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Irregular total labeling of disjoint union of prisms and cycles

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Abstract

We investigate two modifications of the well-known irregularity strength of graphs, namely, a total edge irregularity strength and a total vertex irregularity strength. Recently the bounds and precise values for some families of graphs concerning these parameters have been determined. In this paper, we determine the exact value of the total edge (vertex) irregularity strength for the disjoint union of prisms and the total edge (vertex) irregularity strength for the disjoint union of cycles.

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1 Introduction

Let us consider a simple (without loops and multiple edges) undirected graph $G = (V, E)$ with vertex set V and edge set E. For a graph G we define a labeling $\phi : V \cup E \to \{1, 2, \dots, k\}$ to be a *total k-labeling*. A total k-labeling ϕ is defined to be an *edge irregular total k-labeling* of the graph G if for every two different edges xy and $x'y'$ their weights $\phi(x)+\phi(xy)+\phi(y)$ and $\phi(x')+\phi(x'y')+\phi(y')$ are distinct. Similarly, a total k-labeling ϕ is defined to be a *vertex irregular total k-labeling* of G if for every two distinct vertices x and y of G their weights $\mathrm{wt}(x)$ and $\mathrm{wt}(y)$ are distinct. Here, the weight of a vertex x in G is the sum of the label of x and the labels of all edges incident with the vertex x.

The minimum k for which the graph G has an edge irregular total k-labeling is called the *total edge irregularity strength* of G, $\mathrm{tes}(G)$. Analogously, the minimum k for which the graph G has a vertex irregular total k-labeling is called the *total vertex irregularity strength* of G, $\mathrm{tvs}(G)$.

The notion of total edge (and vertex) irregularity strength was defined by Bača, Jendroľ, Miller and Ryan in [6]. They may be taken as an extension of the irregularity strength of a graph defined by Chartrand, Jacobson, Lehel, Oellermann, Ruiz and Saba in [9]; see also [7, 10, 11, 14, 16, 19].

Simple lower bounds for $\mathrm{tes}(G)$ and $\mathrm{tvs}(G)$ of a (p, q)-graph G in terms of maximum degree $\triangle(G)$ and minimum degree $\delta(G)$, determined in [6], are given by the following theorems.

Theorem 1. [6] *Let G be a (p, q)-graph with maximum degree $\Delta = \Delta(G)$. Then*

$$\mathrm{tes}(G) \geq \max\left\{ \left\lceil \frac{q+2}{3} \right\rceil, \left\lceil \frac{\Delta+1}{2} \right\rceil \right\}.$$

Theorem 2. [6] *Let G be a (p, q)-graph with minimum degree $\delta = \delta(G)$ and maximum degree $\Delta = \Delta(G)$. Then*

$$\left\lceil \frac{p+\delta}{\Delta+1} \right\rceil \leq \mathrm{tvs}(G) \leq p + \Delta - 2\delta + 1. \tag{1}$$

Thus, for an r-regular (p, q)-graph, we obtain

$$\left\lceil \frac{p+r}{r+1} \right\rceil \leq \mathrm{tvs}(G) \leq p - r + 1. \tag{2}$$

For a (p, q)-graph with no component of order less than or equal to 2, in [6] the following is proved:

$$\mathrm{tvs}(G) \leq p - 1 - \left\lceil \frac{p-2}{\Delta+1} \right\rceil.$$

These results were then improved by Przybylo in [18] for sparse graphs and for graphs with large minimum degree. In the latter case the bounds

$$\mathrm{tvs}(G) < 32\,\frac{p}{\delta} + 8$$

in general and

$$\mathrm{tvs}(G) < 8\,\frac{p}{r} + 3$$

for r-regular (p,q)-graphs were proved to hold.

The best upper bound on $\mathrm{tvs}(G)$ is given in [4] in the form

$$\mathrm{tvs}(G) \le 3\left\lceil \frac{p}{\delta} \right\rceil + 1.$$

The exact values of $\mathrm{tes}(G)$ and $\mathrm{tvs}(G)$ are known only for a few families of graphs. For recent results we refer the reader to [1, 3, 2, 5, 8, 13, 15, 17, 20, 21].

In this paper, we deal with irregular total labeling for disjoint union of prisms and cycles.

2 Irregular total labelings for disjoint union of prisms

A prism D_n, $n \ge 3$, is obtained by the Cartesian product of C_n and K_2, i.e., $D_n = C_n \square K_2$. Ivančo and Jendroľ in [12] proved that $\mathrm{tes}(G \square K_2) = q + 1$ for every (p,q)-graph, where $p - 1 \le q \le p$.

In this section, we state the exact value of the total edge irregularity strength and total vertex irregularity strength of the disjoint union of prisms $\bigcup\limits_{j=1}^{m} D_{n_j}$.

The vertex set and the edge set of $\bigcup\limits_{j=1}^{m} D_{n_j}$ are defined as follows:

$$V\left(\bigcup_{j=1}^{m} D_{n_j}\right) = \bigcup_{j=1}^{m}\bigcup_{i=1}^{n_j}\{a_i^j, b_i^j\},$$

$$E\left(\bigcup_{j=1}^{m} D_{n_j}\right) = \bigcup_{j=1}^{m}\bigcup_{i=1}^{n_j}\{a_i^j a_{i+1}^j,\ b_i^j b_{i+1}^j,\ a_i^j b_i^j\},$$

where the subscript $n_j + 1$ is replaced by 1, for $1 \le j \le m$.

