ON THE EXISTENCE OF (v, k, t) TRADES

E.S. MAHMOODIAN and NASRIN SOLTANKHAH

Department of Mathematical Sciences
Sharif University of Technology
P. O. Box 11365–9415
Tehran, I.R. Iran

ABSTRACT

A (v, k, t) trade can be used to construct new designs with various support sizes from a given t-design. H.L. Hwang (1986) showed the existence of (v, k, t) trades of volume 2^t and the non-existence of trades of volumes less than 2^t or of volume $2^t + 1$. In this paper, first we show that there exist (v, k, t) trades of volumes $2^t + 2^{t-1}$ $(t \geq 1)$, $2^t + 2^{t-1} + 2^{t-2}$ $(t \geq 2)$, $2^t + 2^{t-1} + 2^{t-2} + 2^{t-3}$ $(t \geq 3)$, and 2^{t+1}. Then we prove that, given integers $v > k > t \geq 1$, there does not exist a (v, k, t) trade of volume s, where $2^t < s < 2^t + 2^{t-1}$.

1. Introduction

Let V be a set of elements with cardinality v, and let k, t and λ be positive integers such that $v > k > t$. A k-subset of V will be called a *block*. A t-(v, k, λ) *design* (or simply a *t-design*) is a collection of blocks such that every t-subset of V is contained in λ blocks. The number of blocks in a t-design is denoted by b.

A (v, k, t) *trade* of *volume* s consists of two disjoint collections T_1 and T_2 , each of s blocks, such that for every t-subset the number of blocks containing this subset is the same in both T_1 and T_2 . When $s = 0$ the trade is said to be *void*.

The definitions of t-designs and (v, k, t) trades allow repeated blocks. A t-design (or a trade) having no repeated block is called a *uniform* design (or trade). The number of distinct blocks in a t-design is referred to as the *support size* of the design, and is denoted by b^*.

It is an easy exercise to prove that a t-design (or a (v, k, t) trade) is also a t'-design (or a (v, k, t') trade), for all t' with $0 < t' < t$. In a (v, k, t) trade both collections of blocks cover the same set of elements. This set of elements is called the *foundation* of the trade. Note that in a (v, k, t) trade, some of the elements of V may not appear at all. Let D be a t-(v, k, λ) design which contains the collection of blocks in T_1 of a (v, k, t) trade. Then by substituting the blocks of T_2 for the blocks of T_1 in the design, we obtain a new t-(v, k, λ) design, with possibly a new support size. This method, called the *method of trade off*, was introduced in 1979 by A. Hedayat and S.-Y.R. Li [2]. For a reference on the application of this method see, for example, [1] or [5].

The structure of (v,k,t) trades has been studied by H.L. Hwang [3]. She has shown that for each t and any $k \geq t+1$, $v \geq k+t+1$, there exists a (v,k,t) trade of volume 2^t , and there exists no (v,k,t) trade of volume less than 2^t or of volume 2^t+1. In constructing 3-(9,4,λ) and 3-(10,4,λ) designs with various support sizes, one of the present authors noted that there might not exist a $(v,4,3)$ trade of volume 10, 11 or 13 [5]. We shall prove a general theorem, which as a special case implies that there is no $(v,k,3)$ trade of volume 10 or 11.

2. Notation

To avoid triviality, unless otherwise stated, all the trades in this paper are non-void and v, k, t satisfy $k \geq t+1$ and $v \geq k+t+1$. We shall follow the notation used by Hwang [3]:

(i) A (v,k,t) trade of volume s will be represented by

$$T = T_1 - T_2 = \sum_{i=1}^{s} B_{1i} - \sum_{i=1}^{s} B_{2i},$$

where the B_{1i}'s and B_{2i}'s are the blocks contained in T_1 and T_2, respectively.

(ii) The foundation of a trade T will be denoted by found (T).

(iii) Let D be collection of blocks and $\{x_1, ..., x_i\}$ be an i-subset of V, $0 < i < k$. We define $\lambda_{D\{x_1,...,x_i\}}$ = number of blocks in D which contain $\{x_1, ..., x_i\}$.
To avoid messy notation, we shall use $\lambda_{(x_1,...,x_i)}$ for $\lambda_{D\{x_1,...,x_i\}}$ and r_x for $\lambda_{D\{x\}}$.

