AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **60(1)** (2014), Pages 69–78

The maximum product of sizes of cross-t-intersecting uniform families

Peter Borg

Department of Mathematics
University of Malta
Malta
peter.borg@um.edu.mt

Abstract

We verify a conjecture of Hirschorn concerning the maximum product of sizes of cross-t-intersecting uniform families. We say that a set A *t-intersects* a set B if A and B have at least t common elements. Two families $\mathcal{A}$ and $\mathcal{B}$ of sets are said to be *cross-t-intersecting* if each set in $\mathcal{A}$ t-intersects each set in $\mathcal{B}$. For any positive integers n and r, let $\binom{[n]}{r}$ denote the family of all r-element subsets of $\{1, 2, \ldots, n\}$. We prove that for any integers r, s and t with $1 \leq t \leq r \leq s$, there exists an integer $n_0(r, s, t)$ such that for any integer $n \geq n_0(r, s, t)$, if $\mathcal{A} \subseteq \binom{[n]}{r}$ and $\mathcal{B} \subseteq \binom{[n]}{s}$ such that $\mathcal{A}$ and $\mathcal{B}$ are cross-t-intersecting, then $|\mathcal{A}||\mathcal{B}| \leq \binom{n-t}{r-t}\binom{n-t}{s-t}$, and equality holds if and only if for some $T \in \binom{[n]}{t}$, $\mathcal{A} = \{A \in \binom{[n]}{r} \colon T \subseteq A\}$ and $\mathcal{B} = \{B \in \binom{[n]}{s} \colon T \subseteq B\}$.

1 Introduction

Unless otherwise stated, we will use small letters such as x to denote positive integers or elements of a set, capital letters such as X to denote sets, and calligraphic letters such as $\mathcal{F}$ to denote *families* (that is, sets whose elements are sets themselves). Arbitrary sets and families are taken to be finite and may be the empty set $\emptyset$. An *r-set* is a set of size r, that is, a set having exactly r elements. For any $n \geq 1$, $[n]$ denotes the set $\{1, \ldots, n\}$ of the first n positive integers. For any set X, $\binom{X}{r}$ denotes the family $\{A \subseteq X \colon |A| = r\}$ of all r-subsets of X. For any family $\mathcal{F}$, we denote the family $\{F \in \mathcal{F} \colon |F| = r\}$ by $\mathcal{F}^{(r)}$, and for any t-set T, we denote the family $\{F \in \mathcal{F} \colon T \subseteq F\}$ by $\mathcal{F}[T]$, and we call $\mathcal{F}[T]$ a *t-star of* $\mathcal{F}$ if $\mathcal{F}[T] \neq \emptyset$.

Given an integer $t \geq 1$, we say that a set A *t-intersects* a set B if A and B have at least t common elements. A family $\mathcal{A}$ is said to be *t-intersecting* if each set in $\mathcal{A}$ t-intersects all the other sets in $\mathcal{A}$ (i.e. $|A \cap B| \geq t$ for every $A, B \in \mathcal{A}$ with $A \neq B$). A 1-intersecting family is also simply called an *intersecting family.* Note that t-stars are t-intersecting families.

Families $\mathcal{A}_1, \dots, \mathcal{A}_k$ are said to be *cross-t-intersecting* if for every $i, j \in [k]$ with $i \neq j$, each set in $\mathcal{A}_i$ t-intersects each set in $\mathcal{A}_j$ (i.e. $|A \cap B| \geq t$ for every $A \in \mathcal{A}_i$ and every $B \in \mathcal{A}_j$). Cross-1-intersecting families are also simply called *cross-intersecting families*.

The study of intersecting families originated in [13], which features the classical result that says that if $r \leq n/2$, then the size of a largest intersecting subfamily of $\binom{[n]}{r}$ is the size $\binom{n-1}{r-1}$ of every 1-star of $\binom{[n]}{r}$. This result is known as the Erdős-Ko-Rado (EKR) Theorem. There are various proofs of this theorem (see [11, 19, 21, 23]), two of which are particularly short and beautiful: Katona's [21], introducing the elegant cycle method, and Daykin's [11], using the powerful Kruskal-Katona Theorem [22, 25]. Also in [13], Erdős, Ko and Rado proved that for $t \leq r$, there exists an integer $n_0(r, t)$ such that for any $n \geq n_0(r, t)$, the size of a largest t-intersecting subfamily of $\binom{[n]}{r}$ is the size $\binom{n-t}{r-t}$ of every t-star of $\binom{[n]}{r}$. Frankl [15] showed that for $t \geq 15$, the smallest such $n_0(r, t)$ is $(r - t + 1)(t + 1)$. Subsequently, Wilson [32] proved this for all $t \geq 1$. Frankl [15] conjectured that the size of a largest t-intersecting subfamily of $\binom{[n]}{r}$ is $\max\{|\{A \in \binom{[n]}{r} \colon |A \cap [t+2i]| \geq t+i\}| \colon i \in \{0\} \cup [r-t]\}$. A remarkable proof of this conjecture together with a complete characterisation of the extremal structures was obtained by Ahlswede and Khachatrian [1]. The t-intersection problem for $2^{[n]}$ was completely solved in [23]. These are prominent results in extremal set theory. The EKR Theorem inspired a wealth of results of this kind, that is, results that establish how large a system of sets can be under certain intersection conditions; see [8, 12, 14, 16].

