AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **60(1)** (2014), Pages 1–10

Congestion-free routing of linear permutations on Fibonacci and Lucas cubes

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Abstract

In recent years there has been much interest in certain subcubes of hypercubes, namely Fibonacci cubes and Lucas cubes (and their generalized versions). In this article we consider off-line routing of linear permutations on these cubes. The model of routing we use regards edges as bi-directional, and we do not allow queues of length greater than one. Messages start out at different vertices, and in movements synchronized with a clock, move to an adjacent vertex or remain where they are, so that at the next stage there is still exactly one message per vertex. This is the routing model we defined in an earlier paper and used in two later ones.

1 Introduction

It is well-known that the n-dimensional hypercube Q_n is the graph whose vertices are all binary strings of length n, and whose edges join those pairs of vertices which differ in exactly one position. The Fibonacci cube of dimension n, introduced in [1], which we shall denote by F_n is the subgraph consisting of those strings with no two adjacent ones. It is easy to see that $|\,V(F_n)\,| = f_n$, the n^{th} Fibonacci number. For a comprehensive survey of the structure of Fibonacci cubes, see [2]. The Lucas cube of dimension n, which we shall denote by L_n is defined analogously, except that now the first and last positions, x_1 and x_n of the string $(x_1, x_2, \ldots, x_n)$ are considered adjacent, and so may not both be 1. $|\,V(L_n)\,| = l_n$, the n^{th} Lucas number.

Notation We use $\vec{x}$ to mean a row vector of the vector space $\mathbb{Z}_2^n$ over the field $\mathbb{Z}_2$ of two elements, 0 and 1, and $\vec{x}^T$ (where T denotes *transpose*) to be the corresponding vector considered as a column vector.

In [3] the automorphism group of F_n is computed (it is $\mathbb{Z}_2$) and in [4] that of L_n (it is the dihedral group D_{2n}). In the next section we determine the linear permutations of F_n and L_n, i.e. the $n \times n$ 0–1 matrices A such that for all $x \in F_n$ (respectively

L_n), $Ax^T \in F_n$ (respectively L_n), and identify the groups they form. In Section 3 we give routings for these permutations, presented as products of permutations of F_n (respectovely L_n), that move messages to adjacent vertices or leave them fixed, following the protocol and method of [5]. We later [6], [7], used this method to route certain classes of permutations on the hypercube.

2 Identifying the linear permutations of F_n and of L_n

2.1 The linear permutations of F_n

Let A be an $n \times n$ invertible binary matrix. We say that A is F_n-good (respectively L_n-good) if $A(F_n) \subseteq F_n$ (respectively $A(L_n) \subseteq L_n$).

We first determine the F_n-good matrices. From now on, assume that A is an $n \times n$ invertible binary matrix, and assume that A_n is F_n-good. $A_{i,j}$ denotes the entry in row i, column j, A_i the i^{th} row, and $A^{(j)}$ the j^{th} column.

Lemma 1 *Suppose A is F_n-good and has a 1 in row i, column j. Then a 1 in row $i-1$ or row $i+1$ can only occur in columns $j-1$ or $j+1$.*

Proof. Suppose row $i+1$ has a 1 in column k. By definition of F_n, if $\vec{e}_j$ denotes the j^{th} standard basis vector of $\mathbb{Z}_2^n$, then $A\vec{e}_j^{\,T} = A^{(j)}$ does not have adjacent 1's and so $A_{i-1,j} = 0 = A_{i+1,j}$. Suppose $k < j-1$ or $k > j+1$ and $A_{i-1,k}$ or $A_{i+1,k} = 1$. If either $A_{i-1,k} = 1$ or $A_{i+1,k} = 1$ then $A_{i,k} = 0$. In either case, $A(\vec{e}_j^{\,T} + \vec{e}_k^{\,T}) = A\vec{e}_j^{\,T} + A\vec{e}_k^{\,T} = A^{(j)} + A^{(k)}$ has two adjacent ones, contradicting the fact that A is F_n-good unless $\mid k - j \mid = 1$, in which case $\vec{e}_j + \vec{e}_k \notin F_n$. So $k = j-1$ or $j+1$. □