(iv) For a subset B of A, $A \setminus B$ denotes the complement of B with respect to A. For two given (v,k,t) trades $T = T_1 - T_2$ and $T^* = T_1^* - T_1^*$, $T \setminus T^* = (T_1 \setminus T_1^*) - (T_2 \setminus T_2^*)$ and $T + T^* = T_1T_1^* - T_2T_2^*$ are also (v,k,t) trades.

(v) ABC will denote the set union of A, B and C.

Now we state some of the results of Hwang [3] which are needed in the sequel.

Lemma 1 [3]. Let T be a (v,k,t) trade of volume s, and $x \in$ found (T) with $r_x < s$, where $r_x = r_{T_{1x}} = r_{T_{2x}}$. Let

$$T_{1x} = \sum_{i:B_{1i}\ni x} B_{1i} \quad , \quad T_{2x} = \sum_{i:B_{2i}\ni x} B_{2i}$$

and

$$T'_{1x} = \sum_{i:B_{1i}\not\ni x} B_{1i} \quad , \quad T'_{2x} = \sum_{i:B_{2i}\not\ni x} B_{2i}.$$

Then:

(i) $T_x = T_{1x} - T_{2x}$ is a $(v, k, t-1)$ trade of volume r_x;

(ii) $T'_x = T'_{1x} - T'_{2x}$ is a $(v-1, k, t-1)$ trade of volume $s - r_x$.

Theorem 1 [3]. If T is a (v, k, t) trade, then:

(i) $|\text{found } (T)| \geq k + t + 1$;

(ii) the volume of T is at least 2^t.

Lemma 2 [3]. If T is a (v, k, t) trade of volume 2^t, then for any i-subset of V, say $(x_1, ..., x_i)$, $0 < i \leq t$,

$$\lambda_{T_1\{x_1,...,x_i\}} = \lambda_{T_2\{x_1,...,x_i\}} = 2^t, 2^{t-1}, ..., 2^{t-i} \quad \text{or} \quad 0.$$

Lemma 3 [3]. If T is a (v, k, t) trade of volume s, then for any $x \in$ found (T), either $r_x = s$ or $2^{t-1} \leq r_x \leq s - 2^{t-1}$.

Lemma 4 [3]. If T is a (v, k, t) trade of volume s with $2^t < s < 2^t + 2^{t-1}$, then $r_x > 2^{t-1}$ for some $x \in$ found (T). Consider the permutations on $\{1, ..., 2n\}$. For each ℓ, $\ell = 1, ..., n$, let

$$P_\ell = \{(i_1, i_1 + 1)(i_2, i_2 + 1)...(i_\ell, i_\ell + 1) : \{i_1, ..., i_\ell\} \subseteq \{1, 3, ..., 2n-1\}\}.$$

So P_ℓ is a permutation which sends i_j into $i_j + 1$ and $i_j + 1$ to i_j for $j = 1, ..., \ell$, and leaves all other elements fixed. Also, let $P_0 = \{(1)\}$ consist of the identity permutation only. Define

$$\Delta_{2n} = \bigcup_{\ell:\text{ even}} P_\ell \quad \text{and} \quad \overline{\Delta}_{2n} = \bigcup_{\ell:\text{ odd}} P_\ell.$$

Theorem 3 [3]. All (v, k, t) trades of volume 2^t are of the form:
$T = T_1 - T_2 = \sum_{\sigma\in\Delta_{2t+2}} S_{\sigma(1)}S_{\sigma(3)}...S_{\sigma(2t+1)}S_{2t+3} - \sum_{\overline{\sigma}\in\overline{\Delta}_{2t+2}} S_{\overline{\sigma}(1)}S_{\overline{\sigma}(3)}...S_{\overline{\sigma}(2t+1)}S_{2t+3}$ with

$S_i \subseteq V$ for $i = 1, ..., 2t+3$, $S_i \cap S_j = \emptyset$ for $i \neq j$, $|S_{2i-1}| = |S_{2i}| \geq 1$ for $i = 1, ..., t+1$, and $\sum_{i=1}^{t+2} |S_{2i-1}| = k$.

Theorem 5 [3]. There is no $(v, k, 2)$ trade of volume 5.

3. The existence of some trades

In this section we show the existence of some trades of a given volume s. First we state and prove the following useful lemma.

