AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **59(2)** (2014), Pages 311–322

On the k-independence number in graphs*

Ahmed Bouchou

University of Médéa
Algeria
bouchou.ahmed@yahoo.fr

Mostafa Blidia

Department of Mathematics
University of Blida
Algeria
m_blidia@yahoo.fr

Abstract

For an integer $k \geq 1$ and a graph $G = (V, E)$, a subset S of V is k-independent if every vertex in S has at most $k - 1$ neighbors in S. The k-independent number $\beta_k(G)$ is the maximum cardinality of a k-independent set of G. In this work, we study relations between $\beta_k(G)$, $\beta_j(G)$ and the domination number $\gamma(G)$ in a graph G where $1 \leq j < k$. Also we give some characterizations of extremal graphs.

1 Introduction

We consider simple graphs $G = (V(G), E(G))$ of order $|V(G)| = |V| = n(G)$ and size $|E(G)| = m(G)$. The *neighborhood* of a vertex $v \in V$ is $N_G(v) = \{u \in V \mid uv \in E\}$. The *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. If S is a subset of vertices, its *neighborhood* is $N_G(S) = \cup_{v \in S} N_G(v)$. The closed neighborhood of v and S are $N_G[v] = N_G(v) \cup \{v\}$ and $N_G[S] = N_G(S) \cup S$, respectively. The *degree* of a vertex v of G is $d_G(v) = |N_G(v)|$. The *maximum degree* of G is $\Delta(G) = \max\{d_G(v) \mid v \in V\}$ and the *minimum degree* of G is $\delta(G) = \min\{d_G(v) \mid v \in V\}$. The subgraph induced in G by a subset of vertices S is denoted by $G[S]$. The degree of vertex v in the subgraph induced in G by $S \subseteq V$ is denoted by $d_S(v) = |N_G(v) \cap S| = |N_S(v)|$. A graph is *bipartite* if its vertex set can be partitioned in two independent sets. A *matching* in a graph G is a subset of pairwise non-adjacent edges. A *d-regular* graph is a graph with a degree d for each vertex of G. The *subdivision graph* of a graph G

* This work was supported by Programmes Nationaux de Recherche: Code 8/u09/510.

is the graph obtained from G by replacing each edge uv of G by a vertex w and edges uw and vw. The *corona* of a graph $G = (V, E)$, denoted by $G \circ K_1$, is the graph that is obtained by attaching a leaf to each vertex $v \in V$. A *tree* is a connected graph with no cycle. The *path* (the *cycle*, the *clique*, the *star*, respectively) of order n is denoted by P_n (C_n, K_n, $K_{1,n-1}$, respectively).

An *independent set* S is a set of vertices whose induced subgraph has no edge, equivalently $\Delta(G[S]) = 0$. A *dominating set* S is a set of vertices such that every vertex in $V - S$ has at least one neighbor in S, equivalently $N[S] = V$. In [7, 8] Fink and Jacobson defined a generalization of the concepts of independence and domination. For an integer $k \geq 1$ and a graph G, a subset S of V is k*-independent* if $\Delta(G[S]) < k$ and k*-dominating* if every vertex in $V - S$ has at least k neighbors in S. We denote by $\beta_k(G)$ the maximum order of a k-independent set, this parameter is called the k*-independence number* and we denote by $\gamma_k(G)$ the minimum order of a k-dominating set and it is called the k*-domination number*. A k-independence set S with cardinality $\beta_k(G)$ is called a $\beta_k(G)$*-set*. Thus for $k = 1$, the 1-independent and 1-dominating sets are the classical independent and dominating sets. However, $\beta_1(G) = \beta(G)$ is the independence number and $\gamma_1(G) = \gamma(G)$ is the domination number.

More details and results on k-domination and k-independence can be found in [4, 7, 8].

In this paper we present relations between $\beta_k(G)$, $\beta_j(G)$ and $\gamma(G)$ in a graph G where $1 \leq j < k$. Also we give some characterizations of extremal graphs.

First, we recall some known results of k-domination and k-independence that will be useful here.

Theorem 1 (Favaron [5]) *For any graph G and positive integer k, every k-independent set D such that $\varphi_k(D) = k|D| - |E(G[D])|$ is maximum, is a k-dominating set of G.*

Corollary 1 (Favaron [5]) *For any graph G and positive integer k, $\gamma_k(G) \leq \beta_k(G)$.*

Theorem 2 (Jacobson et al. [11]) *If G is a graph of order n, then $\gamma_k(G) + \beta_j(G) \leq n$ for $\delta(G) = k + j - 1$.*

Theorem 3 (Favaron [6]) *If G is a graph of order n, then $\gamma_k(G) + \beta_j(G) \geq n$ for $\Delta(G) = k + j - 1$.*

If moreover G is d-regular with $d = k + j - 1$, then $\gamma_k(G) + \beta_j(G) = n$.

It is well-known (see Ore [12]) that every graph G of order n without isolated vertices satisfies $\gamma(G) \leq \frac{n}{2}$. Extremal graphs achieving equality in Ore's bound have been given independently by Walikar et al. [14], Payan and Xuong [13] and by Fink et al. [9].