Lemma 2 *Suppose A is F_n-good, where $n \geq 3$. If row i of A has two 1's, occurring in columns j and k, with $j < k$, then $k = j+2$ and $i = 1$ or n.*

Proof. By Lemma 1, because of the 1 in row i, column j, a 1 in row $i-1$ (if $i \geq 2$) must occur in either column $j-1$ or column $j+1$, and because of the 1 in row i, column k, any 1 in row $i-1$ must also occur in either column $k-1$ or column $k+1$. To satisfy both of these conditions, since $j < k$ we must have $j+1 = k-1$, i.e. $k = j+2$. If $i \leq n-1$, the same argument holds for row $i+1$. So if $2 \leq i \leq n-1$, row $A_{i-1} = \vec{e}_{j+1} =$ row A_{i+1}, contradicting the assumption that A is invertible. Hence $i = 1$ or n. □

Corollary 1 *Suppose A is F_n-good. Then*
(1) *Any 1 in column 1 must occur in row 1 or row n. Similarly, a 1 in column n must occur in row 1 or row n.*
(2) *$A_2, A_3, \ldots, A_{n-1}$ is a permutation of $n-2$ of the rows of the identity matrix I_n.*
(3) *Each of A_1 and A_n has either one or two 1's.*
(4) *In row 1 there must be a 1 in either column 1 or column n. The same is true for row n. A second 1 (if there is one) in rows A_1 or A_n must occur in column $A^{(3)}$ or $A^{(n-2)}$.*

Proof. (1) Suppose $A_{i,1} = 1$, and $2 \le i \le n-1$. Then since $A^{(1)} \in F_n$, $A_{i-1,1} = A_{i+1,1} = 0$. By Lemma 1, the only 1 in A_{i-1} and the only 1 in A_{i+1} both occur in column $A^{(2)}$. Hence $A_{i-1} = \vec{e}_2 = A_{i+1}$, contradicting the invertibility of A. Thus for $2 \le i \le n-1, A_{i,1} = 0$. Since $A^{(1)} \neq \vec{0}^T$, either $A_{1,1} = 1$ or $A_{n,1} = 1$. The argument for column $A^{(n)}$ is analogous.

(2) By Lemma 2, for $2 \le i \le n-1$, A_i has exactly one 1, and two rows A_{i_1} and A_{i_2} are equal if their 1's occur in the same column. Since A is invertible, this cannot happen. Thus $A_2, A_3, \ldots, A_{n-1}$ is a permutation of $n-2$ of the rows of the identity matrix I_n.

(3) This is part of Lemma 2, since for integers j, k, and l with $j < k < l$ not all 3 of their differences can be 2.

(4) By (1), either $A_{1,1} = 1$ or $A_{1,n} = 1$. By Lemma 2, if both are 1, then $n = 1 + 2 = 3$. If $n \ge 4$, then either $A_1 = \vec{e}_1$, or $A_1 = \vec{e}_1 + \vec{e}_3$, or $A_1 = \vec{e}_n$ or $A_1 = \vec{e}_{n-2} + \vec{e}_n$. The same statements hold for A_n, for analogous reasons, except that $A_1 \neq A_n$. □

Corollary 2 *If $A_{1,1} = 1$, then for $2 \le i \le n-1, A_i = \vec{e}_i$.*
If $A_{1,n} = 1$, then for $2 \le i \le n-1, A_i = \vec{e}_{n-i+1}$.

Proof. Suppose $A_{1,1} = 1$. Then $A_{2,1} = 0$. By Lemma 1, $A_2 = \vec{e}_2$. Let $2 \le i \le n-2$ and assume, inductively, that $A_i = \vec{e}_i$. Then again by Lemma 1, $A_{i+1} = \vec{e}_{i+1}$. So by induction, for $2 \le i \le n-1, A_i = \vec{e}_i$.
The argument when $A_{1,n} = 1$ is entirely analogous. □

Next, we shall exhibit the group of F_n-good linear permutations.