Lemma 3.1. Let $T = T_1 - T_2$ be a $(v, t+1, t)$ trade of volume s. Then, based on T, a $(v+2, t+2, t+1)$ trade T^* of volume $2s$ can be constructed.

Proof. Let x and y be two new elements. Then we can construct the blocks of T^* as follows.

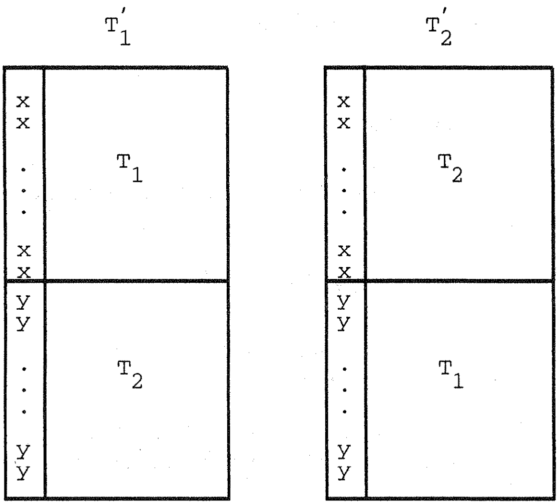

Figure 1

Clearly T^* is a $(v+2, t+2, t+1)$ trade of volume $2s$. ∎

Theorem 3.2. Let $k \geq t+1$. Then:

(i) there exists a (v, k, t) trade of volume 2^t for any $v \geq k+t+1$;

(ii) there exist trades of volumes $2^t + 2^{t-1}$ $(t \geq 1)$, $2^t + 2^{t-1} + 2^{t-2}$ $(t \geq 2)$, $2^t + 2^{t-1} + 2^{t-2} + 2^{t-3}$ $(t \geq 3)$, for any $v \geq k+t+2$;

(iii) there exists a trade of volume 2^{t+1} for any $v \geq k+t+3$.

Proof. The existence of a (v, k, t) trade of volume 2^t is shown in [3]. Observing the following special trades and utilizing Lemma 3.1 the theorem is established.

A (5,2,1) trade of volume 3:

$$T_1 = \{13,\ 14,\ 25\}, \quad T_2 = \{12,\ 15,\ 34\};$$

a (7,3,2) trade of volume 7:

$$\begin{aligned} T_1 &= \{123,\ 147,\ 245,\ 267,\ 347,\ 357,\ 456\}, \\ T_2 &= \{124,\ 137,\ 256,\ 237,\ 345,\ 457,\ 467\}; \end{aligned}$$

a (9,4,3) trade of volume 15:

$$\begin{aligned} T_1 = \{&1238,\ 1245,\ 1357,\ 1478,\ 2359,\ 2367,\ 2489,\ 2569, \\ &2578,\ 3469,\ 3478,\ 3789,\ 4567,\ 4579,\ 4589\}, \\ T_2 = \{&1235,\ 1248,\ 1378,\ 1457,\ 2369,\ 2378,\ 2459,\ 2567, \\ &2589,\ 3467,\ 3489,\ 3579,\ 4569,\ 4578,\ 4789\}; \end{aligned}$$

a (6,2,1) trade of volume 4:

$$T_1 = \{13,\ 14,\ 25,\ 26\}, \quad T_2 = \{15,\ 16,\ 23,\ 24\}. \blacksquare$$

Remark. Construction of the above 1-trades is easy and the 2-trade and 3-trade can be deduced from the tables of [5]. All the above trades are uniform trades. Similar non-uniform trades exist as well; for example, the following (5,2,1) trade of volume 4 is non-uniform:

$$T_1 = \{13,\ 14,\ 23,\ 25\ \}, \quad T_2 = \{12,\ 12,\ 34,\ 35\ \}.$$

4. Non-existence of some trades

In this section we state and prove our main theorem, which shows that, for certain volumes, there does not exist any trade. First we prove the following lemmas.

Lemma 4.1. Let $k \geq t+1$, and let T be a (v, k, t) trade of volume s, such that $s = 2^t + 2^{t-2} + i$, with $0 \leq i \leq 2^{t-2} - 1$, and let x, $y \in$ found (T). Then:

(i) if $r_x = 2^{t-1} + 2^{t-2} + i$ and $r_y = 2^{t-1}$, then $\lambda_{xy} = 0$ or 2^{t-2};

(ii) if $r_x = r_y = 2^{t-1} + 2^{t-2} + i$, then $\lambda_{xy} = 2^{t-1} + i$ or $2^{t-1} + 2^{t-2} + i$;

(iii) if $r_x = r_y = 2^{t-1}$, then $\lambda_{xy} = 2^{t-2}$ or 2^{t-1}.

