $\{3, 4\}$ is green. Then $\{2, 3\}$ and $\{0, 1\}$ are yellow, $\{1, 2\}$ is blue and $\{n-2, 2\}$ is green. Edge $\{n-1, 0\}$ is blue (otherwise $(n-1, 0, 4, 3)$ is a red-green 4-cycle) and thus $\{n-4, 0\}$ is green, $\{n-2, n-1\}$ is yellow, $\{n-5, n-1\}$ is green, $\{n-3, n-2\}$ is blue and $\{n-4, n-3\}$ is yellow (see Figure 2). If $\{n-5, n-4\}$ is blue, then $(n-5, n-4, 0, n-1)$ is a blue-green 4-cycle, a contradiction. If $\{n-5, n-4\}$

is red, then $(n-5, n-4, 0, 4, 3, n-1)$ is a red-green 6-cycle, again a contradiction. Thus $\{3,4\}$ cannot be green.

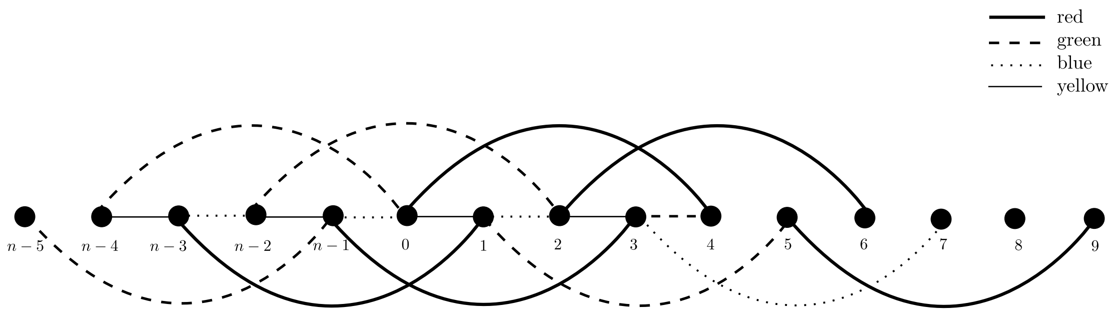

Figure 2: If edge $\{3,4\}$ is green in Case 2.

Therefore $\{3,4\}$ is yellow. It follows that $\{2,3\}$ is green, $\{4,5\}$ is blue, $\{4,8\}$ is green, $\{5,6\}$ is yellow, $\{6,7\}$ is green and $\{6,10\}$ is blue, as in Figure 3.

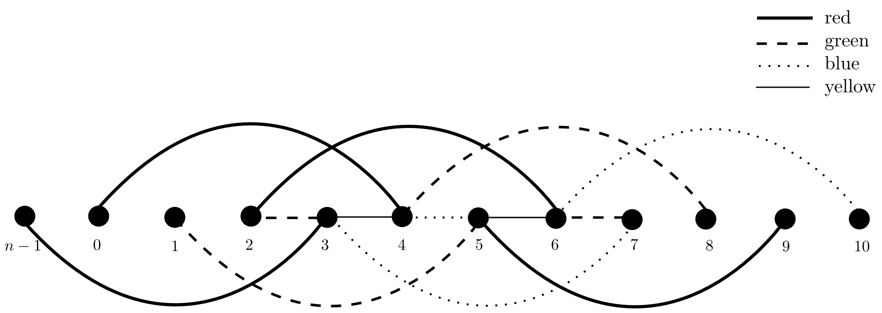

Figure 3: Edge $\{3,4\}$ is yellow in Case 2.

Suppose $\{1,2\}$ is yellow. Then $\{0,1\}$ is blue. If $\{7,8\}$ is yellow, then $(1,2,3,4,8,7,6,5)$ is a green-yellow 8-cycle, which is a contradiction; thus $\{7,8\}$ is red. However, now edge $\{n-1,0\}$ cannot be yellow or green, otherwise $(n-1,0,4,3)$ is a red-yellow 4-cycle, or $(n-1,0,4,8,7,6,2,3)$ is a red-green 8-cycle, respectively, which is a contradiction in either case. Hence $\{1,2\}$ is not yellow.

Therefore $\{1,2\}$ is blue. Now, suppose $\{7,8\}$ is yellow. Then $\{8,9\}$ is blue. However, now edge $\{9,10\}$ cannot be yellow or green, otherwise $(3,4,5,6,10,9,8,7)$ is a blue-yellow 8-cycle, or $(1,2,3,7,6,10,9,8,4,5)$ is a blue-green 10-cycle, respectively, which is a contradiction in either case. Hence $\{7,8\}$ is not yellow, so $\{7,8\}$ is red.

Since $\{1,2\}$ is blue and $\{7,8\}$ is red, it follows that $\{n-2,2\}$, $\{0,1\}$ and $\{7,11\}$ are yellow, as in Figure 4. It is straightforward to check that edge $\{n-1,0\}$ is blue (since otherwise $(n-1,0,4,8,7,6,2,3)$ is a red-green 8-cycle) and edge $\{8,9\}$ is yellow (since otherwise $(n-1,0,4,5,9,8,7,3)$ is a red-blue 8-cycle). It follows that $\{8,12\}$ is blue,$\{9,10\}$ is green, $\{9,13\}$ is blue, $\{10,11\}$ is red and $\{11,12\}$ is green as in Figure 4. However, now edge $\{12, 13\}$ cannot be yellow or red, since otherwise

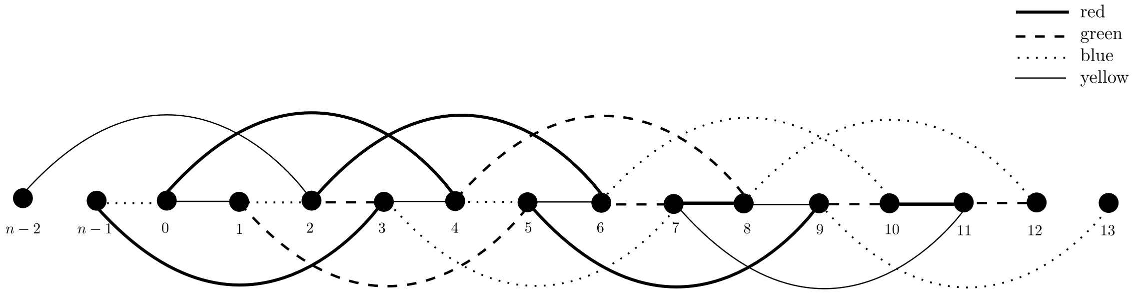

Figure 4: Case 2 is impossible

$(8, 9, 13, 12)$ is a blue-yellow 4-cycle, or $(n-1, 0, 4, 5, 9, 13, 12, 8, 7, 3)$ is a red-blue 10-cycle, respectively, which is a contradiction in either case. Therefore, Case 2 is impossible.

Case 3: Suppose $\{1, 5\}$ and $\{3, 7\}$ are both red. Without loss of generality, suppose that $\{n-1, 3\}, \{2, 3\}$ and $\{3, 4\}$ are green, yellow and blue, respectively. Now $\{1, 2\}$ is either blue or green.