Theorem 4 (Fink et al. [9], Payan and Xuong [13], Walikar [14]) *Let G be a graph of even order n without isolated vertices. Then $\gamma(G) = \frac{n}{2}$ if and only if each component of G is either a cycle C_4 of length four or the corona $J \circ K_1$ of some connected graph J.*

2 Bounds on β_k and its relation with other parameters

We give a relation between β_k and β_j for $1 \leq j < k \leq \Delta(G) + 1$.

Note that the same relation is given independently by Caro and Hansberg in [3] by using the bound $\beta_j(G) \geq \frac{n}{1+\lfloor \frac{\Delta(G)}{j} \rfloor}$ due to Hopkins and Staton [10].

Here, we give a new proof which is useful for some of the following characterizations.

Theorem 5 *Let G be a graph of order n and maximum degree $\Delta(G)$, and let j, k be integers with $1 \leq j < k \leq \Delta(G) + 1$. Then*

$$\beta_k(G) \leq \left\lceil \frac{k}{j} \right\rceil \beta_j(G). \tag{1}$$

Proof. Let I be a $\beta_k(G)$-set of G. Let S_1 be a j-independent and j-dominating set of $G[I]$. In view of Theorem 1, such a set exists. Then every vertex of $I - S_1$ has at least j neighbors in S_1 and thus, $\Delta(G[I - S_1]) \leq k - j - 1$. Let S_2 be a j-independent and j-dominating set of $G[I - S_1]$. Then every vertex of $I - (S_1 \cup S_2)$ has at least j neighbors in S_1 and j neighbors in S_2 and thus, $\Delta(G[I - (S_1 \cup S_2)]) \leq k - 2j - 1$. We continue the process until the choice of a j-independent and j-dominating set S_{p-1} of $G\left[I - \bigcup_{i=1}^{p-2} S_i\right]$ such that the set $S_p = I - \bigcup_{i=1}^{p-1} S_i$ is j-independent. Hence, $\Delta\left(G[I - \bigcup_{i=1}^{p-1} S_i]\right) \leq k - (p-1)j - 1$. Therefore $|S_i| \leq \beta_j(G)$ for $1 \leq i \leq p$. Hence

$$\beta_k(G) = |I| = \sum_{i=1}^{p} |S_i| \leq p\beta_j(G).$$

Now we show that $p \leq \left\lceil \frac{k}{j} \right\rceil$. Let x be a vertex of S_p. Since $\Delta(G[I]) \leq k - 1$ and $d_{S_i}(x) \geq j$ for $1 \leq i \leq p - 1$, then $j(p-1) \leq d_I(x) \leq k - 1$, which means that $p \leq \left\lfloor \frac{k-1}{j} \right\rfloor + 1 = \left\lceil \frac{k}{j} \right\rceil$. Consequently $\beta_k(G) \leq \left\lceil \frac{k}{j} \right\rceil \beta_j(G)$. ■

Setting $k = \Delta(G) + 1$ ($j = 1$, respectively) in Theorem 5, the following known bound of Hopkins and Staton [10] (Blidia et al. [2], respectively) follows.

Corollary 2 (Hopkins, Staton [10]) *If G is a graph of order n, maximum degree $\Delta(G)$ and $j \geq 1$ an integer, then $\beta_j(G) \geq \dfrac{n}{1 + \left\lfloor \frac{\Delta(G)}{j} \right\rfloor}$.*

Corollary 3 (Blidia et al. [2]) *If G is a graph and k a positive integer, then $\beta_k(G) \leq k\beta(G)$.*

Now we give a necessary condition for the equality $\beta_k(G) = \left\lceil \frac{k}{j} \right\rceil \beta_j(G)$ when $j = k-1$.

Theorem 6 *For every graph G of order n and for every integer $k \geq 2$,*

$$\beta_k(G) \leq 2\beta_{k-1}(G). \tag{2}$$

Also if equality holds, then every component of any $\beta_k(G)$-set I is either a clique K_2 and $k = 2$ or a cycle C_4 and $k = 3$.

Proof. Replacing j by $k-1$ in (1), we deduce that $\left\lceil \frac{k}{k-1} \right\rceil = 2$, so we obtain the desired inequality.

Now assume that $\beta_k(G) = 2\beta_{k-1}(G)$. Following the notation used in the proof of Theorem 5, Since I is k-independent, $d_{S_2}(x) \leq k-1$ for every $x \in S_1$, and since S_1 is a $(k-1)$-dominating set of $G[I]$, $d_{S_1}(y) \geq k-1$ for every $y \in S_2$. Hence the number $m(S_1, S_2)$ of edges of G between S_1 and S_2 satisfies $(k-1)|S_2| \leq m(S_1,S_2) \leq (k-1)|S_1|$ and so $|S_2| \leq |S_1|$. Since $2\beta_{k-1}(G) = \beta_k(G) = |I| = |S_1| + |S_2| \leq 2|S_1| \leq 2\beta_{k-1}(G)$, $|S_1| = |S_2|$ and so we obtain that $(k-1)|S_2| = m(S_1,S_2) = (k-1)|S_1|$. Therefore $G[I]$ is a $(k-1)$-regular bipartite graph and $\beta_{k-1}(G) = \frac{|I|}{2}$. Now applying Theorem 3 for the subgraph $G[I]$, we obtain $\gamma(G[I]) = |I| - \beta_{k-1}(G[I]) = \frac{|I|}{2}$, since $G[I]$ is $(k-1)$-regular, and Theorem 4 shows that the only connected regular bipartite graphs with $\gamma(G[I]) = \frac{|I|}{2}$ are K_2 or C_4, so each component of $G[I]$ either is a clique K_2 and $k = 2$ or a cycle C_4 and $k = 3$. ■

The converse of Theorem 6 is not true, as shown by the following examples.