Proof. (i) Let $x, y \in$ found (T) such that $r_x = 2^{t-1} + 2^{t-2} + i$ and $r_y = 2^{t-1}$. Then T'_x is a $(v-1, k, t-1)$ trade of volume 2^{t-1}. By Lemma 2 [3], r'_y, the number of blocks in T'_x containing y, is given by

$$r'_y = 2^{t-1}, \quad 2^{t-2} \quad \text{or} \quad 0.$$

Therefore

$$\lambda_{xy} = 0, \quad 2^{t-2} \quad \text{or} \quad 2^{t-1}.$$

But if $\lambda_{xy} = 2^{t-1}$, then $T_x - T_y$ is a $(v-1, k, t-1)$ trade of volume $r_x - r_y = 2^{t-2} + i < 2^{t-1}$. This is impossible by Theorem 1(ii) [3].

(ii) Let $x, y \in$ found (T), $r_x = r_y = 2^{t-1} + 2^{t-2} + i$. Then T'_x is a $(v-1, k, t-1)$ trade of volume 2^{t-1}. Again by Lemma 2 [3],

$$r'_y = 2^{t-1}, \quad 2^{t-2} \quad \text{or} \quad 0,$$

which implies that

$$\lambda_{xy} = 2^{t-2} + i, \quad 2^{t-1} + i \quad \text{or} \quad 2^{t-1} + 2^{t-2} + i.$$

But if $\lambda_{xy} = 2^{t-2} + i$, then $T - (T'_x + T'_y)$ is a $(v, k, t-1)$ trade of volume $2^{t-2} + i < 2^{t-2} + 2^{t-2} = 2^{t-1}$, which is also impossible by Theorem 1(ii) [3].

(iii) Let $x, y \in$ found (T), $r_x = r_y = 2^{t-1}$. Then T_x is a $(v, k, t-1)$ trade of volume 2^{t-1}. By taking $i = 1$ in Lemma 2 [3],

$$\lambda_{xy} = 2^{t-1}, \quad 2^{t-2} \quad \text{or} \quad 0.$$

But the impossibility of $\lambda_{xy} = 0$ can be seen by considering the $(v-2, k, t-1)$ trade $T - (T_x + T_y)$. ∎

Lemma 4.2. Suppose that $k > t+1$ and that T is a (v, k, t) trade of volume s. If $x, y \in$ found (T), such that $r_x + r_y = s$ and $\lambda_{xy} = 0$, then $T' = T'_1 - T'_2$, where

$$T'_1 = \sum_{i=1}^{s} (B_{1i} - \{x, y\}) \quad \text{and} \quad T'_2 = \sum_{i=1}^{s} (B_{2i} - \{x, y\}),$$

is a $(v-2, k-1, t)$ trade.

Proof. If $\{a_1, \ldots, a_t\}$ is a t-subset of V such that $a_i \notin \{x, y\}$ for $i = 1, \ldots, t$, then we have

$$\lambda_{T'_1(a_1, \ldots, a_t)} = \lambda_{T_1(a_1, \ldots, a_t)},$$

$$\lambda_{T_2'(a_1,\ldots,a_t)} = \lambda_{T_2(a_1,\ldots,a_t)}.$$

Since T is a (v,k,t) trade, $\lambda_{T_1(a_1,\ldots,a_t)} = \lambda_{T_2(a_1,\ldots,a_t)}$. Therefore $\lambda_{T_1'(a_1,\ldots,a_t)} = \lambda_{T_2'(a_1,\ldots,a_t)}$. Thus T' is a $(v-2,k-1,t)$ trade. ■

Note that in Lemma 4.2, the volume of T' is less than or equal to s. Indeed, T' may be a void trade; for example, consider the following (6,3,1) trade:

$$T = (x12) + (y34) - (y12) - (x34).$$

Lemma 4.3. If a $(v,t+1,t)$ trade T contains x, y such that $r_x \neq r_y$ and $r_x + r_y = s$, then $\lambda_{xy} \neq 0$.