<u>Subcase 3.1</u>: Suppose $\{1, 2\}$ is blue. Then $\{n-2, 2\}$ is green. Now $\{0, 1\}$ is either yellow or green.

a) Suppose $\{0, 1\}$ is yellow. Then $\{n-1, 0\}$ is blue. Now $\{n-2, n-1\}$ is red (since otherwise $(n-2, n-1, 3, 2)$ is a green-yellow 4-cycle) and $\{4, 5\}$ is green (since otherwise $(0, 1, 5, 4)$ is a red-yellow 4-cycle), so $\{4, 8\}$ is yellow. Edge $\{5, 6\}$ is yellow (since otherwise $(1, 2, 6, 5)$ is a red-blue 4-cycle), so $\{5, 9\}$ blue; also $\{6, 7\}$ is blue (since otherwise $(n-2, n-1, 3, 7, 6, 2)$ is a red-green 6-cycle). It follows that $\{6, 10\}$ and $\{7, 8\}$ are green, $\{7, 11\}$ is yellow, $\{8, 9\}$ is red and $\{9, 10\}$ is yellow (see Figure 5). If $\{10, 11\}$ is blue then $(5, 6, 7, 11, 10, 9)$ is a blue-yellow 6-cycle, which is

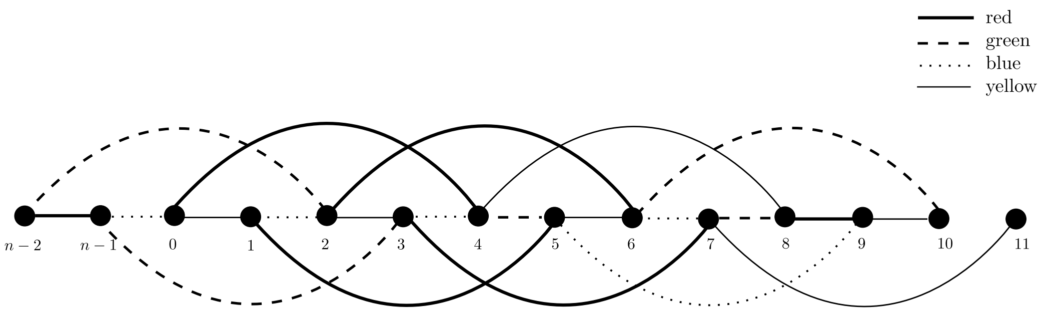

Figure 5: Case 3.1(a) is impossible.

a contradiction; if $\{10, 11\}$ is red, then $(0, 1, 5, 6, 2, 3, 7, 11, 10, 9, 8, 4)$ is a red-yellow 12-cycle, which is again a contradiction, so Case 3.1(a) is impossible.

b) If $\{0, 1\}$ is green then $\{4, 5\}$ is yellow (since otherwise $(0, 1, 5, 4)$ is a red-green 4-cycle), so $\{4, 8\}$ is green. Also $\{5, 6\}$ is green (since otherwise $(1, 2, 6, 5)$ is a red-blue 4-cycle) so $\{5, 9\}$ is blue. Edge $\{6, 7\}$ is blue (since otherwise $(2, 3, 7, 6)$ is a red-yellow 4-cycle), thus it follows that $\{6, 10\}$ and $\{7, 8\}$ are yellow, $\{7, 11\}$ is green, $\{8, 9\}$ is red, $\{8, 12\}$ is blue, $\{9, 10\}$ is green and $\{9, 13\}$ is yellow. Edge $\{10, 11\}$ is red (since otherwise $(5, 6, 7, 11, 10, 9)$ is a green-blue 6-cycle) so $\{10, 14\}$ is blue (see Figure 6). Now edges $\{7, 11\}, \{9, 13\}, \{8, 12\}$ and $\{10, 14\}$ have a Case 2 configuration, which is a contradiction.

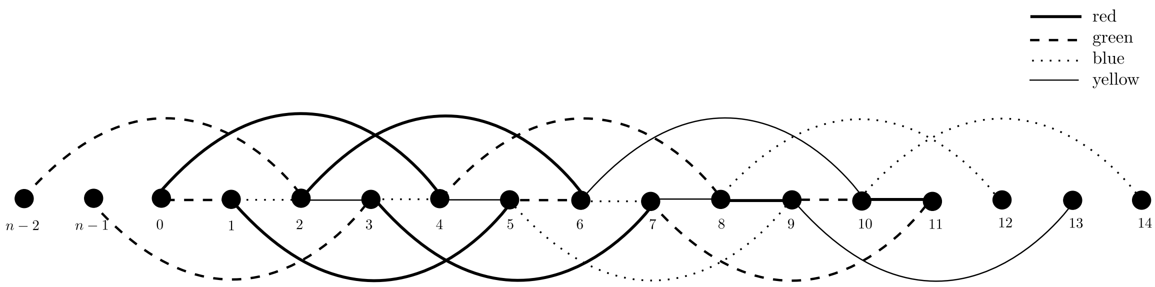

Figure 6: Case 3.1(b) is impossible.

<u>Subcase 3.2</u>: Suppose $\{1, 2\}$ is green. Edge $\{5, 6\}$ is not green (otherwise $(1, 2, 6, 5)$ is a red-green 4-cycle) so $\{5, 6\}$ is either yellow or blue.

a) Suppose $\{5, 6\}$ is yellow. Then $\{4, 5\}$ is green, $\{4, 8\}$ is yellow and $\{5, 9\}$ is blue. Now $\{6, 7\}$ is either blue or green.

i) If $\{6, 7\}$ is blue then $\{6, 10\}$ and $\{7, 8\}$ are green, $\{7, 11\}$ is yellow, $\{8, 9\}$ is red, $\{8, 12\}$ is blue, $\{9, 10\}$ is yellow and $\{9, 13\}$ is green. Edge $\{10, 11\}$ is not blue (otherwise $(5, 6, 7, 11, 10, 9)$ is a blue-yellow 6-cycle) so $\{10, 14\}$ is blue (see Figure 7). Now edges $\{7, 11\}, \{9, 13\}, \{8, 12\}$ and $\{10, 14\}$ have a Case 2 configuration, which is a contradiction.

ii) If $\{6, 7\}$ is green, then $\{6, 10\}$ and $\{7, 8\}$ are blue and $\{7, 11\}$ is yellow. Now edge $\{9, 10\}$ is not yellow (otherwise $(5, 6, 10, 9)$ is a blue-yellow 4-cycle) so $\{9, 13\}$ is yellow. The edge $\{8, 12\}$ is either green or red (see Figure 8), and in both cases the edges $\{6, 10\}, \{8, 12\}, \{7, 11\}$ and $\{9, 13\}$ have a Case 2 configuration, which is a contradiction.

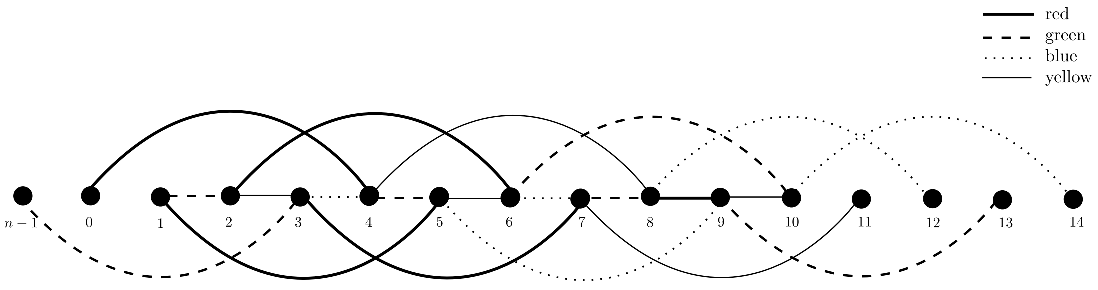

Figure 7: Case 3.2(a)(i) is impossible.