Let P_5 be the path on five vertices, labeled in order x_1, x_2, x_3, x_4, x_5. Let F be the graph obtained from P_5 by adding new edges x_1x_4 and x_2x_5.

For $k = 2$: Let G_1 consist of the disjoint union of $2p$ copies of P_5 plus a path through the central vertices of these copies. It is clear that $n(G_1) = 10p$, $\beta_2(G_1) = 8p$, $\beta(G_1) = 5p$ and each component of any $\beta_2(G_1)$-set is a clique K_2, but $\beta_2(G_2) \neq 2\beta(G_1)$.

For $k = 3$: Let G_2 consist of the disjoint union of $3p$ copies of F plus a path through x_3 of these copies. It is clear that $n(G_2) = 15p$, $\beta_3(G_2) = 12p$, $\beta_2(G_2) = 8p$ and each component of any $\beta_3(G)$-set is a cycle C_4, but $\beta_3(G_2) \neq 2\beta_2(G_2)$.

From Theorem 6 and since $\beta_{\Delta+1}(G) = n$, we obtain the following.

Corollary 4 *If G is a graph of order n and maximum degree $\Delta(G) \geq 1$, then $\beta_\Delta(G) \geq \left\lceil \frac{n}{2} \right\rceil$.*

By Corollary 3, we have $\beta_k(G) \leq k\beta(G)$; this inequality cannot be improved to $\beta_k(G) \leq k\gamma(G)$, even for trees, as shown by the star $K_{1,p}$ with $p \geq k+1$. However the next theorem improves it in the class of graphs with at most one cycle for $k = 2$. We denote by $\lambda(G) = m(G) - n(G) + 1$ the *cyclomatic* number of a connected graph G.

Theorem 7 *Let G be a connected graph of order $n \geq 3$. Then*

$$\beta_2(G) \leq \beta(G) + \gamma(G) + \lambda(G) - 1.$$

Proof. Let I be a $\beta_2(G)$-set and S be a maximal independent set of $G[I]$. If $G[I]$ is independent, then $\beta_2(G) = \beta(G)$ and so $\beta_2(G) \leq \beta(G) + \gamma(G) + \lambda(G) - 1$. If $G[I]$ is not independent, then the edges of $G[I]$ form an induced matching M between $A = I - S$ and a subset A' of S. Let D be a $\gamma(G)$-set, M_1 the edges of M with no endvertex in D, and A_1 (A'_1 respectively) the set of the endvertices of the edges of M_1 in A (A' respectively). If $|M_1| \neq 0$ and $\gamma(G) \leq |M| - \lambda(G)$, then the vertices of $A_1 \cup A'_1$ cannot be dominated by vertices in $D \cap I$, since M is induced. Hence the set $W = D - I$ is not empty and dominates $A_1 \cup A'_1$. Therefore the induced subgraph $G[W \cup A_1 \cup A'_1]$ of order $|W| + 2|M_1|$ contains at least $3|M_1|$ edges. Moreover, since D contains at least one endvertex of each edge in $M - M_1$, $|W| \leq |D| - |M - M_1| = (\gamma(G) - |M|) + |M_1| < |M_1| - \lambda(G) + 1$. So, the connected subgraph G' induced by $W \cup A_1 \cup A'_1$ satisfies $|E(G')| \geq 3|M_1| > 2|M_1| + |W| + \lambda(G) - 1 \geq 2|M_1| + |W| + \lambda(G') - 1$, thus $\lambda(G') = m(G') - n(G') + 1 < |E(G')| - (2|M_1| + |W|) + 1$, a contradiction. Thus $\gamma(G) \geq |M| - \lambda(G) + 1$ or $|M_1| = 0$. If $\gamma(G) \geq |M| - \lambda(G) + 1$, then $\gamma(G) \geq |A| - \lambda(G) + 1$ and $\beta_2(G) = |S| + |A| \leq \beta(G) + \gamma(G) + \lambda(G) - 1$. If $\gamma(G) \leq |M| - \lambda(G)$ and $|M_1| = 0$, then $\lambda(G) = 0$, since $|M| \leq \gamma(G)$ and so $\gamma(G) = |M| = |A|$, $S - A' = \emptyset$ and G is a tree. Hence, we must have $V - I \neq \emptyset$. Let x be a vertex of $V - I$. For any edge e of M, x is adjacent to at most one endvertex of e. Without loss of generality, suppose that the vertices adjacent to x are in A. Then $A' \cup \{x\}$ is an independent set. So, $\beta(G) \geq |A| + 1$ and $\beta_2(G) = 2|A| \leq \beta(G) + \gamma(G) - 1 = \beta(G) + \gamma(G) + \lambda(G) - 1$, and the proof is complete. ■

In general, the bounds of Theorem 6 with $k = 2$ and Theorem 7 are not comparable for $\lambda(G) \geq 2$. Indeed, if G is the graph obtained from $G' = pK_2 + qK_1$ by joining all vertices of G' to a new vertex x, then $\beta_2(G) = 2p + q$, $\beta(G) = p + q$, $\lambda(G) = p$ and $\gamma(G) = 1$. If $p = 0$ and $q \geq 2$, then G is a star and $\beta_2(G) = \beta(G) + \gamma(G) - 1 = 2\beta(G)$. If $p \geq 1$ and $q \geq 0$, then G is a graph with p triangles and $\beta_2(G) = \beta(G) + \gamma(G) + p - 1 < 2\beta(G)$. However, if G is the graph obtained by joining each vertex of p copies of K_3 to a new vertex x, then $\beta_2(G) = 2p$, $\beta(G) = p$, $\gamma(G) = 1$, $\lambda(G) = 3p$ and $\beta_2(G) \leq 2\beta(G) = 2p < \beta(G) + \gamma(G) + \lambda(G) - 1 = 4p + 1$.