Proof. Suppose $\lambda_{xy} = 0$. Each t-subset $\{a_1,\ldots,a_t\}$ such that $a_i \neq x$ for $i = 1,\ldots,t$, which appears with x in the blocks of T_1 must appear with y in the blocks of T_2. This implies $r_x = r_y$, a contradiction. ■

Theorem 4.4. Given integers $v > k > t \geq 1$, there exists no (v,k,t) trade of volume s, where $2^t < s < 2^t + 2^{t-1}$.

Proof. We proceed by induction on t. For $t = 1$ there is nothing to prove. For $t = 2$, by Theorem 5 [3], there exists no $(v,k,2)$ trade of volume 5. Assume that the theorem holds for all values less than t ($t \geq 3$). We show that it holds for t also. Suppose the theorem is not true for t, and let T be a (v,k,t) trade of volume $s = 2^t + i$, where $0 < i < 2^{t-1}$. We show a contradiction.

There are two possible cases:

$$\text{(i)}\ \ 0 < i < 2^{t-2} \quad \text{or} \quad \text{(ii)}\ \ 2^{t-2} \leq i < 2^{t-1}.$$

If $0 < i < 2^{t-2}$, then in any trade of volume s, there exists an element x in found (T) such that $r_x < s$. By Lemma 3 [3], we have $r_x = 2^{t-1}+j$, where $0 \leq j \leq i$. If $j \neq 0$, then T_x is a $(v,k,t-1)$ trade of volume $2^{t-1}+j$, with $0 < j \leq i < 2^{t-2}$, which is contradictory to the induction hypothesis. If $j = 0$, then T_x' is a $(v-1,k,t-1)$ trade of volume $2^{t-1}+i$, where $0 < i < 2^{t-2}$, again contradicting the induction hypothesis.

Now we show that (ii) is also impossible; that is, there exists no (v,k,t) trade of volume

$$s = 2^t + 2^{t-2} + i, \quad \text{where} \quad 0 \leq i \leq 2^{t-2} - 1. \tag{1}$$

There exists an element x in found (T) such that $r_x < s$. By Lemma 3 [3] we have $r_x = 2^{t-2}+j$, with $0 \leq j \leq 2^{t-2}+i$. If $0 < j < 2^{t-2}+i$, then either T_x or T_x' is a $(v,k,t-1)$ trade of volume $2^{t-1}+l$, where $1 \leq l < 2^{t-2}$, and this is impossible by the induction hypothesis. Therefore, if $r_x \neq s$, then either $r_x = 2^{t-1}+2^{t-2}+i$ or $r_x = 2^{t-1}$. For simplicity put

$$a = 2^{t-1} + 2^{t-2} + i \quad \text{and} \quad b = 2^{t-1}.$$

We should note that the existence of a (v,k,t) trade of volume s, in which precisely c elements belong to all the blocks of T, is equivalent to the existence of a $(v-c,k-c,t)$ trade of volume s, having no element x with $r_x = s$. Thus we may assume that, for each element,

$$\text{either} \qquad r_x = a \qquad \text{or} \qquad r_x = b. \tag{2}$$

Therefore, there are three cases to be investigated:
Case 1: $r_x = b$ for all $x \in$ found (T);
Case 2: $r_x = a$ for all $x \in$ found (T);
Case 3: there exist $x, y \in$ found (T), such that $r_x = a$ and $r_y = b$.

Case 1 is impossible by Lemma 4 [3].

For **Case 2**, let $x \in$ found (T). T'_x is a $(v-1,k,t-1)$ trade of volume 2^{t-1}, which by Theorem 3 [3] is of the following form:

$$T'_x = T'_{1x} - T'_{2x} = \textstyle\sum_{\sigma\in\Delta_{2t}} S_{\sigma(1)}S_{\sigma(3)}...S_{\sigma(2t-1)}S_{2t+1} - \sum_{\overline{\sigma}\in\overline{\Delta}_{2t}} S_{\overline{\sigma}(1)}S_{\overline{\sigma}(3)}...S_{\overline{\sigma}(2t-1)}S_{2t+1},$$

where $S_j \subseteq V$ for $j = 1, ..., 2t+1$, $S_j \cap S_\ell = \emptyset$ for $j \neq \ell$,

$$|S_{2j-1}| = |S_{2j}| \geq 1 \text{ for } j = 1,2,...,t \text{ and } \sum_{j=1}^{t+1} |S_{2j-1}| = k.$$

But $S_{2t+1} = \emptyset$, for if $y \in S_{2t+1}$, then in trade T we have $\lambda_{xy} = 2^{t-2} + i$, and this contradicts Lemma 4.1(ii).