For t-intersecting subfamilies of a given family $\mathcal{F}$, the natural question to ask is how large they can be. For cross-t-intersecting families, two natural parameters arise: the sum and the product of sizes of the cross-t-intersecting families (note that the product of sizes of k families $\mathcal{A}_1, \dots, \mathcal{A}_k$ is the number of k-tuples $(A_1, \dots, A_k)$ such that $A_i \in \mathcal{A}_i$ for each $i \in [k]$). It is therefore natural to consider the problem of maximising the sum or the product of sizes of k cross-t-intersecting subfamilies (not necessarily distinct or non-empty) of a given family $\mathcal{F}$. The paper [9] analyses this problem in general and reduces it to the t-intersection problem for k sufficiently large. In this paper we are concerned with the family $\binom{[n]}{r}$. We point out that the maximum product problem for $2^{[n]}$ and $k = 2$ is solved in [26], and the maximum sum problem for $2^{[n]}$ and any k is solved in [9] via the results in [23, 24, 31].

Wang and Zhang [31] solved the maximum sum problem for $\binom{[n]}{r}$ using an elegant combination of the method in [3, 4, 5, 6] and an important lemma that is found in [2, 10] and referred to as the 'no-homomorphism lemma'. The solution for the case $t = 1$ had been obtained by Hilton [18] and is the first result of this kind.

The maximum product problem for $\binom{[n]}{r}$ was first addressed by Pyber [28], who proved that for any r, s and n such that either $r = s \leq n/2$ or $r < s$ and $n \geq 2s + r - 2$, if $\mathcal{A} \subseteq \binom{[n]}{r}$ and $\mathcal{B} \subseteq \binom{[n]}{s}$ such that $\mathcal{A}$ and $\mathcal{B}$ are cross-intersecting, then $|\mathcal{A}||\mathcal{B}| \leq \binom{n-1}{r-1}\binom{n-1}{s-1}$. Subsequently, Matsumoto and Tokushige [27] proved this for any $r \leq s \leq n/2$, and they also determined the optimal structures. This brings us to the main result of this paper, namely Theorem 1.1, which solves the cross-t-

intersection problem for n sufficiently large and consequently verifies a conjecture of Hirschorn [20, Conjecture 3].

For $t \leq r \leq s$, let

$$n_0(r,s,t) = \max\left\{r(s-t)\binom{r+s-t}{t},\ (r-t)\binom{r}{t}\binom{r+s-t}{t+1}\right\} + t + 1.$$

Theorem 1.1 *Let $t \leq r \leq s$ and $n \geq n_0(r,s,t)$. If $\mathcal{A} \subseteq \binom{[n]}{r}$ and $\mathcal{B} \subseteq \binom{[n]}{s}$ such that $\mathcal{A}$ and $\mathcal{B}$ are cross-t-intersecting, then*

$$|\mathcal{A}||\mathcal{B}| \leq \binom{n-t}{r-t}\binom{n-t}{s-t},$$

and equality holds if and only if for some $T \in \binom{[n]}{t}$, $\mathcal{A} = \{A \in \binom{[n]}{r} \colon T \subseteq A\}$ and $\mathcal{B} = \{B \in \binom{[n]}{s} \colon T \subseteq B\}$.

The special case $r = s$ is treated in [17, 29, 30], which establish values of $n_0(r,r,t)$ that are close to best possible. Hirschorn made a conjecture [20, Conjecture 4] for any n, r, s and t.

From Theorem 1.1 we immediately obtain the following generalisation to any number of cross-t-intersecting families.

Theorem 1.2 *Let $k \geq 2$, $t \leq r_1 \leq \cdots \leq r_k$ and $n \geq n_0(r_{k-1}, r_k, t)$. If $\mathcal{A}_1 \subseteq \binom{[n]}{r_1}, \dots, \mathcal{A}_k \subseteq \binom{[n]}{r_k}$, and $\mathcal{A}_1, \dots, \mathcal{A}_k$ are cross-t-intersecting, then*

$$\prod_{i=1}^{k} |\mathcal{A}_i| \leq \prod_{i=1}^{k} \binom{n-t}{r_i - t},$$

and equality holds if and only if for some $T \in \binom{[n]}{t}$, $\mathcal{A}_i = \{A \in \binom{[n]}{r_i} \colon T \subseteq A\}$ for each $i \in [k]$.