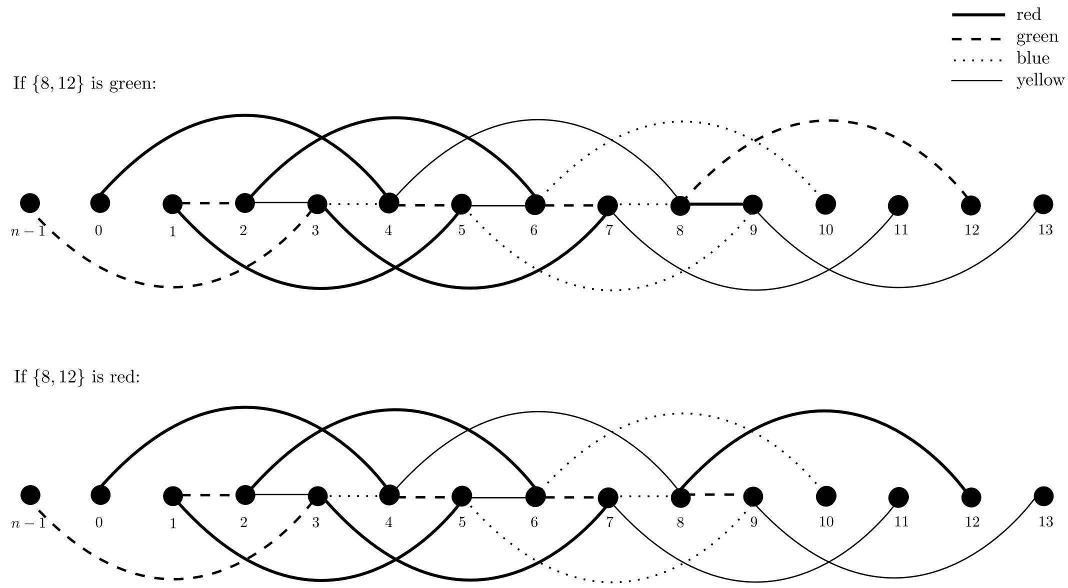

Figure 8: Case 3.2(a)(ii) is impossible.

b) Suppose $\{5, 6\}$ is blue. Then $\{6, 7\}$ is green (since otherwise $(2, 3, 6, 7)$ is a red-yellow 4-cycle) so $\{6, 10\}$ is yellow. Now $\{4, 5\}$ is either green or yellow.

i) If $\{4, 5\}$ is green then $\{4, 8\}$ and $\{5, 9\}$ are yellow, $\{7, 8\}$ is blue and $\{7, 11\}$ is yellow. Also $\{8, 9\}$ is red (since otherwise $(4, 5, 9, 8)$ is a green-yellow 4-cycle), so $\{8, 12\}$ is green. Edge $\{9, 10\}$ is green (since otherwise $(5, 6, 10, 9)$ is a blue-yellow 4-cycle), so $\{9, 13\}$ is blue. Edge $\{10, 11\}$ is blue (since otherwise $(2, 3, 7, 11, 10, 6)$ is a red-yellow 6-cycle) and hence $\{10, 14\}$ and $\{11, 12\}$ are red, $\{12, 13\}$ is yellow and $\{11, 15\}$ and $\{13, 14\}$ are green as in Figure 9.

If $\{14, 15\}$ is blue, then $(9, 10, 11, 15, 14, 13)$ is a blue-green 6-cycle, which is a contradiction; if $\{14, 15\}$ is yellow, then $(4, 5, 9, 10, 6, 7, 11, 15, 14, 13, 12, 8)$ is a green-yellow 12-cycle, which is again a contradiction, so Case 3.2(b)(i) is impossible.

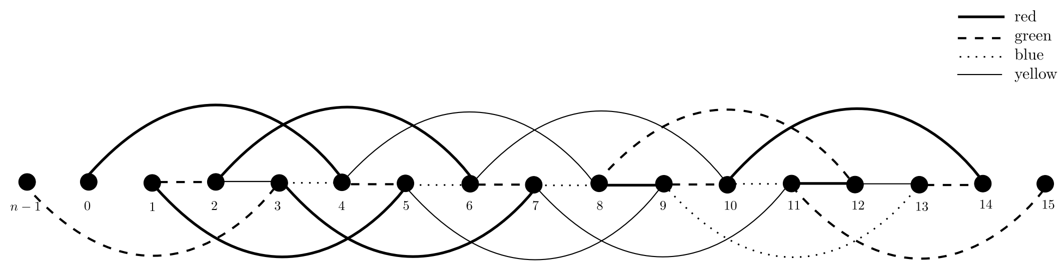

Figure 9: Case 3.2(b)(i) is impossible.

ii) If $\{4,5\}$ is yellow, then $\{4,8\}$ and $\{5,9\}$ are green. Edge $\{8,9\}$ is not yellow (otherwise $(4,5,9,8)$ is a green-yellow 4-cycle) thus $\{9,13\}$ is yellow, as shown in Figure 10.

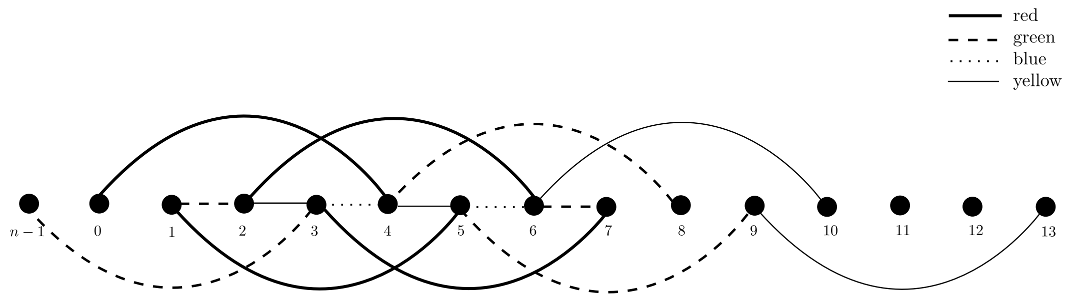

Figure 10: Edge $\{4,5\}$ is yellow in Case 3.2(b)(ii).

Now edge $\{7,8\}$ is either blue or yellow. If $\{7,8\}$ is blue then $\{7,11\}$ is yellow, $\{8,9\}$ is red, $\{8,12\}$ is yellow and $\{9,10\}$ is blue, as in Figure 11. However, if $\{10,11\}$ is green then $(6,7,11,10)$ is a green-yellow 4-cycle, which is a contradiction, and if $\{10,11\}$ is red then $(2,3,7,11,10,6)$ is a red-yellow 6-cycle, which is again a contradiction. Thus $\{7,8\}$ is not blue. Hence $\{7,8\}$ is yellow and thus $\{7,11\}$ is blue.

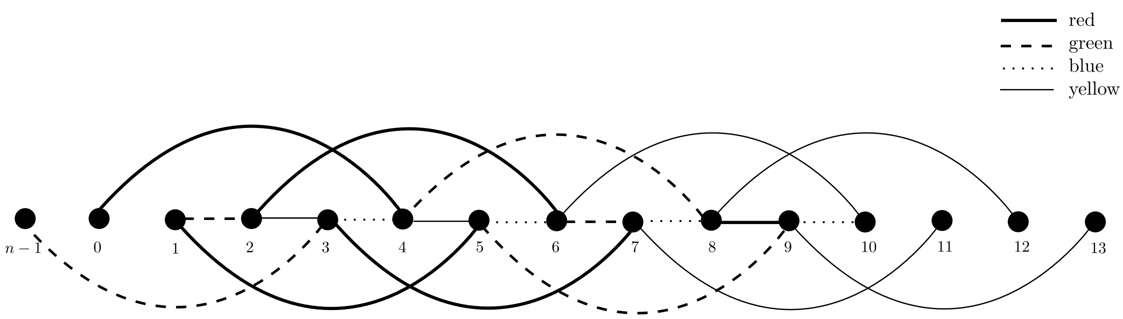

Figure 11: Edge $\{7,8\}$ cannot be blue in Case 3.2(b)(ii).

Now $\{8,9\}$ is either red or blue. If $\{8,9\}$ is red (see the first part of Figure 12), then $\{8,12\}$ and $\{9,10\}$ are blue. Edge $\{10,11\}$ is red (since otherwise $(5,6,7,11,10,9)$ is a blue-green 6-cycle) so $\{10,14\}$ is green. Edge $\{11,12\}$ is green (since otherwise $(7,8,12,11)$ is a blue-yellow 4-cycle) so $\{11,15\}$ is yellow. Now edges $\{8,12\},\{10,14\},\{9,13\}$ and $\{11,15\}$ have a Case 2 configuration, which is a contradiction.