For each pair of $y, z \in$ found (T'_x), by Lemma 2 [3] and from the fact that $S_{2t+1} = \emptyset$, we have

$$\lambda'_{yz} = 2^{t-2}, 2^{t-3}, \text{ or } 0, \tag{3}$$

and

$$r'_y = 2^{t-2} \tag{4}$$

(As earlier λ'_{yz}, for example, denotes $\lambda_{T'_x\{yz\}}$.)

Let $y \in S_{2j-1}$ and $z \in S_\ell$. Then by (3) we conclude that

$$\text{if } \ell = 2j \text{ then } \lambda'_{yz} = 0 \tag{5}$$

and

$$\text{if } \ell \neq 2j \text{ or } 2j-1 \text{ then } \lambda'_{yz} = 2^{t-3}. \tag{6}$$

Let $y \in S_{2j-1}$. By (5) there are as many as $2^{t-1}+i$ blocks in T_{1x} which contain y. By (6) and Lemma 4.1(ii) every element of S_{2j} together with y, must be contained in $2^{t-1}+i$ blocks of T_{1x}. Since this is true for every $y \in S_{2j-1}$, all the elements in S_{2j-1} and S_{2j} must be included in the same blocks of T_{1x}. For simplicity, let $A_j = S_{2j-1} \cup S_{2j}$ for $j = 1, ..., t$ in T_{1x}. Since all the elements in A_j appear in the same blocks of T_{1x}, we can consider every set A_j in T_{1x} as an element.

By (7) and Lemma 4.1(ii), every pair of A_j and A_ℓ, for $(j \neq \ell)$, is included in exactly $3(2^{t-3}) + i$ blocks in T_{1x}. Now T_{xA_1} is a $(v, k, t-2)$ trade of volume $2^{t-1} + i$ (see Figure 2).

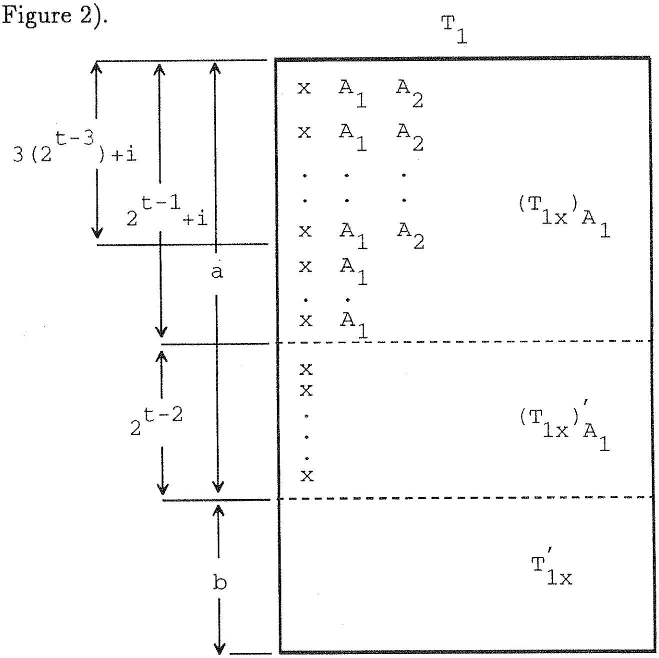

Figure 2

Moreover, $(T_x)'_{A_1}$, that is, the blocks in T_x which do not contain A_1, is also a $(v, k, t-2)$ trade of volume 2^{t-2}. Each A_j, for $j \neq 1$, is included in as many as $2^{t-1} + i - (3(2^{t-3}) + i) = 2^{t-3}$ blocks in $(T_x)'_{A_1}$. By Theorem 3 [3], the trade $(T_x)'_{A_1}$ has the following unique structure:

$$(T_x)'_{A_1} = (T_{1x})'_{A_1} - (T_{2x})'_{A_1} = \sum_{\sigma \in \Delta_{2t-2}} \overline{S}_{\sigma(1)} \overline{S}_{\sigma(3)} \ldots \overline{S}_{\sigma(2t-3)} \overline{S}_{2t-1} - \sum_{\sigma \in \overline{\Delta}_{2t-2}} \overline{S}_{\overline{\sigma}(1)} \overline{S}_{\overline{\sigma}(3)} \ldots \overline{S}_{\overline{\sigma}(2t-3)} \overline{S}_{2t-1} \qquad (7)$$