Proof. For each $i \in [k]$, let $a_i = |\mathcal{A}_i|$, $b_i = \binom{n-t}{r_i-t}$ and $\mathcal{F}_i = \binom{[n]}{r_i}$. Note that $n_0(r_i, r_j, t) \leq n_0(r_{k-1}, r_k, t)$ for every $i, j \in [k]$ with $i < j$. Thus, by Theorem 1.1, for every $i, j \in [k]$ with $i < j$, we have $a_i a_j \leq b_i b_j$, and equality holds if and only if for some $T_{i,j} \in \binom{[n]}{t}$, $\mathcal{A}_i = \mathcal{F}_i[T_{i,j}]$ and $\mathcal{A}_j = \mathcal{F}_j[T_{i,j}]$. We have

$$\left(\prod_{i=1}^{k} a_i\right)^{k-1} = \prod_{i \in [k-1]} \prod_{j \in [k] \backslash [i]} a_i a_j \leq \prod_{i \in [k-1]} \prod_{j \in [k] \backslash [i]} b_i b_j = \left(\prod_{i=1}^{k} b_i\right)^{k-1}.$$

So $\prod_{i=1}^{k} a_i \leq \prod_{i=1}^{k} b_i$. Suppose equality holds. Then, for every $i, j \in [k]$ with $i < j$, $a_i a_j = b_i b_j$ and hence there exists $T_{i,j} \in \binom{[n]}{t}$ such that $\mathcal{A}_i = \mathcal{F}_i[T_{i,j}]$ and $\mathcal{A}_j = \mathcal{F}_j[T_{i,j}]$. Let $h \in [k]\backslash\{1\}$. We have $\mathcal{F}_1[T_{1,2}] = \mathcal{A}_1 = \mathcal{F}_1[T_{1,h}]$ and $\mathcal{A}_h = \mathcal{F}_h[T_{1,h}]$. It is easy to check that $n_0(r_{k-1}, r_k, t) > r_k$. So $n > r_1$. Since $\mathcal{F}_1[T_{1,2}] = \mathcal{F}_1[T_{1,h}]$, it follows that $T_{1,2} = T_{1,h}$. So $\mathcal{A}_h = \mathcal{F}_h[T_{1,2}]$. Hence the result. □

2 The compression operation

For any $i, j \in [n]$, let $\delta_{i,j}\colon 2^{[n]} \to 2^{[n]}$ be defined by

$$\delta_{i,j}(A) = \begin{cases} (A\backslash\{j\}) \cup \{i\} & \text{if } j \in A \text{ and } i \notin A \\ A & \text{otherwise} \end{cases},$$

and let $\Delta_{i,j}\colon 2^{2^{[n]}} \to 2^{2^{[n]}}$ be the *compression operation* (see [13]) defined by

$$\Delta_{i,j}(\mathcal{A}) = \{\delta_{i,j}(A)\colon A \in \mathcal{A}, \delta_{i,j}(A) \notin \mathcal{A}\} \cup \{A \in \mathcal{A}\colon \delta_{i,j}(A) \in \mathcal{A}\}.$$

Note that $|\Delta_{i,j}(\mathcal{A})| = |\mathcal{A}|$. A survey on the properties and uses of compression (also called *shifting*) operations in extremal set theory is given in [16].

If $i < j$, then we call $\Delta_{i,j}$ a *left-compression.* A family $\mathcal{F} \subseteq 2^{[n]}$ is said to be *compressed* if $\Delta_{i,j}(\mathcal{F}) = \mathcal{F}$ for every $i, j \in [n]$ with $i < j$. In other words, $\mathcal{F}$ is compressed if it is invariant under left-compressions.

The following lemma captures some well-known fundamental properties of compressions, and we will prove it for completeness.

Lemma 2.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be cross-t-intersecting subfamilies of $2^{[n]}$.*
(i) For any $i, j \in [n]$, $\Delta_{i,j}(\mathcal{A})$ and $\Delta_{i,j}(\mathcal{B})$ are cross-t-intersecting subfamilies of $2^{[n]}$.
(ii) If $t \leq r \leq s \leq n$, $\mathcal{A} \subseteq \binom{[n]}{r}$, $\mathcal{B} \subseteq \binom{[n]}{s}$, and $\mathcal{A}$ and $\mathcal{B}$ are compressed, then $|A \cap B \cap [r+s-t]| \geq t$ for any $A \in \mathcal{A}$ and any $B \in \mathcal{B}$.