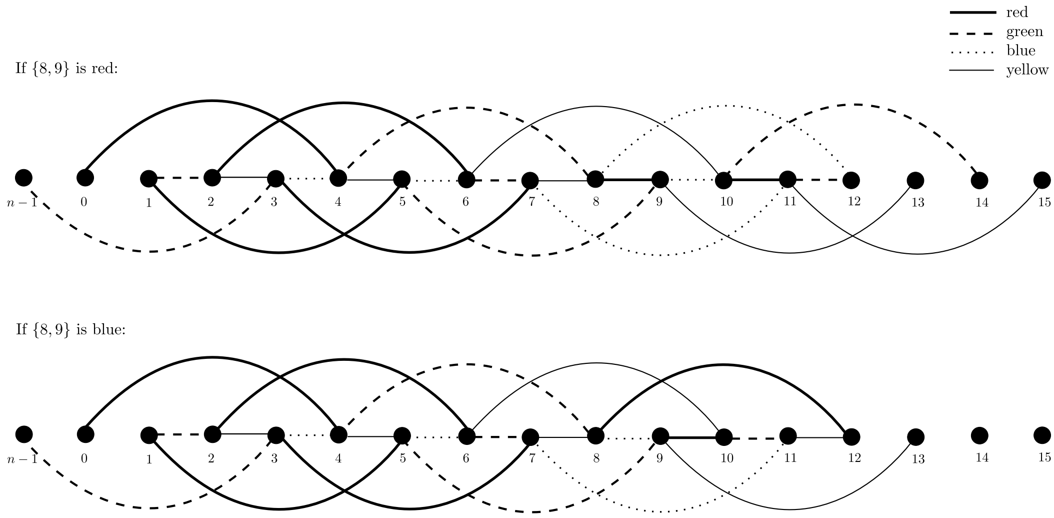

Figure 12: Case 3.2(b)(ii) is impossible.

If $\{8,9\}$ is blue (see the second part of Figure 12), then $\{8,12\}$ and $\{9,10\}$ are red, $\{10,11\}$ is green and $\{11,12\}$ is yellow. Thus $\{12,13\}$ is either blue or green. If $\{12,13\}$ is blue then $(7,8,9,13,12,11)$ is a blue-yellow 6-cycle, which is a contradiction; if $\{12,13\}$ is green then $(4,5,9,13,12,11,10,6,7,8)$ is a green-yellow 10-cycle, which is again a contradiction. Therefore Case 3 is impossible.

Case 4: Suppose $\{1,5\}$ is red and $\{3,7\}$ is not red, say without loss of generality that it is green. Then $\{n-1,3\}$ is red and hence edges $\{n-1,3\},\{1,5\},\{0,4\}$ and $\{2,6\}$ have a Case 3 configuration, which is a contradiction. Therefore Case 4 is impossible.

Case 5: Suppose $\{3,7\}$ is red and $\{1,5\}$ is not red, say without loss of generality that it is green. If $\{n-1,3\}$ is green, edges $\{n-1,3\},\{1,5\},\{0,4\}$ and $\{2,6\}$ have a Case 1 configuration, which is a contradiction; if $\{n-1,3\}$ is yellow or blue, then edges $\{n-1,3\},\{1,5\},\{0,4\}$ and $\{2,6\}$ have a Case 2 configuration, which is again a contradiction. Thus, Case 5 is impossible.

Thus, the edges $\{x,x+4\},\{x+2,x+6\}$ cannot belong to the same 1-factor for any $x \in \mathbb{Z}_n$. $\square$

Theorem 16 *If n is even with $n > 16$ then* $\mathrm{Circ}(n,\{1,4\})$ *does not admit a P1F.*

Proof. Suppose Circ$(n, \{1, 4\})$ admits a P1F and let the four 1-factors be coloured red, blue, green and yellow. By Lemma 15, the edges $\{0, 4\}$, $\{2, 6\}$ and $\{4, 8\}$ must all be distinct colours, so without loss of generality, suppose that $\{0, 4\}$ is red, $\{2, 6\}$ is green and $\{4, 8\}$ is blue. We now consider edge $\{1, 5\}$, which could belong to any one of the four 1-factors, and derive a contradiction in each of the four cases.

Case 1: Suppose $\{1, 5\}$ is yellow. Then $\{4, 5\}$ is green, and $\{3, 4\}$ and $\{n-2, 2\}$ are yellow. Now edge $\{3, 7\}$ is red, blue or green.

a) Suppose $\{3, 7\}$ is red. Then $\{2, 3\}$ is blue, $\{n-1, 3\}$ is green and $\{1, 2\}$ is red. By Lemma 15, $\{5, 9\}$ is not red, so $\{5, 6\}$ is red. Similarly, by Lemma 15, $\{n-3, 1\}$ is not green, so $\{0, 1\}$ is green and thus $(0, 1, 2, 6, 5, 4)$ is a red-green 6-cycle, which is a contradiction.

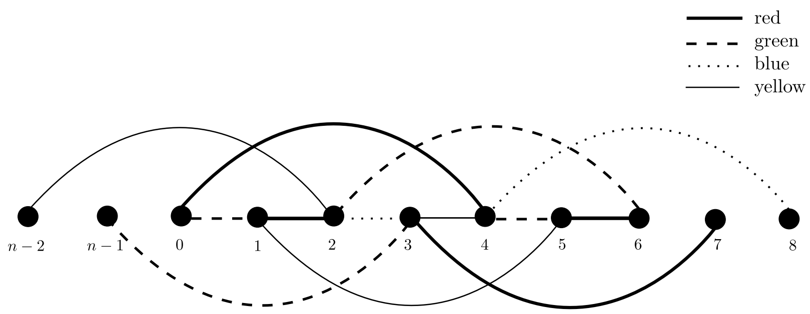

Figure 13: Case 1(a) is impossible.

b) Suppose $\{3, 7\}$ is blue. Then $\{2, 3\}$ is red and $\{n-1, 3\}$ is green. As in the previous case, $\{0, 1\}$ is green. It follows that $\{1, 2\}$ is blue and $\{n-3, 1\}$ is red. By Lemma 15, $\{n-1, 0\}$ is yellow and now $(n-1, 0, 1, 5, 4, 3)$ is a green-yellow 6-cycle, which is a contradiction.

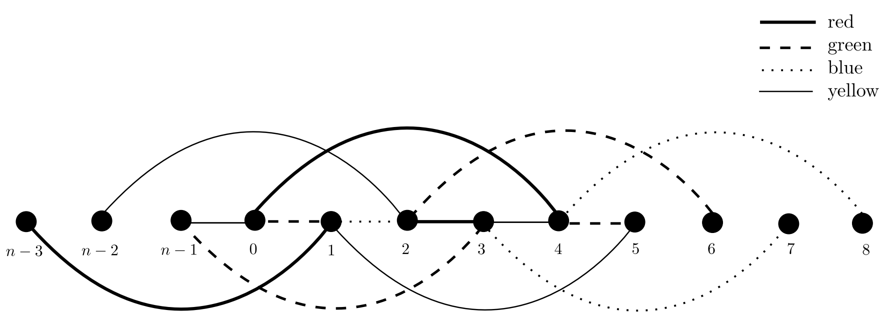

Figure 14: Case 1(b) is impossible.

c) Suppose $\{3, 7\}$ is green. Edge $\{n-1, 3\}$ is either red or blue. If $\{n-1, 3\}$ is red (see the first part of Figure 15), then $\{2, 3\}$ is blue and $\{1, 2\}$ is red. By Lemma 15, $\{n-4, 0\}$ is not yellow, hence $\{n-1, 0\}$ is yellow. Then $(n-1, 0, 4, 3)$ is a red-yellow 4-cycle, which is a contradiction. If $\{n-1, 3\}$ is blue (see the second part of Figure 15), then $\{2, 3\}$ is red, $\{1, 2\}$ is blue, $\{0, 1\}$ is green and $\{n-1, 0\}$ is yellow. It follows that $\{n-3, 1\}$ is red, so by Lemma 15, $\{n-5, n-1\}$ is not red and hence $\{n-1, n-2\}$ is red. Thus $(n-2, n-1, 0, 4, 3, 2)$ is a red-yellow 6-cycle, which is again a contradiction.