where $\overline{S}_j \subseteq V$ for $j = 1, 2, ..., 2t-1$, $\overline{S}_j \cap \overline{S}_\ell = \emptyset$ for $j \neq \ell$, $|\overline{S}_{2j-1}| = |\overline{S}_{2j}| \geq 1$

for $j = 1, 2, \ldots, t-1$ and $\sum_{j=1}^{t} |\overline{S}_{2j-1}| = k$. Since $\bigcup_{j=1}^{2t-2} \overline{S}_j \subseteq \bigcup_{\ell=2}^{t} A_\ell$, $|\overline{S}_j| \geq 1$ for $j = 1, 2, \ldots, 2t-2$, and $\overline{S}_j \cap \overline{S}_\ell = \emptyset$ for $j \neq \ell$. Therefore, by the pigeonhole principle, some of the $\overline{S}_j$'s must be empty, which is a contradiction.

For **Case 3**, we will call an element of type a (or b) if $r_x = a$ (or $r_x = b$) respectively. First we show that, for $k = t+1$, this case is impossible. For, let T be a $(v, t+1, t)$ trade and $r_x = a$, $r_y = b$, for some $x, y \in$ found (T). Then by Lemma 4.3, $\lambda_{xy} \neq 0$ and, consequently by Lemma 4.1(i), $\lambda_{xy} = 2^{t-2}$.

Thus the number of appearances of y in the blocks of T'_{1x} is also 2^{t-2}.

We denote by α_b the number of elements of type b in T. These elements occupy $\alpha_b.2^{t-2}$ places in all the blocks of T'_{1x}. Since T'_{1x} is of volume $b = 2^{t-1}$, the remaining $k.b - \alpha_b 2^{t-2}$ places, are filled with elements of type a. By Lemma 4.1(ii), every element $z \in$ found (T'_{1x}) of type a appears 2^{t-2} times in the blocks of T'_{1x}. Therefore, the number of such elements is equal to $\frac{k.b-\alpha_b.2^{t-2}}{2^{t-2}}$. This implies that

$$\alpha_a > \frac{k.b - \alpha_b.2^{t-2}}{2^{t-2}} \tag{8}$$

where α_a denotes the number of elements of type a in found (T).

We know that

$$a.\alpha_a + b.\alpha_b = s.k. \tag{9}$$

Hence

$$\alpha_a = \frac{s.k - b.\alpha_b}{a}.$$

Therefore,

$$\frac{s.k - b.\alpha_b}{a} > \frac{k.b - \alpha_b.2^{t-2}}{2^{t-2}}. \tag{10}$$

Now by substituting the values of a, b, and s in (11) we obtain

$$\alpha_b > k. \tag{11}$$

With a similar argument we can deduce that, at the same time

$$\alpha_a > k. \tag{12}$$

Inequalities (12) and (13) contradict (10).

Thus there does not exist a $(v, t+1, t)$ trade satisfying the condition s of case 3.

Now by induction on k, we show that case 3 is always impossible. Suppose that for $k-1$, there does not exist a $(v, k-1, t)$ trade $(k > t+1)$ satisfying the conditions of case 3, and of volume s, where $2^t < s < 2^t + 2^{t-1}$. Let T be a (v, k, t)

trade satisfying conditions of case 3. And let $x, y \in$ found (T), such that $r_x = a$ and $r_y = b$. We consider two cases.

(i) Suppose $\lambda_{xy} = 0$.

By Lemma 4.2, T' is a $(v-2, k-1, t)$ trade of volume s', where $0 \leq s' \leq s$. By Theorem 1(ii) [3], it is impossible to have $0 < s' < 2^t$.

If $s' = 2^t$, then T must be of the form given in Figure 3 below, with A and B identical.