Theorem 1 *For $n > 3$, there are precisely 8 F_n-good linear permutations and they form the dihedral group D_4. For $n = 3, F_3$ has exactly 6 F_3-good linear permutations, and they form the permutation group $\mathfrak{S}_3$.*

Proof. Denote by $E_{i,j}$ the $n \times n$ matrix whose single non-zero entry is a 1 in the $(i,j)^{\text{th}}$ position. Let C denote the matrix such that for $1 \le i \le n, C_{i,n-i+1} = 1$ and for $1 \le j \le n, C_{i,j} = 0$ if $i + j \neq n + 1$. Suppose that $n \neq 3$. We claim that the following set of 8 matrices constitutes the set of F_n-good matrices:

$$\{I, I + E_{1,3}, I + E_{n,n-2}, I + E_{1,3} + E_{n,n-2}\}$$

$$\bigcup \{C, C + E_{n,3}, C + E_{1,n-2}, C + E_{1,n-2} + E_{n,3}\}.$$

From the previous corollary we see that any F_n-good matrix must be one of these 8. We claim that each *is* F_n-good. We shall demonstrate this for three representative A's. Let $\vec{x} = (x_1, x_2, \ldots, x_n)$. $C\vec{x}^T = (x_n, x_{n-1}, \ldots, x_2, x_1)^T$. If $C\vec{x}^T \notin F_n$ then for some i with $n \ge i \ge 2, x_i$ and x_{i-1} are both 1. But then $\vec{x} \notin F_n$. Thus C is F_n-good. Next, consider $A = I + E_{1,3}$. $A\vec{x}^T = (x_1 + x_3, x_2, x_3, x_4, \ldots, x_n)^T$. Since $\vec{x}^T \in F_n$, if $A\vec{x}^T \notin F_n$ then $x_1 + x_3 = 1$ and $x_2 = 1$. For $x_1 + x_3$ to be

1, exactly one of x_1 and x_3 is 1. Thus either $x_1 = x_2 = 1$ or $x_2 = x_3 = 1$, either of which implies that $\vec{x}^T \notin F_n$. This contradiction shows that $I + E_{1,3}$ is F_n-good. We'll demonstrate the proof for one more example: $A = C + E_{1,n-2} + E_{n,3}$. $A\vec{x}^T = C\vec{x}^T + E_{1,n-2}\vec{x}^T + E_{n,3}\vec{x}^T = (x_n + x_{n-2}, x_{n-1}, x_{n-2}, \ldots, x_3, x_2, x_1 + x_3)^T$. If $A\vec{x}^T \notin F_n$ then either $x_n + x_{n-2} = x_{n-1} = 1$ or $x_2 = x_1 + x_3 = 1$. In the first case, exactly one of x_n and x_{n-2} is 1, and so together with x_{n-1} we have two adjacent 1's in $\vec{x}$, or, in the second case, exactly one of x_1 and x_3 is 1, and so together with x_2 we again have two adjacent 1's in $\vec{x}$. In either case, $\vec{x} \notin F_n$. Hence A must be F_n-good.

The following three identities are easily checked and will be quite useful.

$$E_{i,j}E_{k,l} = \begin{cases} E_{i,l} & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \tag{1}$$

$$CE_{i,j} = E_{n-i+1,j} \tag{2}$$

$$E_{i,j}C = E_{i,n-j+1} \tag{3}$$

Using identities (1)-(3) we shall show that the 8 F_n-good matrices form the dihedral group D_4. To do so we will exhibit F_n-good matrices A and B such that $A^4 = B^2 = I, A^2 \neq I$ and $BAB = A^3$.

Let $A = C + E_{1,n-2}$ and $B = I + E_{1,3}$. Since $C^2 = I$ and $n \neq 3, A^2 = I + CE_{1,n-2} + E_{1,n-2}C$. From (2) and (3) we have $A^2 = I + E_{n,n-2} + E_{1,3} \neq I$. From (1) and the assumption that $n \neq 3$ (and thus $n - 2 \neq 1$) we get $A^4 = I$. Clearly $B^2 = I$ since $1 + 1 = 0$ in $\mathbb{Z}_2$.