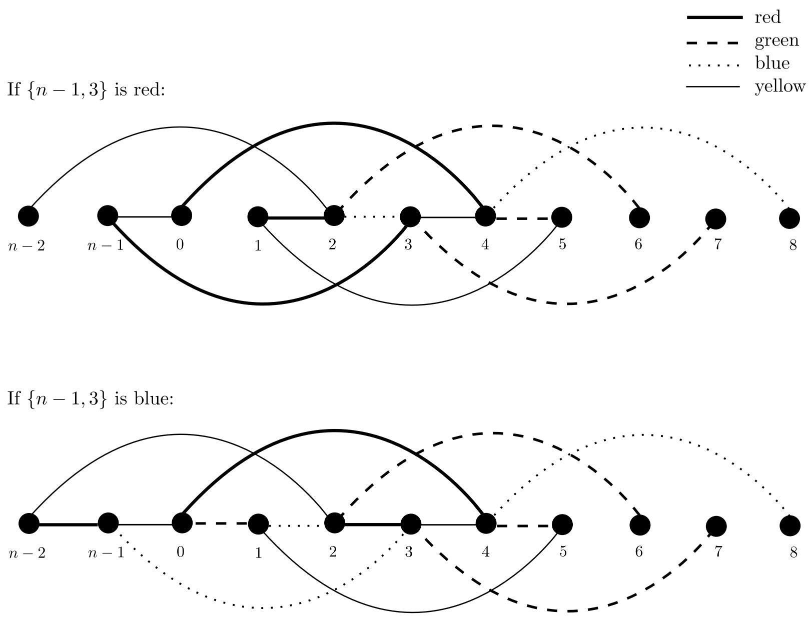

Figure 15: Case 1(c) is impossible.

Thus Case 1 is impossible.

Case 2: Suppose $\{1, 5\}$ is blue. Then, by Lemma 15, neither $\{3, 7\}$ nor $\{n-1, 3\}$ is blue, hence $\{2, 3\}$ is blue. Since $\{0, 4\}$ is red, it follows from Lemma 15 that $\{n-2, 2\}$ is not red, hence $\{1, 2\}$ is red. Also by Lemma 15, $\{6, 10\}$ is not blue, so $\{6, 7\}$ is blue. If $\{3, 7\}$ is green, then $(2, 3, 7, 6)$ is a blue-green 4-cycle, which is a contradiction. Thus $\{3, 7\}$ is either red or yellow.

If $\{3, 7\}$ is red, then, by Lemma 15, $\{5, 6\}$ is red (see the first part of Figure 16). Thus $(1, 2, 3, 7, 6, 5)$ is a red-blue 6-cycle, which is a contradiction. If $\{3, 7\}$ is yellow (see the second part of Figure 16), then $\{3, 4\}$ is green, $\{4, 5\}$ is yellow, $\{5, 6\}$ is red and $\{5, 9\}$ is green. By Lemma 15, $\{7, 11\}$ is not green, so $\{7, 8\}$ is green and hence $(2, 3, 4, 8, 7, 6)$ is a blue-green 6-cycle, which is again a contradiction. Thus Case 2 is impossible.

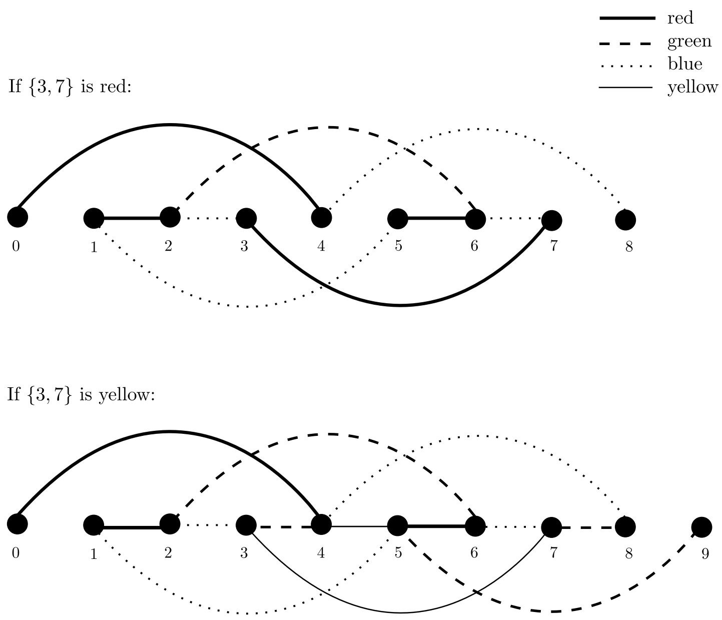

Figure 16: Case 2 is impossible.

Case 3: Suppose $\{1, 5\}$ is green. By Lemma 15, neither $\{n-1, 3\}$ nor $\{3, 7\}$ is green, hence $\{3, 4\}$ is green and $\{4, 5\}$ is yellow. We consider the three possible colours of $\{3, 7\}$.

a) Suppose $\{3, 7\}$ is red. Then, by Lemma 15, $\{5, 9\}$ is not red, so $\{5, 6\}$ is red. Also, by Lemma 15, $\{n-2, 2\}$ is not red so it follows that $\{1, 2\}$ is red (see Figure 17). Thus $(1, 2, 6, 5)$ is a red-green 4-cycle, which is a contradiction.

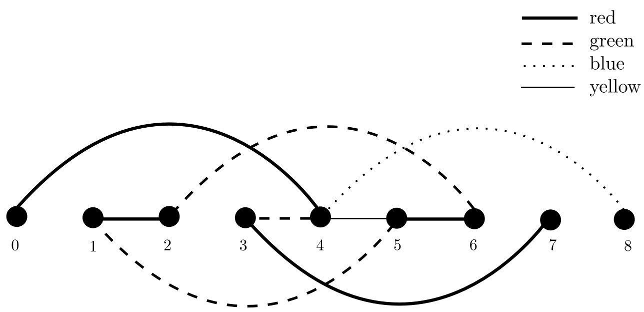

Figure 17: Case 3(a) is impossible.

b) Suppose $\{3, 7\}$ is yellow. Then $\{5, 9\}$ is either blue or red. If $\{5, 9\}$ is blue (see the first part of Figure 18), then $\{5, 6\}$ is red and $\{6, 7\}$ is blue, so $\{6, 10\}$ is yellow. Then, by Lemma 15, $\{8, 12\}$ is not yellow, so $\{8, 9\}$ is yellow. Thus $(4, 5, 9, 8)$ is a blue-yellow 4-cycle, which is a contradiction. If $\{5, 9\}$ is red (see the second part of Figure 18), then $\{5,6\}$ is blue and $\{6,7\}$ is red, so $\{6,10\}$ is yellow, $\{7,8\}$ is green, $\{8,9\}$ is yellow and $\{7,11\}$ is blue. Then, by Lemma 15, $\{9,13\}$ is not blue, so $\{9,10\}$ is blue and thus $(4,5,6,10,9,8)$ is a blue-yellow 6-cycle, which is again a contradiction.