3 Characterizations of some special graphs

In this section we give some characterizations of special graphs for inequality $\beta_k(G) \leq \left\lceil \frac{k}{j} \right\rceil \beta_j(G)$.

We begin by giving a characterization of extremal graphs attaining the bound in Corollary 4. We need the following known result.

Theorem 8 (Fink, Jacobson [7]) *If G is a graph with $\Delta(G) \geq k \geq 2$, then $\gamma_k(G) \geq \gamma(G) + k - 2$.*

Theorem 9 *Let G be a connected graph of order n and maximum degree $\Delta(G) \geq 1$. Then*

$$\beta_\Delta(G) = \left\lceil \frac{n}{2} \right\rceil$$

if and only if $G \in \{P_2, P_3, C_3, C_4, C_5, C_7\}$.

Proof. It is easy to see that $\beta_\Delta(G) = \lceil \frac{n}{2} \rceil$ for $G \in \{P_2, P_3, C_3, C_4, C_5, C_7\}$.

Now, assume that $\beta_\Delta(G) = \lceil \frac{n}{2} \rceil$. If n is even, then from Theorem 6, we have G is a P_2 or a C_4. If n is odd, then $n = 2\beta_\Delta(G) - 1 \geq 3$ and $\Delta(G) \geq 2$. Now applying Theorem 3, we obtain $\gamma(G) \geq n - \beta_\Delta(G) = \frac{n-1}{2}$ and by Ore's bound [12], we deduce that $\gamma(G) = \frac{n-1}{2}$, on the other hand, from Corollary 1, we have $\gamma_\Delta(G) \leq \beta_\Delta(G)$, so $\gamma_\Delta(G) - \gamma(G) \leq \beta_\Delta(G) - \gamma(G) = 1$ which is only possible when $\Delta(G) \leq 3$, since $\gamma_p(G) \geq \gamma(G) + p - 2$ for any $2 \leq p \leq \Delta(G)$ (see Theorem 8). We distinguish between two cases :

Case 1. $\Delta(G) = 2$.

Then G is a path or a cycle. If G is a path with $n \geq 5$ or a cycle with $n \geq 9$, then $\beta_2(P_n) = \lceil \frac{2n}{3} \rceil > \frac{n+1}{2}$ and $\beta_2(C_n) = \lfloor \frac{2n}{3} \rfloor > \frac{n+1}{2}$.

Case 2. $\Delta(G) = 3$.

Then $1 \leq \delta(G) \leq 3$. As in the proof of Theorem 5, $I = V$, S_1 is a 3-independent and a 3-dominating set of G and S_2 is independent. Hence $3|S_2| = m(S_1, S_2) \leq 3|S_1|$. So, $|S_2| \leq |S_1| - 1$, since $n = |S_1| + |S_2|$ is odd. Then $2\beta_3(G) - 1 = |S_1| + |S_2| \leq 2|S_1| - 1 \leq 2\beta_3(G) - 1$, we deduce that $\beta_3(G) = |S_1| = |S_2| + 1 = \frac{n+1}{2}$ and $m(S_1, S_2) = 3|S_2| = 3|S_1| - 3$. So the subgraph induced by S_2 has at most one edge. We have to examine three possibilities:

Subcase 2.1. S_1 has a vertex x with $d_{S_2}(x) = 0$. Then every vertex v of $S_1 - \{x\}$ satisfies $d_{S_2}(v) = 3$ and S_1 is independent, and so $d_G(x) = 0$, contradicting $\delta(G) \geq 1$.

Subcase 2.2. S_1 has two vertices x and $x'$with $d_{S_2}(x) = 2$ and $d_{S_2}(x') = 1$. Then every vertex v of $S_1 - \{x, x'\}$ satisfies $d_{S_2}(v) = 3$. Let $y, y' \in N_{S_2}(x)$. Then $S' = (S_1 - \{x\}) \cup \{y, y'\}$ is 3-independent with $|S'| = |S_1| + 1$, a contradiction.