Next, $BAB = (I + E_{1,3})\,[(C + E_{1,n-2})(I + E_{1,3})]$

$$= (I + E_{1,3})(C + E_{1,n-2} + CE_{1,3} + 0) = (I + E_{1,3})(C + E_{1,n-2} + E_{n,3})$$

$$= C + E_{1,n-2} + E_{1,n-2} + E_{n,3} = C + E_{n,3}.$$

On the other hand, $A^3 = AA^2 = (C + E_{1,n-2})(I + E_{n,n-2} + E_{1,3})$

$$= C + E_{1,n-2} + CE_{n,n-2} + CE_{1,3} + E_{1,n-2}E_{1,3} = C + E_{1,n-2} + E_{1,n-2} + E_{n,3}$$

$$= C + E_{n,3}.$$

Thus $BAB = A^3$. It follows that for $n > 3, F_n \cong D_4$.

For $n = 3$, the first and third rows of $I + E_{1,3} + E_{3,1}$ are equal and so the matrix is not invertible. The same is true for $C + E_{n,3} + E_{1,n-2}$. Thus the order of G_3 is 6. Finally, G_3 is non-abelian, since, for example, $(I + E_{1,3})(I + E_{3,1}) = I + E_{1,3} + E_{3,1} + E_{1,1} \neq I + E_{3,1} + E_{1,3} + E_{3,3} = (I + E_{3,1})(I + E_{1,3})$. Hence $G_3 \cong \mathfrak{S}_3$. □

2.2 The linear permutations of L_n

In L_n entries in the first and last postions are considered adjacent, in addition to those in positions i and $i + 1$, for $1 \leq i \leq n - 1$. Thus $\vec{x} \in L_n \iff \vec{x} \in F_n$ and not both x_1 and x_n are 1. Thus for an $n \times n$ binary matrix A that is invertible, A is

L_n-good provided that for all $\vec{x} \in \mathbb{Z}_2^n$ with no 1's in adjacent positions, $A\vec{x}$ also has no 1's in adjacent positions.

We will determine the L_n-good matrices A as we did for the F_n-good matrices.

Remark Unlike F_n, L_n has the property that if $\vec{x} \in L_n$ and $\vec{y}$ is obtained from $\vec{x}$ by a cyclic permutation of the coordinates of $\vec{x}$, then $\vec{y} \in L_n$.

Corollary 3 *If A is an L_n-good matrix, then so is any matrix obtained from A by a cyclic permutation of its rows. In particular, C is L_n-good.*

Proof. Let B be the matrix such that for $1 \le i \le n-1$, $B_i = A_{i+1}$, and $B_n = A_1$. Then $B = A\left(\vec{e_2}^{\,T}, \vec{e_3}^{\,T}, \ldots, \vec{e_n}^{\,T}, \vec{e_1}^{\,T}\right)$. Let $\vec{x} \in L_n$. So $B\vec{x}^T = A\left(\vec{e_2}^{\,T}, \vec{e_3}^{\,T}, \ldots, \vec{e_n}^{\,T}, \vec{e_1}^{\,T}\right)\vec{x}^T = A\vec{y}^T$, where $\vec{y} = (x_n, x_1, x_2, \ldots, x_{n-1})$. Since $\vec{x} \in L_n$, by the Remark, $\vec{y} \in L_n$. But A is L_n-good, so $A\vec{y}^T \in L_n$, i.e. $B\vec{x}^T \in L_n$. Hence B is L_n-good. The second statement follows from the fact that C is obtained from I by a cyclic permutation of its rows. □

Lemma 3 *If A is L_n-good then no column of A has 1's in both the first and last rows.*

Proof. Since $\vec{e_j}^{\,T} \in L_n$, if A is L_n-good, $A\vec{e_j}^{\,T} = A^{(j)} \in L_n$. Since the first and last positions are considered adjacent, this means that the first and last entries of $A^{(j)}$ can not *both* be 1's. □

Lemma 4 *If A is L_n-good and $A_{i,j} = 1$ then a 1 in row A_{i-1} or A_{i+1} can occur only in column $A^{(j-1)}$ or $A^{(j+1)}$. Here addition of subscripts or superscripts is modulo n, so that when $i = 1$ or $j = 1$, by $i-1$ or $j-1$ we mean n, and if $i = n$ or $j = n$ we mean 1. Hence either $A_{i-1} = \vec{e}_{j-1}$ and $A_{i+1} = \vec{e}_{j+1}$ or $A_{i-1} = \vec{e}_{j+1}$ and $A_{i+1} = \vec{e}_{j-1}$.*