Proof. (i) Let $i, j \in [n]$. Suppose $A \in \Delta_{i,j}(\mathcal{A})$ and $B \in \Delta_{i,j}(\mathcal{B})$. If $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then $|A \cap B| \geq t$ since $\mathcal{A}$ and $\mathcal{B}$ are cross-t-intersecting. Suppose that either $A \notin \mathcal{A}$ or $B \notin \mathcal{B}$; we may assume that $A \notin \mathcal{A}$. Then $A = \delta_{i,j}(A') \neq A'$ for some $A' \in \mathcal{A}$. So $i \notin A'$, $j \in A'$, $i \in A$ and $j \notin A$. Suppose $|A \cap B| \leq t-1$. Then $i \notin B$ and hence $B \in \mathcal{B}$. So $B \in \mathcal{B} \cap \Delta_{i,j}(\mathcal{B})$ and hence $B, \delta_{i,j}(B) \in \mathcal{B}$. So $|A' \cap B| \geq t$ and $|A' \cap \delta_{i,j}(B)| \geq t$. From $|A \cap B| \leq t-1$ and $|A' \cap B| \geq t$ we get $A' \cap B = (A \cap B) \cup \{j\}$ and hence $A' \cap \delta_{i,j}(B) = A \cap B$, but this contradicts $|A \cap B| \leq t-1$ and $|A' \cap \delta_{i,j}(B)| \geq t$. So $|A \cap B| \geq t$. Therefore, $\Delta_{i,j}(\mathcal{A})$ and $\Delta_{i,j}(\mathcal{B})$ are cross-t-intersecting.

(ii) Suppose $t \leq r \leq s \leq n$, $A \in \mathcal{A} \subseteq \binom{[n]}{r}$, $B \in \mathcal{B} \subseteq \binom{[n]}{s}$, and $\mathcal{A}$ and $\mathcal{B}$ are compressed. Since $\mathcal{A}$ and $\mathcal{B}$ are cross-t-intersecting, $|A \cap B| \geq t$. Let $X = (A \cap B) \cap [r+s-t]$, $Y = (A \cap B)\backslash[r+s-t]$ and $Z = [r+s-t]\backslash(A \cup B)$. If $Y = \emptyset$, then $X = A \cap B$ and hence $|X| \geq t$. Now consider $Y \neq \emptyset$. Let $p = |Y|$. Since

$$\begin{aligned} |Z| &= r+s-t-|(A \cup B) \cap [r+s-t]| \geq r+s-t-|X|-|A\backslash B|-|B\backslash A| \\ &= r+s-t-|X|-|A\backslash(X \cup Y)|-|B\backslash(X \cup Y)| \\ &= r+s-t-|X|-(r-|X|-|Y|)-(s-|X|-|Y|) = 2|Y|+|X|-t \\ &= |Y|+|Y \cup X|-t = p+|A \cap B|-t \geq p, \end{aligned}$$

$\binom{Z}{p} \neq \emptyset$. Let $W \in \binom{Z}{p}$. Let $C := (B\backslash Y) \cup W$. Let $y_1, \dots, y_p$ be the elements of Y, and let $w_1, \dots, w_p$ be those of W. So $C = \delta_{w_1,y_1} \circ \cdots \circ \delta_{w_p,y_p}(B)$. Note that

$\delta_{w_1,y_1}, \ldots, \delta_{w_p,y_p}$ are left-compressions as $W \subseteq [r+s-t]$ and $Y \subseteq [n]\backslash[r+s-t]$. Since $\mathcal{B}$ is compressed, $C \in \mathcal{B}$. So $|A \cap C| \geq t$. Now clearly $|A \cap C| = |X|$. So $|X| \geq t$. □

Suppose a subfamily $\mathcal{A}$ of $2^{[n]}$ is not compressed. Then $\mathcal{A}$ can be transformed to a compressed family through left-compressions as follows. Since $\mathcal{A}$ is not compressed, we can find a left-compression that changes $\mathcal{A}$, and we apply it to $\mathcal{A}$ to obtain a new subfamily of $2^{[n]}$. We keep on repeating this (always applying a left-compression to the last family obtained) until we obtain a subfamily of $2^{[n]}$ that is invariant under every left-compression (such a point is indeed reached, because if $\Delta_{i,j}(\mathcal{F}) \neq \mathcal{F} \subseteq 2^{[n]}$ and $i < j$, then $0 < \sum_{G \in \Delta_{i,j}(\mathcal{F})} \sum_{b \in G} b < \sum_{F \in \mathcal{F}} \sum_{a \in F} a$).

Now consider $\mathcal{A}, \mathcal{B} \subseteq 2^{[n]}$ such that $\mathcal{A}$ and $\mathcal{B}$ are cross-t-intersecting. Then, by Lemma 2.1(i), we can obtain $\mathcal{A}^*, \mathcal{B}^* \subseteq 2^{[n]}$ such that $\mathcal{A}^*$ and $\mathcal{B}^*$ are compressed and cross-t-intersecting, $|\mathcal{A}^*| = |\mathcal{A}|$ and $|\mathcal{B}^*| = |\mathcal{B}|$. Indeed, similarly to the above procedure, if we can find a left-compression that changes at least one of $\mathcal{A}$ and $\mathcal{B}$, then we apply it to both $\mathcal{A}$ and $\mathcal{B}$, and we keep on repeating this (always performing this on the last two families obtained) until we obtain $\mathcal{A}^*, \mathcal{B}^* \subseteq 2^{[n]}$ such that both $\mathcal{A}^*$ and $\mathcal{B}^*$ are invariant under every left-compression.