Subcase 2.3. S_1 has three vertices x, x', x'' with $d_{S_2}(x) = d_{S_2}(x') = d_{S_2}(x'') = 2$. Then every vertex v of $S_1 - \{x, x', x''\}$ satisfies $d_{S_2}(v) = 3$. Let $y, y' \in N_{S_2}(x)$. If S_1 is independent, then $S' = (S_1 - \{x\}) \cup \{y, y'\}$ is 3-independent with $|S'| = |S_1| + 1$, a contradiction. If S_1 is not independent, then $G[S_1]$ has exactly one edge e. Since $\Delta(G) = 3$, without loss of generality, let $e = xx'$ and $y, y' \in N_{S_2}(x)$, then $S' = (S_1 - \{x\}) \cup \{y, y'\}$ is 3-independent with $|S'| = |S_1| + 1$, a contradiction too. Thus $\beta_\Delta(G) = \frac{n+1}{2}$ is not possible in this case. ■

Now, we give a characterization of extremal graphs attaining the bound in Theorem 6 for $k = \Delta(G)$. Moreover, we improve this upper bound and characterize all graphs attaining the new bound. We recall that $K_4 - e$ is the graph obtained from K_4 by deleting one edge of K_4. Let H be the graph obtained from C_5 by joining three nonconsecutive vertices of C_5 to a new vertex.

Theorem 10 *Let G be a connected graph with maximum degree $\Delta(G) \geq 2$ and $\epsilon \in \{0, 1\}$. Then*

$$\beta_\Delta(G) = 2\beta_{\Delta-1}(G) - \epsilon$$

if and only if G is C_3 and $\epsilon = 0$, or $G \in \{K_4, K_4 - e, H\}$ and $\epsilon = 1$.

Proof. It is clear that $\beta_\Delta(C_3) = 2\beta_{\Delta-1}(C_3)$ and $\beta_\Delta(G) = 2\beta_{\Delta-1}(G) - 1$ when $G \in \{K_4, K_4 - e, H\}$.

For the converse, assume that $\beta_\Delta(G) = 2\beta_{\Delta-1}(G) - \epsilon$. As in the proof of Theorem 6, I is Δ-independent and S_1 is a $(\Delta - 1)$-independent and $(\Delta - 1)$-dominating set of $G[I]$. Now applying Theorem 9 for the subgraph $G[I]$, we obtain that each component of $G[I]$ is P_2, P_3, C_3, C_4, C_5 or C_7 and $\Delta(G) \leq 3$.

Case 1. $\beta_\Delta(G) = 2\beta_{\Delta-1}(G)$.

Then each component of $G[I]$ is P_2 and $\Delta(G) = 2$, or $G[I] = C_4$ and $\Delta(G) = 3$. If $\Delta(G) = 2$, then G is a path or a cycle. If G is a path with $n \geq 3$ or a cycle with $n \neq 3$, then $\beta_2(P_n) = \left\lceil \frac{2n}{3} \right\rceil < 2\left\lceil \frac{n}{2} \right\rceil = 2\beta(P_n)$ and $\beta_2(C_n) = \left\lfloor \frac{2n}{3} \right\rfloor < 2\left\lfloor \frac{n}{2} \right\rfloor = 2\beta(C_n)$. If $\Delta(G) = 3$, then every vertex of $V - I$ is adjacent to at most three vertices of C_4, and we can easily find a 2-independent set S with $|S| = |S_1| + 1$ a contradiction. Thus $\beta_3(G) = 2\beta_2(G)$ is not possible in this case.

Case 2. $\beta_\Delta(G) = 2\beta_{\Delta-1}(G) - 1$.

Then $G[I]$ consists of P_3, C_3, C_5 or C_7 and so $\Delta(G) = 3$. If $G[I]$ is P_3, then every vertex of $V - I$ is adjacent to each vertex of P_3, for otherwise we can find a 2-independent set S with $|S| > |S_1|$. Since $\Delta(G) = 3$, $V - I$ contains exactly one vertex, and so G is $K_4 - e$. If $G[I]$ is C_3, then, by the same argument above, $V - I$ has exactly one vertex which is adjacent to at least two vertices of C_3 and so G is $K_4 - e$ or K_4. If $G[I]$ is C_5, then $V - I$ consists of one vertex which is adjacent to three nonconsecutive vertices of C_5 and so G is the graph H. Finally, if $G[I]$ is C_7, then for every vertex v of $V - I$, we can find a 2-independent set S containing v with $|S| = |S_1| + 1$ contradicting $\beta_2(G) = |S_1|$. Thus $\beta_3(G) = 2\beta_2(G) - 1$ is not possible in this case. ■

Corollary 5 *If T is a tree of order $n \geq 3$, then*

$$\beta_\Delta(T) \leq 2\beta_{\Delta-1}(G) - 2.$$

Now, we give a characterization of extremal bipartite graphs which reach the bound (1) in Theorem 5, when j divides $k - 1$ (i.e.: $\left\lceil \frac{k}{j} \right\rceil = \frac{k+j-1}{j}$).

Proposition 11 *Let G be a bipartite graph of order n and j, k integers with $1 \leq j < k \leq \Delta(G) + 1$. Then*

$$\beta_k(G) = \frac{k + j - 1}{j} \beta_j(G),$$

if and only if G is $\frac{n}{2}K_2$, with $j = 1$ and $k = 2$, or G is $\frac{n}{4}C_4$, with $j = 2$ and $k = 3$.

Proof. Assume that $\beta_k(G) = \frac{k+j-1}{j}\beta_j(G)$. We have $\beta_j(G) \geq \frac{n}{2}$ for bipartite graphs and $\beta_k(G) \leq n$ for any graph G. Thus

$$n \geq \beta_k(G) = \frac{k+j-1}{j}\beta_j(G) \geq \frac{k+j-1}{j}\frac{n}{2} \geq n,$$

so we have equality throughout the previous inequality chain. In particular, $\beta_k(G) = n$, $j = k-1$ and $\beta_j(G) = \frac{n}{2}$. It follows that $k = \Delta(G)+1$ and $j = \Delta(G)$, and so by Theorem 9, G is $\frac{n}{2}K_2$ or $\frac{n}{4}C_4$.

