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Maximum packings of K_n with k-stars

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Abstract

Given graphs G and H, we define an *H-packing of G* to be a partition of the edges of G into some copies of H along with a set of edges L, called the *leave*. An H-packing is called *maximum* when $|L|$ is minimum, or equivalently, when the H-packing contains as many copies of H as possible. A *k-star*, denoted S_k, is defined to be the complete bipartite graph $K_{1,k}$. In this paper we characterize the number of k-stars in a maximum S_k-packing of K_n, as well as investigate the configuration of the leave.

1 Introduction and background

Let $\mathbb{Z}^+$ denote the set $\{1, 2, \dots\}$. Given graphs G and H, we define an *H-packing of G* to be a partition of the edges of G into some copies of H along with a set of edges L, called the *leave*. An H-packing is called *maximum* when $|L|$ is minimum, or equivalently, when the H-packing contains as many copies of H as possible. A *k-star*, denoted S_k, is defined to be the complete bipartite graph $K_{1,k}$. In a k-star, the vertex of maximum degree is referred to as the *center* of the star.

An *H-decomposition of G* is an H-packing of G in which $L = \emptyset$. Necessary and sufficient conditions for an S_k-decomposition of K_n were given by Yamamoto et al. [3], and independently by Tarsi [2]. The following theorem generalizes these results to give necessary and sufficient conditions for decomposing K_n into stars of (possibly) different sizes. This result will be very useful for constructing S_k-packings of K_n.

Theorem 1.1. *(Lin and Shyu [1]) Let $m_1 \geq m_2 \geq \cdots \geq m_\ell$ be nonnegative integers. Necessary and sufficient conditions for K_n to be decomposed into stars $S_{m_1}, S_{m_2}, \ldots, S_{m_\ell}$ are*

(i) $\displaystyle\sum_{i=1}^{\ell} m_i = \binom{n}{2}$ *and*

(ii) $\displaystyle\sum_{i=1}^{p} m_i \leq \sum_{i=1}^{p}(n-i)$ *for* $p = 1, 2, \ldots, n-1$.

2 A quick note on orientations

An *orientation* of a graph G is defined to be an assignment of a direction to each one of its edges. For a vertex, v, in an oriented graph, G, the *outdegree of* v, denoted $d^+(v)$, is defined to be the number of edges incident with v which are directed away from v. An orientation of K_n is called a *tournament*; and furthermore, a *regular tournament* is a tournament for which every pair of vertices $u, v \in V(K_n)$ have the property that $d^+(u) = d^+(v)$. The following is a folklore theorem:

Theorem 2.1. *For every odd positive integer, n, there is a regular tournament on n vertices.*

An orientation is a powerful tool when it comes to constructing S_k-packings. This is due to the fact that a k-star is equivalent to an orientation of $K_{1,k}$ in which every edge is directed away from the center. So constructing the k-stars of an S_k-packing of a graph G is equivalent to orienting the edges of G and reducing the outdegrees of the vertices modulo k. What remains of the outdegrees after reduction modulo k corresponds to the leave of the S_k-packing.

3 Results

We first note that when constructing S_k-packings of K_n, the task becomes trivial when $n \leq k$, since $|V(S_k)| = k + 1$. Thus, we have the following obvious fact:

Proposition 3.1. *Let $n, k \in \mathbb{Z}^+$ with $n \leq k$. Then there are 0 stars in a maximum packing of K_n with k-stars. Moreover, the leave graph must be K_n.*

When n is large enough as compared to k we are able to employ the result of Lin and Shyu (Theorem 1.1) to obtain a maximum S_k-packing of K_n.

Theorem 3.2. *Let $n, k \in \mathbb{Z}^+$ where $n \geq 2k$. Then there are $\left\lfloor \frac{\binom{n}{2}}{k} \right\rfloor$ k-stars in a maximum S_k-packing of K_n. Moreover, it is possible to have the leave graph be a star of size less than k.*

Proof. We will show that conditions (i) and (ii) hold from Theorem 1.1 with $\ell = \left\lfloor \frac{\binom{n}{2}}{k} \right\rfloor + 1$, and $m_1 = m_2 = \cdots = m_{\ell-1} = k$ and $m_\ell = \binom{n}{2} - k(\ell-1)$. To verify (i) we see that

$$\sum_{i=1}^{\ell} m_i = \sum_{i=1}^{\ell-1} m_i + m_\ell = k(\ell-1) + \binom{n}{2} - k(\ell-1) = \binom{n}{2}$$