3 Proof of Theorem 1.1

We will need the following lemma only when dealing with the characterisation of the extremal structures in the proof of Theorem 1.1.

Lemma 3.1 *Let r, s, t and n be as in Theorem 1.1, and let $i, j \in [n]$. Let $\mathcal{H} = 2^{[n]}$. Let $\mathcal{A} \subseteq \mathcal{H}^{(r)}$ and $\mathcal{B} \subseteq \mathcal{H}^{(s)}$ such that $\mathcal{A}$ and $\mathcal{B}$ are cross-t-intersecting. Suppose $\Delta_{i,j}(\mathcal{A}) = \mathcal{H}^{(r)}[T]$ and $\Delta_{i,j}(\mathcal{B}) = \mathcal{H}^{(s)}[T]$ for some $T \in \binom{[n]}{t}$. Then $\mathcal{A} = \mathcal{H}^{(r)}[T']$ and $\mathcal{B} = \mathcal{H}^{(s)}[T']$ for some $T' \in \binom{[n]}{t}$.*

We prove the above lemma using the following special case of [7, Lemma 5.6].

Lemma 3.2 *Let $r \geq t+1$ and $n \geq 2r-t+2$. Let $\mathcal{H} = 2^{[n]}$. Let $\mathcal{G}$ be a t-intersecting subfamily of $\mathcal{H}^{(r)}$. Let $i, j \in [n]$. Suppose $\Delta_{i,j}(\mathcal{G})$ is a largest t-star of $\mathcal{H}^{(r)}$. Then $\mathcal{G}$ is a largest t-star of $\mathcal{H}^{(r)}$.*

Proof of Lemma 3.1. We are given that $t \leq r \leq s$.

Consider first $r = t$. Then $\Delta_{i,j}(\mathcal{A}) = \{T\}$. So $\mathcal{A} = \{T'\} = \mathcal{H}^{(r)}[T']$ for some $T' \in \binom{[n]}{t}$. Since $\mathcal{A}$ and $\mathcal{B}$ are cross-t-intersecting, $T' \subseteq B$ for all $B \in \mathcal{B}$. So $\mathcal{B} \subseteq \mathcal{H}^{(s)}[T']$. Since $\binom{n-t}{s-t} = |\mathcal{H}^{(s)}[T]| = |\Delta_{i,j}(\mathcal{B})| = |\mathcal{B}| \leq |\mathcal{H}^{(s)}[T']| = \binom{n-t}{s-t}$, $|\mathcal{B}| = \binom{n-t}{s-t}$. So $\mathcal{B} = \mathcal{H}^{(s)}[T']$.

Now consider $r \geq t+1$. Note that $T\backslash\{i\} \subseteq E$ for all $E \in \mathcal{A} \cup \mathcal{B}$.

Suppose $\mathcal{A}$ is not t-intersecting. Then there exist $A_1, A_2 \in \mathcal{A}$ such that $|A_1 \cap A_2| \leq t-1$. So $T \not\subseteq A_l$ for some $l \in \{1, 2\}$; we may (and will) assume that $l = 1$. Thus, since $\Delta_{i,j}(\mathcal{A}) = \mathcal{H}^{(r)}[T]$, we have $A_1 \notin \Delta_{i,j}(\mathcal{A})$, $A_1 \neq \delta_{i,j}(A_1) \in \Delta_{i,j}(\mathcal{A})$, $\delta_{i,j}(A_1) \notin \mathcal{A}$ (because otherwise $A_1 \in \Delta_{i,j}(\mathcal{A})$), $i \in T$, $j \notin T$, $j \in A_1$ and $A_1 \cap T = T\backslash\{i\}$. Since $T\backslash\{i\} \subseteq A_1 \cap A_2$ and $|A_1 \cap A_2| \leq t-1$, we have $A_1 \cap A_2 = T\backslash\{i\}$. So $j \notin A_2$ and hence $A_2 = \delta_{i,j}(A_2)$. Since $\delta_{i,j}(A_2) \in \Delta_{i,j}(\mathcal{A}) = \mathcal{H}^{(r)}[T]$, $T \subseteq A_2$. Let $X = [n]\backslash(A_1 \cup A_2)$. We have

$$\begin{aligned}|X| &= n - |A_1 \cup A_2| = n - (|A_1| + |A_2| - |A_1 \cap A_2|) = n - 2r + t - 1 \\ &\geq n_0(r,s,t) - 2(r-t) - t - 1 \geq r(s-t)\binom{r+s-t}{t} - 2(r-t).\end{aligned}$$