The converse is obvious. ■

As a consequence of Proposition 11, we deduce the following result which provides a sufficient condition in Theorem 6.

Corollary 6 *If G is a bipartite graph of order n and $2 \leq k \leq \Delta(G)+1$ is an integer, then $\beta_k(G) = 2\beta_{k-1}(G)$ if and only if G is $\frac{n}{2}K_2$ and $k = 2$ or G is $\frac{n}{4}C_4$ and $k = 3$.*

From Proposition 11 we deduce that $\beta_k(G) \leq \frac{k+j-1}{j}\beta_j(G) - 1$ for trees of order $n \geq 3$. However, we improve this upper bound for $k \geq 3$. Also we characterize all trees attaining this bound. We need an observation for the equality $\beta_k(G) = n-1$, and a constructive characterization of trees T for which $\beta_j(T) = \frac{jn}{j+1}$ due to Blidia et al. [1].

Observation 12 *Let G be a graph of order n and k a positive integer. Then $\beta_k(G) = n-1$ if and only if G has a vertex w such that every neighbor of w has degree at most k, at least w or one of its neighbors has degree k or more, and every vertex in $V(G) - N[w]$, if any, has degree less than k in G.*

We introduce the following operation.

Operation $\mathcal{O}$: For a positive integer j, let v be any vertex of the star $K_{1,j}$. The tree T_{i+1} is obtained from T_i by joining any vertex of T_i with the vertex v.

We now define the family $\mathcal{T}$ as follows:

$T \in \mathcal{T}$ if and only if $T = K_{1,j}$ or T is obtained from $K_{1,j}$ by a finite sequence of the above operation.

Theorem 13 (Blidia et al. [1]) *Let T be a tree of order n and maximum degree Δ. Then for every integer j with $1 \leq j \leq \Delta(G)$, $\beta_j(G) \geq \dfrac{jn}{j+1}$, with equality if and only if $T \in \mathcal{T}$.*

Theorem 14 *Let T be a tree of order $n \geq 3$ and let k be a positive integer with $k \leq \Delta(G)$. Then*

$$\beta_k(T) \leq \frac{k+j-1}{j}\beta_j(T) - \frac{(k-2)\,n}{j+1} - 1$$

with equality if and only if

(i) $T \in \mathcal{T}$, *and*

(ii) T *has a vertex* w *such that every neighbor of* w *has degree at most* k, *at least* w *or one of its neighbors has degree* k *or more, and every vertex in* $V(T) - N[w]$, *if any, has degree less than* k *in* T.

Proof. We first prove the upper bound. Since $\beta_k(T) \leq n-1$ for $k \leq \Delta(T)$, and $\beta_j(T) \geq \frac{jn}{j+1}$ for trees (see Theorem 13), we deduce that $\beta_k(T) - \frac{k+j-1}{j}\beta_j(T) \leq -(k-2)\frac{n}{j+1} - 1$, and the bound is proved.

If $T \in \mathcal{T}$ and T satisfies Condition (ii), then by Theorem 13 and Observation 12, $\beta_j(T) = \frac{jn}{j+1}$ and $\beta_k(T) = n-1$, respectively. So $\beta_k(T) - \frac{k+j-1}{j}\beta_j(T) = -\frac{(k-2)n}{j+1} - 1$.

Now assume that $\beta_k(T) = \frac{k+j-1}{j}\beta_j(T) - \frac{(k-2)n}{j+1} - 1$. Then we have equality throughout the previous inequality chain. In particular, $\beta_j(T) = \frac{jn}{j+1}$ and $\beta_k(T) = n-1$. From Theorem 13, the first equality implies that $T \in \mathcal{T}$, and by Observation 12, the second equality implies that T satisfies Condition (ii), and the proof is complete. ■

As a consequence of Theorem 14, we deduce the following result which improves bounds of Corollary 3 and Theorem 6 for trees.

Corollary 7 *If* T *is a tree of order* $n \geq 3$ *and* k *is an integer with* $2 \leq k \leq \Delta(G)$, *then*

(i) $\beta_k(T) \leq k\beta(T) - \frac{(k-2)n}{2} - 1$.

(ii) $\beta_k(T) \leq 2\beta_{k-1}(T) - \frac{(k-2)n}{k} - 1$.

From Theorem 14 we deduce a descriptive characterization of the class of trees achieving the bound of Theorem 14 for $k = 2$ and $j = 1$.

Corollary 8 *If* T *is a tree, then* $\beta_2(T) = 2\beta(T) - 1$ *if and only if* $T = K_1$, *or* T *is a corona of a star.*

Finally, we characterize the class of trees T achieving the bound of Theorem 7. To this end, we give some more definitions. For a positive integer p, a tree obtained from a star $K_{1,t}, t \geq 1$ such that each of p edges is subdivided once is denoted by $S_p(K_{1,t})$, a vertex of degree t will be called the *center* vertex. If $p = 0$, then $S_p(K_{1,t})$ is the *star* $K_{1,t}$. If $p = t$, then $S_p(K_{1,t})$ is the *healthy spider*. If $1 \leq p \leq t-1$, then $S_p(K_{1,t})$ is a *wounded spider*. For P_2, we will consider both vertices to be center vertices, and in the case of P_4, we will consider both endvertices as leaves and both interior vertices as center vertices.