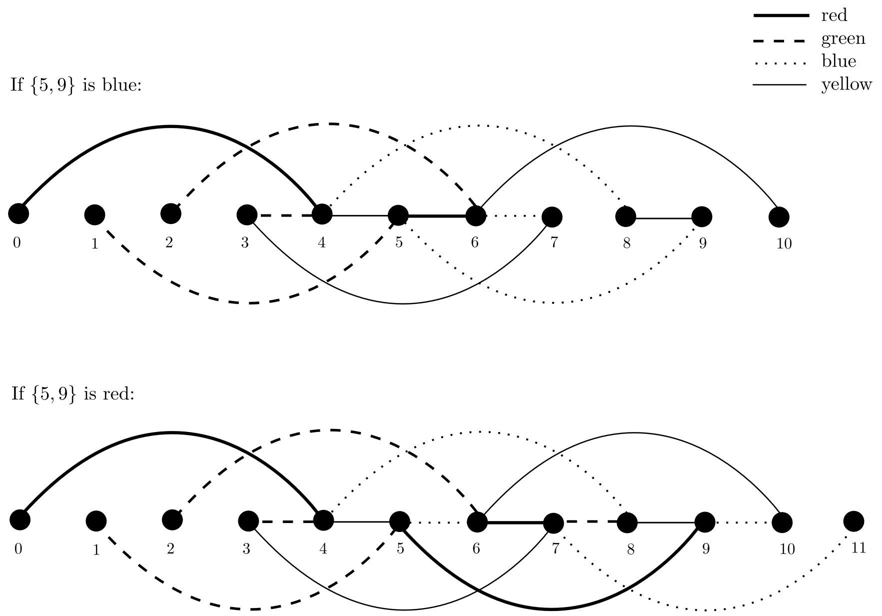

Figure 18: Case 3(b) is impossible.

c) Suppose $\{3,7\}$ is blue. By Lemma 15, $\{5,9\}$ is red and so $\{5,6\}$ is blue. Observe that $\{7,8\}$ is not green (otherwise $(3,4,8,7)$ is a blue-green 4-cycle). Now $\{6,10\}$ is either yellow or red.

i) If $\{6,10\}$ is yellow, then $\{6,7\}$ is red. Since $\{7,8\}$ is not green, it is yellow. It follows that $\{7,11\}$ and $\{8,9\}$ are green, $\{8,12\}$ is red, $\{9,10\}$ is blue, $\{9,13\}$ is yellow, $\{10,11\}$ is red and $\{10,14\}$ is green. By Lemma 15, $\{11,12\}$ is yellow and $\{12,13\}$ is green (see Figure 19). Thus $(7,8,9,13,12,11)$ is a green-yellow 6-cycle, which is a contradiction.

ii) If $\{6,10\}$ is red, then $\{6,7\}$ is yellow. Since $\{7,8\}$ is not green, it is red, and so $\{7,11\}$ is green. Now $\{8,9\}$ is either green or yellow.

If $\{8,9\}$ is green (see the first part of Figure 20), then $\{8,12\}$ is yellow. Edge $\{9,10\}$ is yellow (otherwise $(5,6,10,9)$ is a red-blue 4-cycle) so $\{9,13\}$ and $\{10,11\}$ are blue. It follows that $\{10,14\}$ is green, $\{11,12\}$ is red, $\{11,15\}$ is yellow and $\{12,13\}$ is green. By Lemma 15, $\{13,14\}$ is yellow, and thus $(8,9,10,14,13,12)$ is a green-yellow 6-cycle, which is a contradiction.

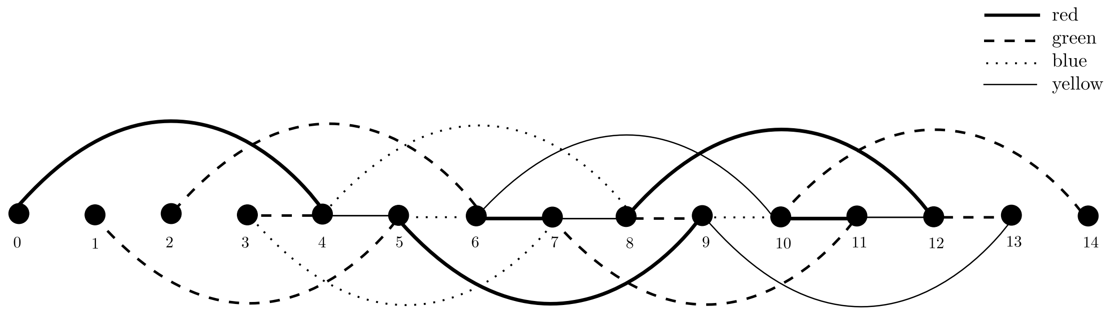

Figure 19: Case 3(c)(i) is impossible.

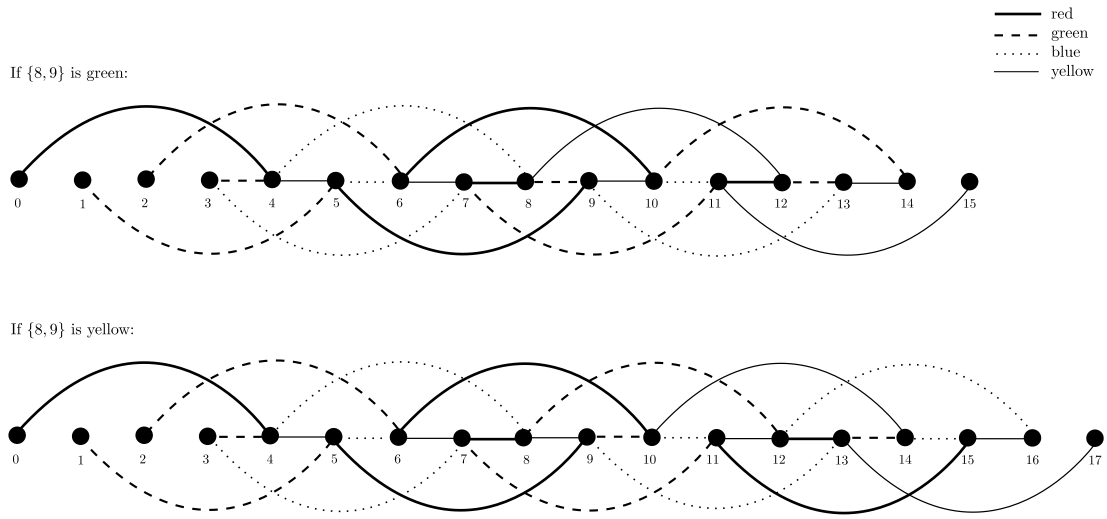

Figure 20: Case 3(c)(ii) is impossible.

If $\{8, 9\}$ is yellow (see the second part of Figure 20), then $\{8, 12\}$ is green. Edge $\{9, 10\}$ is green (otherwise $(5, 6, 10, 9)$ is a red-blue 4-cycle) so $\{9, 13\}$ is blue. If $\{10, 11\}$ is yellow, then $\{11, 12\}$ is neither red nor blue (otherwise $(7, 8, 12, 11)$ is a red-green 4-cycle or $(3, 4, 8, 12, 11, 7)$ is a blue-green 6-cycle, respectively), which is impossible. Thus $\{10, 11\}$ is blue and $\{10, 14\}$ is yellow. Now $\{11, 12\}$ is yellow (otherwise $(7, 8, 12, 11)$ is a red-green 4-cycle), so $\{11, 15\}$ and $\{12, 13\}$ are red, $\{12, 16\}$ is blue, $\{13, 14\}$ is green, $\{13, 17\}$ is yellow and $\{14, 15\}$ is blue. By Lemma 15, $\{15, 16\}$ is yellow, hence $(10, 11, 12, 16, 15, 14)$ is a blue-yellow 6-cycle, which is a contradiction. Therefore Case 3 is impossible.