Thus, since $t+1 \leq r \leq s$, we have $|X| > s-t$ and hence $\binom{X}{s-t} \neq \emptyset$. Let $C \in \binom{X}{s-t}$ and $D = C \cup T$. So $D \in \mathcal{H}^{(s)}[T]$ and $D \cap A_1 = T\backslash\{i\}$, meaning that $D \in \Delta_{i,j}(\mathcal{B})$ and $|D \cap A_1| = t-1$. Thus, since $\mathcal{A}$ and $\mathcal{B}$ are cross-t-intersecting, we obtain $D \notin \mathcal{B}$ and $(D\backslash\{i\}) \cup \{j\} \in \mathcal{B}$, which is a contradiction since $|((D\backslash\{i\}) \cup \{j\}) \cap A_2| = |T\backslash\{i\}| = t-1$.

Therefore, $\mathcal{A}$ is t-intersecting. Similarly, $\mathcal{B}$ is t-intersecting. Now $\mathcal{H}^{(r)}[T]$ and $\mathcal{H}^{(s)}[T]$ are largest t-stars of $\mathcal{H}^{(r)}$ and $\mathcal{H}^{(s)}$, respectively. So $\Delta_{i,j}(\mathcal{A})$ and $\Delta_{i,j}(\mathcal{B})$ are largest t-stars of $\mathcal{H}^{(r)}$ and $\mathcal{H}^{(s)}$, respectively. Since $t+1 \leq r \leq s$, $n_0(r,s,t) \geq (t+1)(s-t)\binom{t+2}{t} + t + 1 \geq 6(s-t) + t + 1 = 2s + 4(s-t) - t + 1 \geq 2s - t + 5$. Since $n \geq n_0(r,s,t)$, we obtain $n \geq 2s - t + 5$ and $n \geq 2r - t + 5$. By Lemma 3.2, for some $T', T^* \in \binom{[n]}{t}$, $\mathcal{A} = \mathcal{H}^{(r)}[T']$ and $\mathcal{B} = \mathcal{H}^{(s)}[T^*]$.

Suppose $T' \neq T^*$. Let $z \in T^*\backslash T'$. Since $n \geq 2r - t + 5 > r$, we can choose $A' \in \mathcal{H}^{(r)}[T']$ such that $z \notin A'$. Since $n \geq 2s - t + 5 \geq r + s - t + 5 > r + s - t$ and $z \in T^*\backslash A'$, we can choose $B^* \in \mathcal{H}^{(s)}[T^*]$ such that $|A' \cap B^*| \leq t-1$; however, this is a contradiction since $\mathcal{A} = \mathcal{H}^{(r)}[T']$, $\mathcal{B} = \mathcal{H}^{(s)}[T^*]$, and $\mathcal{A}$ and $\mathcal{B}$ are cross-t-intersecting. Therefore, $T' = T^*$. □

Proof of Theorem 1.1. Let $\mathcal{H} = 2^{[n]}$. Then $\binom{[n]}{r} = \mathcal{H}^{(r)}$ and $\binom{[n]}{s} = \mathcal{H}^{(s)}$. If either $\mathcal{A} = \emptyset$ or $\mathcal{B} = \emptyset$, then $|\mathcal{A}||\mathcal{B}| = 0$. Thus we assume that $\mathcal{A} \neq \emptyset$ and $\mathcal{B} \neq \emptyset$.

As explained in Section 2, we apply left-compressions to $\mathcal{A}$ and $\mathcal{B}$ simultaneously until we obtain two compressed families $\mathcal{A}^*$ and $\mathcal{B}^*$, respectively, and we know that $\mathcal{A}^*$ and $\mathcal{B}^*$ are cross-t-intersecting, $\mathcal{A}^* \subseteq \mathcal{H}^{(r)}$, $\mathcal{B}^* \subseteq \mathcal{H}^{(s)}$, $|\mathcal{A}^*| = |\mathcal{A}|$ and $|\mathcal{B}^*| = |\mathcal{B}|$. In view of Lemma 3.1, we may therefore assume that $\mathcal{A}$ and $\mathcal{B}$ are compressed.

By Lemma 2.1(ii),

$$|A \cap [r+s-t]| \geq t \text{ for each } A \in \mathcal{A}. \tag{1}$$

Case 1: $|A^* \cap [r+s-t]| = t$ for some $A^* \in \mathcal{A}$. Then $A^* \cap [r+s-t] = T^*$ for some $T^* \in \binom{[r+s-t]}{t}$. By Lemma 2.1(ii), $t \leq |A^* \cap B \cap [r+s-t]| = |B \cap T^*| \leq t$ for each $B \in \mathcal{B}$. Thus, for each $B \in \mathcal{B}$, $|B \cap T^*| = t$ and hence $T^* \subseteq B$. Therefore, $\mathcal{B} \subseteq \mathcal{H}^{(s)}[T^*]$.