To show that (ii) holds let $p \in \{1, \ldots, n-1\}$. First, note that since $n \geq 2k$ we have

$$\ell = \left\lfloor \frac{\binom{n}{2}}{k} \right\rfloor + 1 \geq n.$$

In particular, we have that $\ell \geq p+1$. Upon examining condition (ii), we see that for any fixed p it is equivalent to the following inequality:

$$\frac{p(p+1)}{2} \leq pn - pk.$$

We have

$$\frac{p(p+1)}{2} \leq \frac{pn}{2} \leq p(n-k) = pn - pk$$

where the first inequality holds because $p+1 \leq n$ and the second inequality holds because $n/2 \leq n-k$. Thus, (ii) holds and we conclude that K_n can be decomposed into $\ell - 1$ k-stars and one star of size smaller than k. □

Note that Theorem 3.2 supplies a maximum packing in which the leave graph is a star, but it doesn't guarantee that this is the only possibility. In the following theorem, however, we characterize the leave graph.

Theorem 3.3. *Let $n, k \in \mathbb{Z}^+$ with $k < n < 2k$. There are $2n - 2k - 1$ stars in a maximum S_k-packing of K_n. Moreover, the leave graph must be K_{2k-n+1}.*

Proof. The crucial observation is that when $n < 2k$, we can have at most one star centered at any given vertex. Let $\alpha \in V(K_n)$. Having two stars centered at α would require $|V(K_n) \setminus \{\alpha\}| \geq 2k$, which is not true. Thus, there can be at most one star centered at vertex α.

Now, let b be the number of k-stars in a maximum S_k-packing of K_n. For notational purposes, we partition the vertices of K_n into two sets B and N, where B is the set of vertices that are the center of a star, and N is the set of vertices that are not the center of any star. Let $v \in B$, then there are k edges used in the star centered at v. At most $n - b$ of these edges can have one endpoint at v and the other

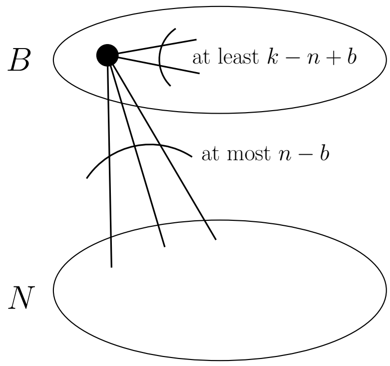

Figure 1: Bounds on the different types of edges.

endpoint in N, which means that at least $k-(n-b)=k-n+b$ of these edges have both endpoints in B. (See Figure 1).

This argument holds for every $v \in B$, so we get the inequality $b(k-n+b) \leq \binom{b}{2}$ which is equivalent to $b \leq 2n-2k-1$.

In a maximum packing we must have b as large as possible. Therefore, if we can find an S_k-packing of K_n with $b=2n-2k-1$, then we know it must be maximum. We will devise such a construction by finding an orientation of the edges within B as well as the edges with one endpoint in B and the other endpoint in N. The edges with both endpoints in N cannot be used in any star, as there are no stars centered in N, so we need not orient them.

The construction: First, place a regular tournament on B, which is guaranteed by Theorem 2.1. For each $v \in B$ we have $d^+(v) = \dfrac{b-1}{2} = n-k-1$. Notice that $k-n+b = k-n+(2n-2k-1) = n-k-1$, so for any given $v \in B$ we have used the least possible number of edges in B for the star centered at v. Therefore, for this star we must use all $n-b$ edges with one endpoint at v and the other endpoint in N. So the next step in the construction is to orient all of the edges with one endpoint in B and the other endpoint in N so that they are directed towards the endpoint in N.

We have that $n-b = 2k-n+1$, so for each $v \in B$, $d^+(v) = (n-k-1)+(n-b) = n-k-1+2k-n+1 = k$. Thus, we have exactly one star centered at each vertex in B. We have oriented all edges in K_n except those with both endpoints in N, and hence the leave is K_{2k-n+1}. □

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