Case 4: Suppose $\{1, 5\}$ is red. By Lemma 15, neither $\{3, 7\}$ nor $\{n - 1, 3\}$ is red, hence $\{2, 3\}$ is red. We consider the three possible colours of edge $\{3, 7\}$.

a) Suppose $\{3, 7\}$ is yellow. Then $\{3, 4\}$ is green, $\{4, 5\}$ is yellow, $\{5, 6\}$ is blue, $\{5, 9\}$ is green, $\{6, 7\}$ is red, $\{6, 10\}$ is yellow and $\{7, 8\}$ is green. By Lemma 15, $\{8, 9\}$ is yellow (see Figure 21). Thus $(3, 4, 5, 9, 8, 7)$ is a green-yellow 6-cycle, which is a contradiction.

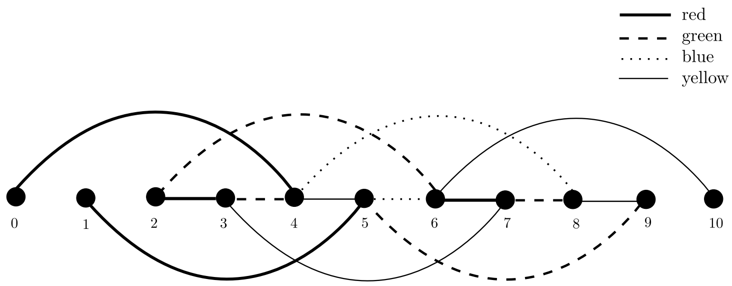

Figure 21: Case 4(a) is impossible.

b) Suppose $\{3,7\}$ is blue. Then, by Lemma 15, $\{5,6\}$ is blue. If $\{3,4\}$ is yellow then $\{4,5\}$ and $\{n-1,3\}$ are green, but then, by Lemma 15, $\{0,1\}$ is green and thus $(0,1,5,4)$ is a red-green 4-cycle, which is a contradiction, so $\{3,4\}$ is not yellow. Thus $\{3,4\}$ is green, $\{4,5\}$ is yellow and $\{5,9\}$ is green, as in Figure 22. By Lemma 15, $\{7,11\}$ is not green, so it follows that $\{7,8\}$ is green and thus $(3,4,8,7)$ is a blue-green 4-cycle, which is a contradiction.

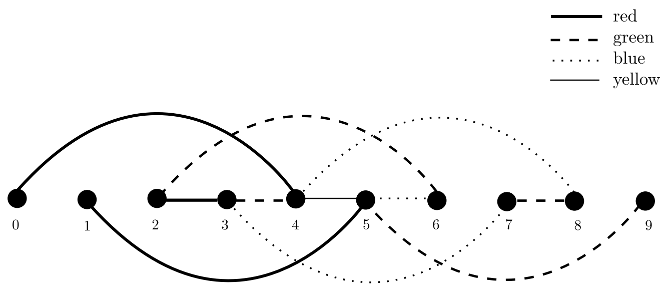

Figure 22: Case 4(b) is impossible.

c) Suppose $\{3,7\}$ is green. Then $\{3,4\}$ is yellow, $\{n-1,3\}$ is blue and $\{4,5\}$ is green. Observe that $\{6,7\}$ is not red (otherwise $(2,3,7,6)$ is a red-green 4-cycle). Now $\{5,9\}$ is either yellow or blue.

i) Suppose $\{5,9\}$ is yellow. Then $\{5,6\}$ is blue. Since $\{6,7\}$ is not red, it is yellow. It follows that $\{6,10\}$ and $\{7,8\}$ are red, $\{7,11\}$ is blue, $\{8,9\}$ is green and $\{8,12\}$ is yellow. By Lemma 15, $\{9,10\}$ is blue and $\{10,11\}$ is yellow (see Figure 23). Thus $(5,6,7,11,10,9)$ is a blue-yellow 6-cycle, which is a contradiction.

ii) Suppose $\{5,9\}$ is blue. Then $\{5,6\}$ is yellow. Since $\{6,7\}$ is not red, it is blue. Observe that $\{0,1\}$ is not green (otherwise $(0,1,5,4)$ is a red-green 4-cycle), so it follows that $\{n-3,1\}$ is green. Now $\{1,2\}$ is either blue or yellow.

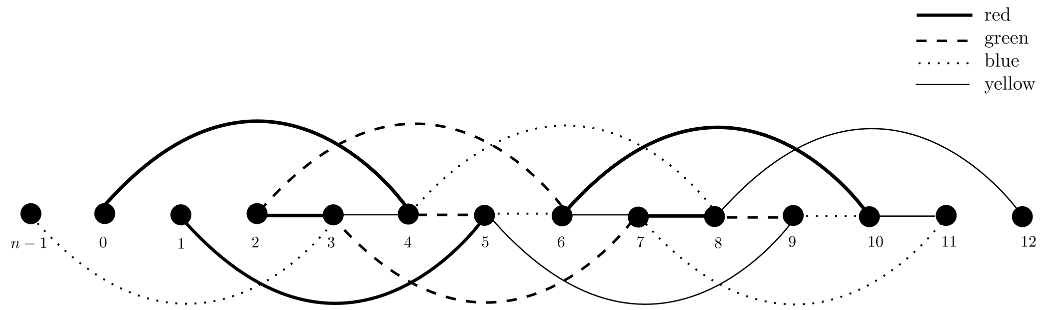

Figure 23: Case 4(c)(i) is impossible.

If $\{1, 2\}$ is blue (see the first part of Figure 24), then $\{n-2, 2\}$ and $\{0, 1\}$ are yellow. It follows that $\{n-1, 0\}$ is green, $\{n-4, 0\}$ is blue, $\{n-2, n-1\}$ is red, $\{n-5, n-1\}$ is yellow and $\{n-3, n-2\}$ is blue. By Lemma 15, $\{n-4, n-3\}$ is yellow. Then $(n-4, n-3, n-2, 2, 1, 0)$ is a blue-yellow 6-cycle, which is a contradiction.

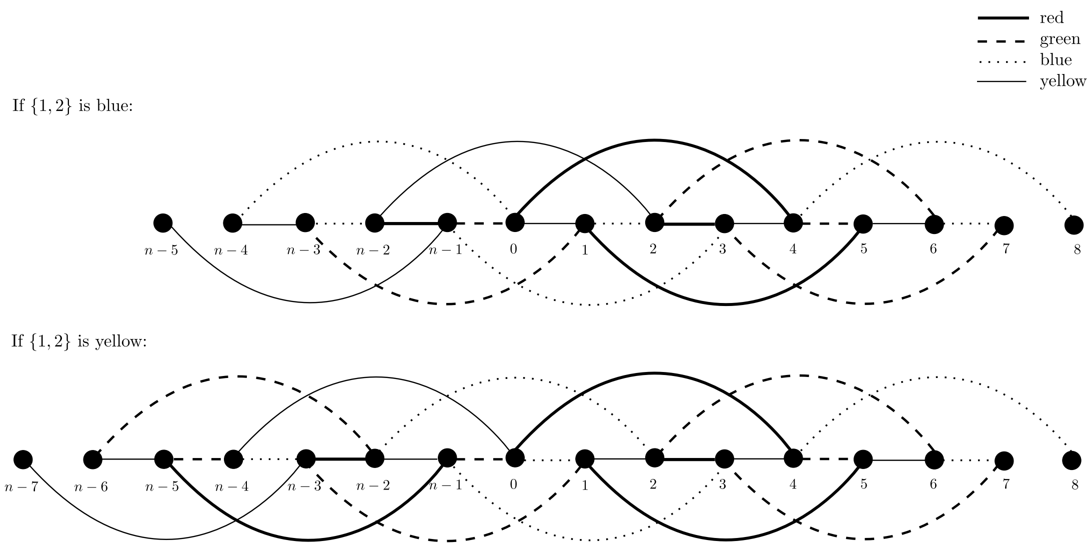

Figure 24: Case 4(c)(ii) is impossible.