If $T^* \subseteq A$ for each $A \in \mathcal{A}$, then $|\mathcal{A}||\mathcal{B}| \leq |\mathcal{H}^{(r)}[T^*]||\mathcal{H}^{(s)}[T^*]| = \binom{n-t}{r-t}\binom{n-t}{s-t}$, and equality holds if and only if $\mathcal{A} = \mathcal{H}^{(r)}[T^*]$ and $\mathcal{B} = \mathcal{H}^{(s)}[T^*]$.

Suppose $T^* \nsubseteq A'$ for some $A' \in \mathcal{A}$. Then $|A' \cap T^*| \le t-1$. Let $C = A' \cap T^*$ and $D = A' \backslash C$. For each $B \in \mathcal{B}$, we have $t \le |B \cap A'| = |B \cap C| + |B \cap D| = |C| + |B \cap D| \le t - 1 + |B \cap D|$ and hence $|B \cap D| \ge 1$. So $\mathcal{B} \subseteq \{B \in \mathcal{H}^{(s)}[T^*]\colon |B \cap D| \ge 1\} = \bigcup_{X \in \binom{D}{1}} \mathcal{H}^{(s)}[T^* \cup X]$ and hence

$$\begin{aligned} |\mathcal{B}| &\le \sum_{X \in \binom{D}{1}} |\mathcal{H}^{(s)}[T^* \cup X]| \\ &= \sum_{X \in \binom{D}{1}} \binom{n-t-1}{s-t-1} \\ &= \binom{|D|}{1}\binom{n-t-1}{s-t-1} \le r\binom{n-t-1}{s-t-1}. \end{aligned}$$

Now, by (1), $\mathcal{A} = \bigcup_{T \in \binom{[r+s-t]}{t}} \mathcal{A}[T] \subseteq \bigcup_{T \in \binom{[r+s-t]}{t}} \mathcal{H}^{(r)}[T]$ and hence

$$|\mathcal{A}| \le \sum_{T \in \binom{[r+s-t]}{t}} |\mathcal{H}^{(r)}[T]| = \sum_{T \in \binom{[r+s-t]}{t}} \binom{n-t}{r-t} = \binom{r+s-t}{t}\binom{n-t}{r-t}.$$

Therefore,

$$\begin{aligned} |\mathcal{A}||\mathcal{B}| &\le r\binom{r+s-t}{t}\binom{n-t}{r-t}\binom{n-t-1}{s-t-1} \\ &= r\binom{r+s-t}{t}\binom{n-t}{r-t}\frac{s-t}{n-t}\binom{n-t}{s-t} \\ &\le \frac{r(s-t)}{n_0(r,s,t)-t}\binom{r+s-t}{t}\binom{n-t}{r-t}\binom{n-t}{s-t} \\ &< \binom{n-t}{r-t}\binom{n-t}{s-t}. \end{aligned}$$

Case 2: $|A \cap [r+s-t]| \ge t+1$ for all $A \in \mathcal{A}$. So $\mathcal{A} = \bigcup_{Z \in \binom{[r+s-t]}{t+1}} \mathcal{A}[Z] \subseteq \bigcup_{Z \in \binom{[r+s-t]}{t+1}} \mathcal{H}^{(r)}[Z]$. Let $A^* \in \mathcal{A}$. Since $|A^* \cap B| \ge t$ for all $B \in \mathcal{B}$, we have $\mathcal{B} = \bigcup_{T \in \binom{A^*}{t}} \mathcal{B}[T] \subseteq \bigcup_{T \in \binom{A^*}{t}} \mathcal{H}^{(s)}[T]$. Therefore,

$$\begin{aligned} |\mathcal{A}||\mathcal{B}| &\le \binom{r+s-t}{t+1}\binom{n-t-1}{r-t-1}\binom{r}{t}\binom{n-t}{s-t} \\ &= \binom{r+s-t}{t+1}\frac{r-t}{n-t}\binom{n-t}{r-t}\binom{r}{t}\binom{n-t}{s-t} \\ &\le \frac{r-t}{n_0(r,s,t)-t}\binom{r}{t}\binom{r+s-t}{t+1}\binom{n-t}{r-t}\binom{n-t}{s-t} \\ &< \binom{n-t}{r-t}\binom{n-t}{s-t}. \end{aligned}$$

This completes the proof of the theorem. □

Acknowledgements

The author is indebted to the anonymous referees for checking the paper carefully and providing remarks that led to an improvement in the presentation.

References

[1] R. Ahlswede and L.H. Khachatrian, The complete intersection theorem for systems of finite sets, *European J. Combin.* 18 (1997), 125–136.

[2] M.O. Albertson and K.L. Collins, Homomorphisms of 3-chromatic graphs, *Discrete Math.* 54 (1985), 127–132.

[3] P. Borg, A short proof of a cross-intersection theorem of Hilton, *Discrete Math.* 309 (2009), 4750–4753.