Proof. This is much the same as Lemma 1 for F_n-good matrices. For example, suppose that $A_{1,1} = 1$. We shall show that a 1 in A_n occurs only in column $A^{(n)}$ or column $A^{(2)}$. Suppose $A_{n,k} = 1$. We know from Lemma 3 that $k \neq 1$. Suppose that $A_{n,k} = 1$ for some k with $3 \le k \le n-1$. Then $\vec{x}^T = \vec{e}_1^T + \vec{e}_k^T$ has no 1's in adjacent positions, so it belongs to L_n. But since, by Lemma 3, $A_{1,n} = 0 = A_{k,1}$, $A^{(1)} + A^{(k)}$ has 1 in both the first and last positions, so that $A\vec{x}^T \notin L_n$. This contradiction means that if $A_{n,k} = 1$ then $k = 2$ or n. The cases $A_{1,n} = 1, A_{n,1} = 1$, and $A_{n,n} = 1$ are handled similarly. The last statement follows from the fact that A is invertible and so $A_{i-1} \neq A_{i+1}$. □

Next, we have the analogue of Lemma 2, for L_n.

Lemma 5 *Let $n \ge 4$. Suppose that A is L_n-good, and $A_{i,j} = 1 = A_{i,k}$, where $j < k$.*
(i) If $2 \le j$ and $k \le n-1$, then $k = j+2$ and $A_{i+1,j+1} = 1 = A_{i-1,j+1}$. Thus $A_{i+1} = \vec{e}_{j+1} = A_{i-1}$. Thus A is not invertible.
(ii) If $j = 1$ then $k = 3$ and $A_{i+1,2} = 1 = A_{i-1,2}$, so that $A_{i+1} = \vec{e}_2 = A_{i-1}$, or else $k = n-1$ and $A_{i+1,n} = 1 = A_{i-1,n}$, and hence $A_{i+1} = \vec{e}_n = A_{i-1}$. Thus in either

case, A is not invertible.
(iii) If $k = n$, then $j = 2$ or $j = n-2$. If $j = 2$, then $A_{i+1,1} = 1 = A_{i-1,1}$, so that $A_{i+1} = \vec{e_1} = A_{i-1}$. If $j = n-2$ then $A_{i+1,n-1} = 1 = A_{i-1,n-1}$, and thus $A_{i+1} = \vec{e}_{n-1} = A_{i-1}$. Again, in either case, A is not invertible.
(iv) Similarly, in row A_{i-1} a 1 can occur only in column $j+1$. Thus $A_{i-1} = \vec{e}_{j+1}$.
(v) Hence at most one row of A has more than one 1.
(vi) If $n \geq 5$, then every *row of A has exactly one 1.*

Proof. (i) By Lemma 4, if $2 \leq j \leq n-2$ and $j < k \leq n-1$, then since $A_{i,j} = 1$, a 1 in row A_{i+1} can occur only in column $A^{(j-1)}$ or column $A^{(j+1)}$. Similarly, since $A_{i,k} = 1$, a 1 in row A_{i+1} can occur only in column $A^{(k-1)}$ or column $A^{(k+1)}$. Thus $j+1 = k-1$ and so $k = j+2$. Furthermore, since A is invertible, *some* entry in row A_{i+1} must be 1. Thus we must have $A_{i+1,j+1} = 1$, and so $A_{i+1} = \vec{e}_{j+1}$. The same argument holds for row A_{i-1}. Therefore $A_{i-1} = A_{i+1}$, contradicting the invertibility of A.

The proofs of (ii), $(iii)$and (iv) are similar.

(v) This follows from $(i) - (iv)$.