If $\{1, 2\}$ is yellow (see the second part of Figure 24), then $\{n-2, 2\}$ and $\{0, 1\}$ are blue. Edge $\{n-2, n-1\}$ is not red (otherwise $(n-2, n-1, 3, 2)$ is a red-blue 4-cycle) so it follows that $\{n-5, n-1\}$ is red; also $\{n-2, n-1\}$ is not green (otherwise $(n-2, n-1, 3, 7, 6, 2)$ is a blue-green 6-cycle) so $\{n-2, n-1\}$ is yellow. Thus $\{n-1, 0\}$ is green, $\{n-4, 0\}$ is yellow, $\{n-3, n-2\}$ is red, $\{n-6, n-2\}$ is green, $\{n-4, n-3\}$ is blue, $\{n-7, n-3\}$ is yellow and $\{n-5, n-4\}$ is green. By Lemma 15, $\{n-6, n-5\}$ is yellow and thus $(n-6, n-5, n-4, 0, n-1, n-2)$ is a green-yellow 6-cycle, which is a contradiction.

Therefore, Case 4 is also impossible, and the theorem holds. $\square$

The circulant graph $\mathrm{Circ}(n, \{1, 4\})$ does not admit a P1F for $n > 16$ (Theorem 16), and does not admit a P1F for $6 < n \leq 30$ by computer search [11]. Thus, this family of graphs does not admit a P1F for $n > 6$.

Corollary 17 *If $n > 6$ is even then any 4-regular circulant graph isomorphic to* $\mathrm{Circ}(n, \{1, 4\})$ *does not admit a P1F.*

4 Conclusion

It remains an open problem to characterise the connected bipartite 4-regular circulant graphs that admit a P1F, which are necessarily of order $n \equiv 2 \pmod 4$. In [11], $\mathrm{Circ}(n, \{1, 3\})$ was shown to have a P1F for all even $n > 6$ with $n \equiv 2 \pmod 4$. Several other existence results were given in [12] and, most importantly, it was shown that there is an infinite family of bipartite 4-regular circulants of order $2 \pmod 4$ that do not admit a P1F. In Section 2 of this paper, we proved many new existence results for P1Fs of bipartite 4-regular circulant graphs of order $2 \pmod 4$. In particular, Theorem 7 demonstrates the existence of P1Fs for infinitely many bipartite 4-regular circulant graphs of the form $\mathrm{Circ}(n, \{1, b\})$ where b is odd and $n \equiv 2 \pmod 4$. The smallest bipartite 4-regular circulant graphs of order $2 \pmod 4$ for which it is unknown whether a P1F is admissable are $\mathrm{Circ}(42, S)$ for $S = \{1, 7\}, \{1, 9\}, \{1, 15\}, \{3, 7\}$.

It was shown in [11] that Conjecture 6 holds for $n < 30$ and for $\mathrm{Circ}(n, \{1, 2\})$ where $n > 6$ [11]. In [12], Conjecture 6 was also shown to hold for $\mathrm{Circ}(n, \{1, \frac{n}{2} - 1\})$ where $n > 6$ and $n \equiv 2 \pmod 4$. In Section 3 of the current paper, we provided further support for this conjecture by proving that there is no P1F of $\mathrm{Circ}(n, \{1, 4\})$ with $n > 6$ (see Corollary 17).

The results of this paper and of [11, 12] support the conjecture that the connected 4-regular circulant graphs of order $n > 6$ that admit a P1F are a subset of the bipartite 4-regular circulant graphs of order $n \equiv 2 \pmod 4$.

Acknowledgements

The author thanks Barbara Maenhaut for helpful suggestions during this research.

References

[1] A. Adam, Research problems 2-10, *J. Combin. Theory*, 2 (1967) 393.

[2] B.A. Anderson, Finite topologies and Hamiltonian paths, *J. Combin. Theory Ser. B* 14 (1973), 87–93.

[3] J. Barajas and O. Serra, On the chromatic number of circulants, *Discrete Math.* 309 (2009), 5687–5696.

[4] J.C. Bermond, O. Favaron and M. Maheo, Hamiltonian decomposition of Cayley graphs of degree 4, *J. Combin. Theory Ser. B* 46 (1989), 142–153.

[5] S. Bonvicini and G. Mazzuoccolo, A new description of perfectly one-factorable cubic graphs, *Atti Semin. Mat. Fis. Univ. Modena Reggio Emilia* 54 (2006), 167–173.

[6] S. Bonvicini and G. Mazzuoccolo, Perfect one-factorizations in generalized Petersen graphs, *Ars Combin.* 99 (2011), 33–43.

[7] D. Bryant, B.M. Maenhaut and I.M. Wanless, A family of perfect factorisations of complete bipartite graphs, *J. Combin. Theory Ser. A* 98 (2002), 328–342.

[8] S.J. Curran and J.A. Gallian, Hamiltonian cycles and paths in Cayley graphs and digraphs—a survey, *Discrete Math.* 156 (1996), 1–18.

[9] C. Delorme, O. Favaron and M. Maheo, Isomorphisms of Cayley multigraphs of degree 4 on finite ablelian groups, *European J. Combin.* 13 (1992), 59–61.

[10] X. Fang and M. Xu, On isomorphisms of Cayley graphs of small valency, *Algebra Colloq.* 1 (1994), 67–76.

[11] S. Herke and B. Maenhaut, Perfect 1-Factorisations of Circulants with Small Degree, *Electronic J. Combin.* 20(1) (2013), #P58.

[12] S. Herke and B. Maenhaut, Perfect 1-Factorisations of a Family of Cayley Graphs, *J. Combin. Des.* (to appear).

[13] A. Kotzig, Construction of third-order Hamiltonian graphs, *Časop. Pěst. Mat.* 87 (1962), 148–168.

[14] A. Kotzig, Hamilton graphs and Hamilton circuits, *Theory of Graphs and its Applications (Proc. Sympos. Smolenice 1963)* Nakl. CSAV, Praha (1964), 63–82.

[15] A. Kotzig, Problem 20, *Theory of Graphs and its Applications (Proc. Sympos. Smolenice 1963)* Nakl. CSAV, Praha (1964), 162.

[16] A. Kotzig and J. Labelle, Strongly Hamiltonian graphs, *Utilitas Math.* 14 (1978), 99–116.

[17] P.J. Laufer, On strongly Hamilton complete bipartite graphs, *Ars. Combin.* 9 (1980), 43–46.

[18] C.C. Lindner and A.P. Street, The Stern-Lenz Theorem: background and applications, *Congr. Numer.* 80 (1991), 5–21.

[19] G. Mazzuoccolo, Perfect one-factorizations in line-graphs and planar graphs, *Australas. J. Combin.* 41 (2008), 227–233.

[20] M. Muzychuk, A solution of the isomorphism problem for circulant graphs, *Proc. London. Math. Soc.* 88 (2004), 1–41.

[21] E. Seah, Perfect one-factorizations of the complete graph—a survey, *Bull. Inst. Combin. Appl.* 1 (1991), 59–70.

[22] G. Stern and H. Lenz, Steiner triple systems with given subspaces: another proof of the Doyen-Wilson theorem, *Boll. Un. Mat. Ital. Ser. A* 17 (1980), 109–114.

[23] I.M. Wanless, New perfect 1-factorisations, `http://users.monash.edu.au/~iwanless/data/P1F/newP1F.html`.

[24] I.M. Wanless, Perfect factorisations of bipartite graphs and Latin squares without proper subrectangles, *Electron. J. Combin.* 6 (1999), R9.

[25] D. Witte and J.A. Gallian, A survey: Hamiltonian cycles in Cayley graphs, *Discrete Math.* 51 (1984), 293–304.

(Received 23 Jan 2014; revised 15 May 2014)