Let $\mathcal{F}'$ be a family of trees obtained from two healthy spiders by joining their centre vertices, $\mathcal{F}''$ ($\mathcal{F}'''$, respectively) be a family of trees obtained from two healthy spiders (one healthy spider and K_1, respectively) and one edge xy by joining x to the

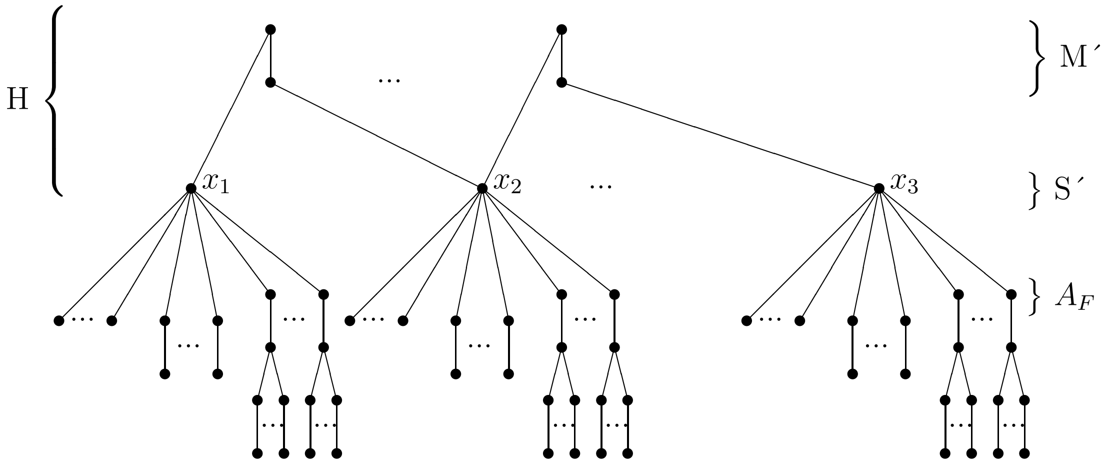

Figure 1: Example of a graph in $\mathcal{F}$

centre of the first healthy spider and y to the centre of the second healthy spider (by joining x to the centre of the healthy spider and y to the centre of K_1, respectively).

Let F_i be a tree obtained from the wounded spider $S_{p_i}(K_{1,t_i})$ and $q_i \leq p_i$ healthy spiders by identifying q_i leaves of the wounded spider $S_{p_i}(K_{1,t_i})$ which are nonadjacent to its center with the centers of these q_i healthy spiders. Let H be a tree obtained by connecting an induced matching M' of size $h-1$ with an independent set $S' = \{x_1, x_2, ..., x_h\}$ such that every endvertex of edges of M' has exactly one neighbor in S'. Now, define $\mathcal{F}$ as the family of trees obtained from F_1, F_2, ..., F_h and H by identifying a vertex x_i of S' with the center of F_i for $i \in \{1, 2, ..., h\}$. For a tree $F \in \mathcal{F}$, let L_F and S_F be the sets of leaves and support vertices of F, respectively, and let A_F be the set of vertices of F adjacent to $\sum_{i=1}^{h} q_i$ selected leaves in $\bigcup_{i=1}^{h} S_{p_i}(K_{1,t_i})$ (see Figure 1).

Now, we are ready to characterize trees T such that $\beta_2(T) = \beta(T) + \gamma(T) - 1$.

Theorem 15 *Let T be a tree of order $n \geq 3$. Then*

$$\beta_2(T) = \beta(T) + \gamma(T) - 1$$

if and only if T is a star, a healthy spider, a wounded spider or $T \in \mathcal{F}' \cup \mathcal{F}'' \cup \mathcal{F}''' \cup \mathcal{F}$.

Proof. It is a simple mater to check that if T is a star, a healthy spider, a wounded spider, or $T \in \mathcal{F}' \cup \mathcal{F}'' \cup \mathcal{F}'''$, then $\beta_2(T) = \beta(T) + \gamma(T) - 1$. If $T \in \mathcal{F}$, then it is easy to see that $\gamma(T) = |S_F|$, $\beta(T) = |L_F \cup A_F| + |M'| = |L_F| + |A_F| + h - 1$ and

$$\beta_2(T) = |(S_F - S') \cup L_F \cup A_F| + 2\,|M'| = |S_F| + |L_F| + |A_F| + h - 2.$$

Hence, $\beta_2(T) = \beta(T) + \gamma(T) - 1$.

Conversely, assume that T is a tree of order $n \geq 3$ with $\beta_2(T) = \beta(T) + \gamma(T) - 1$. We follow the notation used in the proof of Theorem 7. If $G[I]$ is independent, then $\beta_2(T) = \beta(T)$ and so $\gamma(T) = 1$. Therefore T is a star. If $G[I]$ is not independent, then $\gamma(T) \geq |A| = |M|$, since T is a tree, we have to distinguish two cases :

Case 1. $\gamma(T) = |M| = |A|$.