(vi) We know from (v) that at most one row has more than one 1. With no loss of generality we may assume that the row with at least two 1's is A_2, and $A_{2,j} = 1 = A_{2,k}$, where $j < k$. By Lemma 4, since $A_1 \neq A_3$, either (a) $A_1 = \vec{e}_{j+1} = \vec{e}_{k-1}$ and $A_3 = \vec{e}_{j-1} = \vec{e}_{k+1}$ or, (b) $A_1 = \vec{e}_{j-1} = \vec{e}_{k+1}$ and $A_3 = \vec{e}_{j+1} = \vec{e}_{k-1}$. In case (a), $j+1 = k-1$ and $j-1 \equiv k+1 \pmod{n}$. Thus $k = j+2$ and so $j-1 \equiv j+3$ (mod n), i.e. $4 \equiv 0 \pmod{n}$. But then 4 is a multiple of n, contradicting the assumption that $n \geq 5$. In case (b), $j-1 \equiv k+1 \pmod{n}$ and $j+1 = k-1$. So again, $k = j+2$ and so $j-1 \equiv j+3 \pmod{n}$. Once again, this contradicts the assumption that $n \geq 5$. $\square$

Theorem 2 (*Classification of G_n, the group of L_n-good matrices*)
(i) G_2 is the group of order 2, consisting of I and $C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
(ii) G_3 is the group of 3×3 permutation matrices, i.e. the dihedral group D_3 (or equivalently, the symmetric group $\mathcal{S}_3$).
(iii) G_4 is the group of 4×4 matrices consisting of I, $I + E_{1,3}$, $I + E_{3,1}$, $I + E_{2,4}$, $I + E_{1,3} + E_{2,4}$, $I + E_{4,2}$, $I + E_{1,3} + E_{4,2}$, C, $C + E_{1,2}$, $C + E_{2,1}$, $C + E_{1,2} + E_{2,1}$, $C + E_{3,4}$, $C + E_{4,3}$, $C + E_{2,1} + E_{3,4}$, $C + E_{1,2} + E_{3,4}$, $C + E_{1,2} + E_{4,3}$ and all matrices obtained from these by cyclic permutations of the rows. The order of G_4 is 72.
(iv) For $n \geq 5$, each L_n-good matrix has exactly one 1 in each row and exactly one 1 in each column. G_n consists of all the cyclic permutations of the rows of I and so $G_n \cong \mathbb{Z}_n$.

Proof. (i) None of the other 2×2 matrices is invertible.
(ii) For $n = 3$, any two positions in $\vec{x}$ are adjacent. If, say, the j^{th} column contains two 1's then $A\vec{e}_j^{\,T} \notin L_3$, and A is not L_3-good. Thus each column of A, and therefore each row of A, contains exactly one 1. So A is a permutation matrix, i.e. is obtained from I by a permutation of its rows. If π is a non-cyclic permutation, then π is a

transposition. But for $n = 3$, any two rows may be considered adjacent. So with no loss of generality, we may assume that $\pi = (1, 2)$. Then $\pi(I)\vec{x}^T = (x_2, x_1, x_3)^T$. Let $\vec{x} \in L_3$. If any two 1's occur in (x_2, x_3, x_1) then $\vec{x}$ has two adjacent 1's, which is a contradiction. Thus $\pi(I)\vec{x}^T \in L_3$ and so $\pi(I)$ is L_3-good. Since the symmetric group is generated by the transpositions $G_3 \cong \mathfrak{S}_3$.

(iii) It is easy to check that each of these matrices is F_4-good. By the *weight* of a row we mean the number of 1's in it. There are three cases for an F_4-good matrix A: (1) no row of A has weight 2, (2) exactly one row has weight two, or (3) exactly two rows have weight two. In case (1) we have the 8 cyclic permutations of I. In cases (2) and (3), it follows from Lemma 5, (i) and (ii) that there are just two choices for the row of weight 2: [1010] or [0101]. Now in case (2), there are 4 possible positions for the row of weight 2. Assume that [1010] is row 1 of A. Then $A_{2,1} = A_{2,3} = A_{4,1} = A_{4,3} = 0$. A_2 must be either $\vec{e_2}$ or $\vec{e_4}$, and A_4 must be either $\vec{e_4}$ or $\vec{e_2}$ (and $A_2 \neq A_4$). Thus there are 4 L_4-good matrices with $A_1 = [1010]$. Since L_n-good matrices are closed under cyclic permutations of rows, we have 4×4 L_4-good matrices with the single row of weight 2 being [1010]. Similarly, there are 4×4 L_4-good matrices whose single row of weight 2 is [0101]. Thus, in case (2) we have $2 \times 4 \times 4 = 32$ L_4-good matrices.