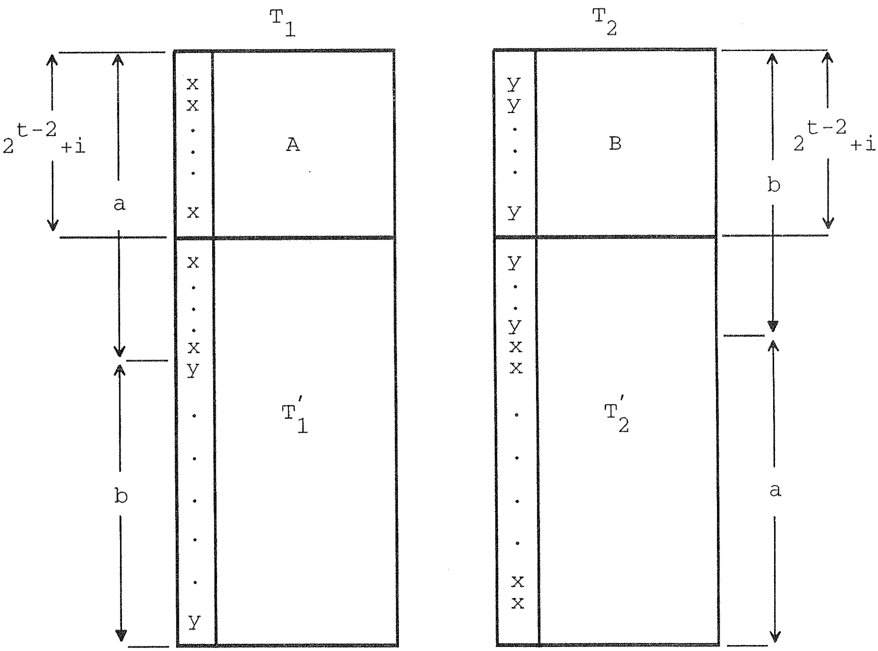

Figure 3

Next, we show that the rows of A (also B) are all identical. Since T' is a $(v-2, k-1, t)$ trade of volume 2^t, then by Lemma 2 [3], for each $z \in$ found (T') we have

$$r'_z = 2^t, 2^{t-1} \text{ or } 0. \tag{13}$$

Since $z \in$ found (T), if $r_z = 2^{t-1} = b$ then it is obviously impossible to have $r'_z = 2^t$. Also, $r'_z \neq 0$, since if $r'_z = 0$, then z must belong only to the set of blocks in A or B (Figure 3), each of which have $2^{t-2} + i$ blocks, where $i < 2^{t-2}$. Therefore $r_z \leq 2^{t-2} + i < 2^{t-1}$, a contradiction. Thus if $r_z = b$, then $r'_z = 2^{t-1} = r_z$, implying that z does not appear either in the blocks of A or in the blocks of B. If $r_z = a$, then a similar argument yields that z belongs to 2^{t-1} blocks of T' and therefore to all $2^{t-2} + i$ blocks of A and also of B.

Thus each set of blocks T_1 and T_2 containing $2^{t-2}+i$ identical blocks. Therefore T'_x, which is a $(v-1, k, t-1)$ trade of volume 2^{t-1}, must have $2^{t-2}+i$ identical blocks in T'_{2x}, which is a contradiction of Theorem 3 [3].

If $s' = 0$, then any $(k-1)$-subset S of elements ($x \notin S$) which appears with x in the blocks of T_1, it must appear with y in the blocks of T_2. Thus $r_x = r_y$, a contradiction.

Therefore, $2^t < s' \leq s$. This implies that T' is a $(v, k-1, t)$ trade of volume s' $(2^t < s' < 2^t + 2^{t-1})$, contradicting the induction hypothesis.

(ii) Suppose $\lambda_{xy} \neq 0$.

By Lemma 4.1(i), we have $\lambda_{xy} = 2^{t-2}$. The non-existence of T can be proved in the same way as in the case $k = t+1$. So we have established the impossibility of case 3. The proof given for the three cases establish the theorem. ■

Remark. Now that the existence and non-existence of some trades have been proven, one can ask a general question: For what values of s does there exist a (v, k, t) trade of volume s? The answer for this question will help to attack the open question on the existence of t-designs with various support sizes (see questions 22 and 24 in [4]). ¿From the results of section 3 and 4, we conjecture that:

1. for every $s_i = 2^t + 2^{t-1} + \ldots + 2^{t-i}$, $i = 0, 1, \ldots, t$, there exists a (v, k, t) trade of volume s_i;

2. for every s, $s_i < s < s_{i+1}$, $i = 0, 1, \ldots, t-1$, there does not exist a (v, k, t) trade of volume s.

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