Then $S - A' = \emptyset$ and so $M_1 = \emptyset$, since T is a tree. Therefore, $\beta_2(T) = 2|A| = 2\gamma(T) = \beta(T) + \gamma(T) - 1$, which means that $\beta(T) = |A| + 1$. Since T is connected with $n \geq 3$, we must have $|V - I| \in \{1, 2\}$, for otherwise we have $\beta(T) > |A| + 1$. If $|V - I| = 1$, let $\{w\} = V - I$, then w is adjacent to exactly one endvertex of each edge of M, since T is a tree. Therefore, T is a healthy spider of center w. If $|V - I| = 2$, let $\{u, v\} = V - I$, then we have to examine possibilities for T depending on whether the edge uv exists or not. If $uv \in E(T)$, then u is adjacent to exactly one endvertex of each edge of a matching $M_u \subset M$ with $M_u \neq \emptyset$, and v is adjacent to exactly one endvertex of each edge of $M_v = M - M_u$ with $M_v \neq \emptyset$, since T is a tree and $M_1 = \emptyset$. Therefore, $T \in \mathcal{F}'$. If $uv \notin E(T)$, then u and v have exactly one common neighbor in M, or u is adjacent to an endvertex of an edge of M and v is adjacent to the other endvertex, otherwise we have a cycle or T is not connected. Since $\beta(T) = |A| + 1$, the first situation cannot occur. The second situation leads to the tree $T \in \mathcal{F}'' \cup \mathcal{F}'''$.

Case 2. $\gamma(T) \geq |M| + 1 = |A| + 1$.

Then $\beta_2(T) = |S| + |A| = \beta(T) + \gamma(T) - 1 \geq |S| + |A|$ and so $\gamma(T) = |A| + 1$ and $\beta(T) = |S|$. Thus $S - A' \neq \emptyset$ and $|S - A'| \geq |V - I|$, since $\beta(T) \geq \frac{n}{2}$ for trees. Without loss of generality we can suppose that $A - A_1 \subset D$ and so $|D \cap (V - I)| = |M_1| + 1$ and the vertices of $S - A'$ are dominated by $D \cap (V - I)$. Since $\gamma(T) = |A| + 1$, $|V - I| \geq 1$. Each vertex of $V - I$ is adjacent to exactly one vertex of $S - A'$, since $D \cap (V - I)$ dominates $(S - A') \cup A_1 \cup A'_1$, for otherwise we have a cycle or $\beta(T) > |S|$. Also, with the same argument, the subset $V - I$ is independent and for any two vertices x and y of $V - I$, x is adjacent to endvertices of edges of $M_x \subseteq M$ and y is adjacent to endvertices of edges of $M_y \subseteq M - M_x$ and the vertices of $(V - I) - D$ are dominated by the vertices of $M - M_1$. Thus $T \in \mathcal{F}$. Note that if $|(V - I) \cap D| = 1$, then T is the tree F_1 and if $|V - I| = 1$, then T which is the tree F_1 is a wounded spider. ■

References

[1] M. Blidia, M. Chellali, O. Favaron and N. Meddah, On k-independence in graphs with emphasis on trees, *Discrete Math.* **307** (2007), 2209–2216.

[2] M. Blidia, M. Chellali, O. Favaron and N. Meddah, Maximal k-independent sets in graphs, *Discuss. Math. Graph Theory* **28** (2008), 151–163.

[3] Y. Caro and A. Hansberg. New approach to the k-independence number of a graph, arXiv: 1208. 4734v1 [math.CO]. 23 Aug 2012.

[4] M. Chellali, O. Favaron, A. Hansberg and L. Volkmann, k-Domination and k-Independence in Graphs: A Survey, *Graphs Combin.* **28** (2012), 1–55.

[5] O. Favaron, On a conjecture of Fink and Jacobson concerning k-domination and k-dependene, *J. Combin. Theory Ser. B* **39** (1) (1985), 101–102.

[6] O. Favaron, Graduate course, University of Blida, Algeria (2005).

[7] J. F. Fink and M. S. Jacobson, n-domination in graphs, in: *Graph Theory with Applications to Algorithms and Computer*, John Wiley and Sons, New York (1985), 283–300.

[8] J. F. Fink and M. S. Jacobson, n-domination, n-dependence and forbidden subgraphs, in: *Graph Theory with Applications to Algorithms and Computer*, John Wiley and Sons, New York (1985), 301–311.

[9] J. F. Fink, M. S. Jacobson, L. F. Kinch and J. Roberts, On graphs having domination number half their order, *Period. Math. Hungar.* **16** (1985), 287–293.

[10] G. Hopkins and W. Staton, Vertex partitions and k-small subsets of graphs, *Ars Combin.* **22** (1986), 19–24.

[11] M. S. Jacobson, K. Peters and D. F. Rall, On n-irredundance and n-domination, *Ars Combin.* **29** B (1990), 151–160.

[12] O. Ore, *Theory of Graphs*, Amer. Math. Soc. Colloq. Publ. **38** (1962).

[13] C. Payan and N. H. Xuong, Domination-balanced graphs, *J. Graph Theory* **6** (1982), 23–32.

[14] H. B.Walikar, B. D. Acharya and E. Sampathkumar, Recent developments in the theory of domination in graphs, in: *MRI Lec. Notes in Math.*, Mahta Research Institute, Allahabad, vol. **1** (1979).

(Received 17 July 2013; revised 30 Mar 2014)