Case (3) has 2 subcases: (a) the 2 rows of weight 2 are consecutive, and (b) they are not.

For subcase (a): first suppose that $A_1 = [1010]$ and $A_2 = [0101]$. We obtain the following 4 L_4-good matrices:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Similarly, if we interchange the first two rows, we get the following 4 L_4-good matrices:

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Since there are 4 pairs of consecutive rows, there are a total of $(4 + 4) \times 4 = 32$ L_4-good matrices in case 3(a).

For subcase (b): If the 2 rows of weight 1 are not consecutive, then each of the rows between them must be the zero row, since no column can have two consecutive 1's. Thus there are no L_4-good matrices of this type.

Hence the total number of L_4-good matrices is $8 + 32 + 32 + 0 = 72$.

(iv) Let $n \geq 5$, and suppose that for some i, $1 \leq i \leq n$, row A_i has at least two 1's. Say that $A_{i,j} = 1 = A_{i,k}$, where $j < k$. Then $A_{i+1} = \vec{e}_{j+1} = A_{i-1}$. This contradicts the fact that A is invertible. So each row of A is $\vec{e}_j$, for a unique j. Thus A is obtained from I by a permutation of its rows. We must show that this permutation is *cyclic*. Suppose that $A_1 = \vec{e}_j$. Then $A_2 =$ either $\vec{e}_{j+1}$ or $\vec{e}_{j-1}$. First suppose it is $\vec{e}_{j+1}$. We claim that for all $1 \le i \le n, A_i = \vec{e}_{j+i-1}$ (remember that subscripts are computed mod n). Assume, inductively, that for $1 \le q \le i-1, A_q = \vec{e}_{j+q-1}$. Then $A_i =$ either $\vec{e}_{j+i+1}$ or $\vec{e}_{j+i-1}$. If it is $\vec{e}_{j+i-1}$, then for $i \ge 3, A_{i-2} = \vec{e}_{j+(i-2)+1} = \vec{e}_{j+i-1} = A_i$, contradicting the fact that no two rows of A can be equal. Hence $A_i = \vec{e}_{j+i+1}$, and so, by induction, for all $1 \le q \le n, A_q = \vec{e}_{j+q+1}$. Thus A is obtained from I via the cyclic permutation $i \mapsto j+i-1$. A similar inductive argument shows that if $A_2 = \vec{e}_{j-1}$ then for all $1 \le q \le n, A_q = \vec{e}_{j+q+1}$, and so A is obtained from I via the cyclic permutation $i \mapsto j+i+1$. $\square$

3 Routings

For a connected graph G, if π is a permutation of G, we define

$$t(\pi) = \max\{d_G(\pi(x), x) \mid x \in G\}.$$

We then consider the group $\mathrm{Perm}(G)$ to be the Cayley graph whose generating set is $\Delta = \{\pi \mid t(\pi) \le 1\}$. A t-fold product of elements of Δ equal to the permutation σ is said to be a "t-step routing of σ". As discussed in [5], since we consider each edge to be doubled, i.e. one in each direction, every element τ of $\mathrm{Perm}(G)$ *is* a finite product of elements of this generating set, and such a factoring we call a *routing* of the permutation τ. We will be interested in the two cases, $G = F_n$ and $G = L_n$. For an $n \times n$ matrix A which is G-good, we define the permutation τ_A by $\tau_A(\vec{x}) = A\vec{x}$.