[4] P. Borg, Cross-intersecting families of permutations, *J. Combin. Theory Ser. A* 117 (2010), 483–487.

[5] P. Borg, Cross-intersecting families of partial permutations, *SIAM J. Disc. Math.* 24 (2010), 600–608.

[6] P. Borg, Cross-intersecting sub-families of hereditary families, *J. Combin. Theory Ser. A* 119 (2012), 871–881.

[7] P. Borg, Extremal t-intersecting sub-families of hereditary families, *J. London Math. Soc.* 79 (2009), 167–185.

[8] P. Borg, Intersecting families of sets and permutations: a survey, in: Advances in Mathematics Research (A.R. Baswell Ed.), Volume 16, Nova Science Publishers, Inc., 2011, pp. 283–299.

[9] P. Borg, The maximum sum and the maximum product of sizes of cross-intersecting families, *European J. Combin.* 35 (2014), 117–130.

[10] P.J. Cameron and C.Y. Ku, Intersecting families of permutations, *European J. Combin.* 24 (2003), 881–890.

[11] D.E. Daykin, Erdős-Ko-Rado from Kruskal-Katona, *J. Combin. Theory Ser. A* 17 (1974), 254–255.

[12] M. Deza and P. Frankl, The Erdős-Ko-Rado theorem—22 years later, *SIAM J. Algebr. Discrete Methods* 4 (1983), 419–431.

[13] P. Erdős, C. Ko and R. Rado, Intersection theorems for systems of finite sets, *Quart. J. Math. Oxford (2)* 12 (1961), 313–320.

[14] P. Frankl, Extremal set systems, in: R.L. Graham, M. Grötschel and L. Lovász (Eds.), Handbook of Combinatorics, Vol. 2, Elsevier, Amsterdam, 1995, pp. 1293–1329.

[15] P. Frankl, The Erdős-Ko-Rado Theorem is true for $n = ckt$, Proc. Fifth Hung. Comb. Coll., North-Holland, Amsterdam, 1978, pp. 365–375.

[16] P. Frankl, The shifting technique in extremal set theory, in: C. Whitehead (Ed.), Surveys in Combinatorics, Cambridge Univ. Press, London/New York, 1987, pp. 81–110.

[17] P. Frankl, S.J. Lee, M. Siggers and N. Tokushige, An Erdős-Ko-Rado theorem for cross t-intersecting families, arXiv:1303.0657.

[18] A.J.W. Hilton, An intersection theorem for a collection of families of subsets of a finite set, *J. London Math. Soc. (2)* 15 (1977), 369–376.

[19] A.J.W. Hilton and E.C. Milner, Some intersection theorems for systems of finite sets, *Quart. J. Math. Oxford* (2) 18 (1967), 369–384.

[20] J. Hirschorn, Asymptotic upper bounds on the shades of t-intersecting families, arXiv:0808.1434.

[21] G.O.H. Katona, A simple proof of the Erdős-Chao Ko-Rado theorem, *J. Combin. Theory Ser. B* 13 (1972), 183–184.

[22] G.O.H. Katona, A theorem of finite sets, in: *Theory of Graphs*, Proc. Colloq. Tihany, Akadémiai Kiadó, 1968, pp. 187–207.

[23] G.O.H. Katona, Intersection theorems for systems of finite sets, *Acta Math. Acad. Sci. Hungar.* 15 (1964), 329–337.

[24] D.J. Kleitman, On a combinatorial conjecture of Erdős, *J. Combin. Theory Ser. A* 1 (1966), 209–214.

[25] J.B. Kruskal, The number of simplices in a complex, in: *Mathematical Optimization Techniques*, University of California Press, Berkeley, California, 1963, pp. 251–278.

[26] M. Matsumoto and N. Tokushige, A generalization of the Katona theorem for cross t-intersecting families, *Graphs Combin.* 5 (1989), 159–171.

[27] M. Matsumoto and N. Tokushige, The exact bound in the Erdős-Ko-Rado theorem for cross-intersecting families, *J. Combin. Theory Ser. A* 52 (1989), 90–97.

[28] L. Pyber, A new generalization of the Erdős-Ko-Rado theorem, *J. Combin. Theory Ser. A* 43 (1986), 85–90.

[29] N. Tokushige, On cross t-intersecting families of sets, *J. Combin. Theory Ser. A* 117 (2010), 1167–1177.

[30] N. Tokushige, The eigenvalue method for cross t-intersecting families, *J. Algebr. Comb.* 38 (2013), 653–662.

[31] J. Wang and H. Zhang, Cross-intersecting families and primitivity of symmetric systems, *J. Combin. Theory Ser. A* 118 (2011), 455–462.

[32] R.M. Wilson, The exact bound in the Erdős-Ko-Rado theorem, *Combinatorica* 4 (1984), 247–257.

(Received 12 Dec 2013; revised 4 June 2014)