3.1 Routings of permutations of F_n

Lemma 6 *For $A = I, I + E_{1,3}$, and $I + E_{n,n-2}, t(\pi_A) = 1$. For $n \ne 3$, $t(I + E_{1,3} + E_{n,n-2}) = 2$. For $n = 3, I + E_{1,3} + E_{n,n-2} = I + E_{1,3} + E_{3,1}$ is not F_3-good, nor is $C + E_{n,3} + E_{1,n-2}$.*

Proof. $d(A\vec{x}^T, \vec{x}^T) = \mathrm{weight}(A\vec{x}^T + \vec{x}^T)$. For $A = I, I\vec{x}^T + \vec{x}^T = \vec{0}^T$ and $\mathrm{weight}(\vec{0}^T) = 0$. For $A = I + E_{1,3}, A\vec{x}^T = \vec{x}^T + x_3\vec{e_1}^T$. Thus $A\vec{x}^T + x^T = x_3\vec{e_1}^T$, whose weight is 0 if $x_3 = 0$ and 1 if $x_3 = 1$. Therefore $t(A) = 1$. If $A = I + E_{n,n-2}$ then $A\vec{x}^T + x^T = E_{n,n-2}\,\vec{x}^T = x_{n-2}\vec{e}_n^T$, so again, $t(A) = 1$. If $A = I + E_{1,3} + E_{n,n-2}$, then $A\vec{x}^T + \vec{x}^T = x_3\vec{e}_1^T + x_{n-2}\vec{e}_n^T$, whose weight is 2. For $n \ne 3, E_{1,3}E_{n,n-2} = 0$, and so $A = (I + E_{1,3})(I + E_{n,n-2})$ is a 2-step routing of A. $\square$

Lemma 7 *If $\vec{x}, \vec{y}, \vec{z}$ is a path in F_n, then $(\vec{x}, \vec{z}) = (\vec{x}, \vec{y})(\vec{y}, \vec{z})(\vec{x}, \vec{y})$ is a 3-step routing of $(\vec{x}, \vec{z})$.*

Corollary 4 *C is obtained from I by the row permutation*

$$(1, n)(2, n-1)\ldots(k, k+1) \quad \text{if } n = 2k, \quad \text{and by}$$

$$(1, n)(2, n-1)\ldots(k, k+2) \quad \text{if } n = 2k+1.$$

Each row transposition (i, j) (where $|\,j - i\,| > 1$) can be routed in 3 steps. Hence C can be routed in $3\lfloor n/2 \rfloor$ steps.

Corollary 5 *$C + E_{n,3} = (I + E_{n,n-2})C$ and $C + E_{n,n-2} = (I + E_{n,3})C$. Hence each of these can be routed in $1 + 3\lfloor n/2 \rfloor$ steps.*

3.2 Routings of permutations of L_n

Lemma 8 *Let A be the matrix obtained from I by the row permutation $(1, 2, \ldots, n)$. Then the permutation τ_A corresponds to the product of transpositions $(1, n)(1, n-1) \ldots (1, 3)(1, 2)$. Each transposition $(1, i)$ corresponds to an element of Δ, and hence τ_A has an n-step routing.*

Corollary 6 *If A is obtained from I by* any *cyclic permutation of the rows of I, then τ_A can be routed in at most n steps.*

Proof. Any cyclic permutation corresponds to a power of $(1, 2, \ldots, n)$. This, in turn, is a product of disjoint cycles. Each cycle of length k is the product of k transpositions, and therefore can be routed in k steps. Since the cycles are disjoint, the transpositions in one cycle are disjoint from those in the other cycles. Hence the routings of these cycles can be carried out simultaneously, and so the number of steps in the routing is the maximum length of a cycle in this product, and thus is at most n. □

Lemma 9 *C can be routed in $3\lfloor n/2 \rfloor$ steps.*

Proof. The routing for τ_C as a permutation of F_n given in Corollary 4 works equally well for τ_C as a permutation of L_n.

Corollary 7 *If A is obtained from C by a cyclic permutation of the rows of C then τ_C can be routed in at most $5n/2$ steps.*

Proof. Let π be a cyclic permutation of the rows of C. Then $A = \pi(C) = C\pi(I)$. By Lemma 8 $\tau_{\pi(I)}$ has can be routed in at most n steps, and then by Lemma 9 τ_C can be routed in an additional $3\lfloor n/2 \rfloor$ steps, for a total of $n + 3\lfloor n/2 \rfloor \leq 5n/2$ steps. □

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(Received 12 July 2012; revised 11 June 2013, 29 Nov 2013)