In the next theorem, we prove that the total edge irregularity strength of the disjoint union of prisms is equal to the lower bound from Theorem 1.

Theorem 3. *Let $m \ge 2$ and $n_j \ge 3$ for every $1 \le j \le m$. Then*

$$\mathrm{tes}\left(\bigcup_{j=1}^{m} D_{n_j}\right) = \sum_{j=1}^{m} n_j + 1.$$

Proof. Let $3 \le n_1 \le n_2 \le \cdots \le n_m$ and $k = \sum\limits_{j=1}^{m} n_j + 1$. Since $|E(\bigcup\limits_{j=1}^{m} D_{n_j})| = 3\sum\limits_{j=1}^{m} n_j$, by Theorem 1, $\mathrm{tes}(\bigcup\limits_{j=1}^{m} D_{n_j}) \ge \sum\limits_{j=1}^{m} n_j + 1$. To prove the equality, we describe an optimal edge irregular total k-labeling $\phi_1 : V(\bigcup_{j=1}^{m} D_{n_j}) \cup E(\bigcup_{j=1}^{m} D_{n_j}) \rightarrow \{1, 2, \ldots, k\}$ as follows: for $1 \leq i \leq n_j, 1 \leq j \leq m$

$$\phi_1(a_i^j) = 1, \quad \phi_1(b_i^j) = k,$$

$$\phi_1(a_i^j b_i^j) = \phi_1(a_i^j a_{i+1}^j) = \phi_1(b_i^j b_{i+1}^j) = \sum_{s=1}^{j-1} n_s + i.$$

Since

$$\begin{aligned}
\mathrm{wt}(a_i^j a_{i+1}^j) &= \sum_{s=1}^{j-1} n_s + i + 2, \\
\mathrm{wt}(a_i^j b_i^j) &= \sum_{s=1}^{m} n_s + \sum_{s=1}^{j-1} n_s + i + 2, \\
\mathrm{wt}(b_i^j b_{i+1}^j) &= 2\sum_{s=1}^{m} n_s + \sum_{s=1}^{j-1} n_s + i + 2,
\end{aligned}$$

for $1 \leq i \leq n_j$, $1 \leq j \leq m$, the weights of the edges under the function ϕ_1 successively attain values $3, 4, \ldots, 3\sum_{j=1}^{m} n_j + 2$.

It is not difficult to see that all vertex and edge labels are at most k and the edge weights are distinct for all pairs of distinct edges. Therefore, the total labeling ϕ_1 is an optimal edge irregular total k-labeling. This completes the proof. ■

Next we determine the exact value of the total vertex irregularity strength of the disjoint union of prisms.

Theorem 4. *Let $m \geq 2$ and $n_j \geq 3$ for every $1 \leq j \leq m$. Then*

$$\mathrm{tvs}\left(\bigcup_{j=1}^{m} D_{n_j}\right) = \left\lceil \frac{2\sum_{j=1}^{m} n_j + 3}{4} \right\rceil.$$

Proof. Suppose that $k = \left\lceil \frac{2\sum_{j=1}^{m} n_j + 3}{4} \right\rceil$ and $3 \leq n_1 \leq n_2 \leq \cdots \leq n_m$. From (2) it follows that $\mathrm{tvs}\left(\bigcup_{j=1}^{m} D_{n_j}\right) \geq k$. To prove the equality, it is sufficient to show the existence of a vertex irregular total k-labeling for disjoint union of prisms.

We define a labeling $\phi_2 : V(\bigcup_{j=1}^{m} D_{n_j}) \cup E(\bigcup_{j=1}^{m} D_{n_j}) \rightarrow \{1, 2, \ldots, k\}$ in the following way: for $1 \leq i \leq n_j, 1 \leq j \leq m$

$$\phi_2(a_i^j) = \phi_2(b_i^j) = \max\left\{1, i - k + 1 + \sum_{s=1}^{j-1} n_s\right\},$$

$$\phi_2(a_i^j a_{i+1}^j) = 1,\ \phi_2(b_i^j b_{i+1}^j) = k,\ \phi_2(a_i^j b_i^j) = \min\left\{\sum_{s=1}^{j-1} n_s + i, k\right\}.$$

For $1 \le i \le n_j$ and $1 \le j \le m$, the weights of vertices of the disjoint union of prisms are as follows:

$$\begin{aligned} \mathrm{wt}(a_i^j) &= \sum_{s=1}^{j-1} n_s + i + 3, \\ \mathrm{wt}(b_i^j) &= \sum_{s=1}^{j-1} n_s + i + 1 + 2k. \end{aligned}$$

Thus, the weights of vertices a_i^j, $1 \le i \le n_j$, $1 \le j \le m$, successively attain values $4, 5, \ldots, \sum_{j=1}^{m} n_j + 3$ and the weights of vertices b_i^j, $1 \le i \le n_j$, $1 \le j \le m$, receive distinct values from $\sum_{j=1}^{m} n_j + 4$ up to $\sum_{j=1}^{m} n_j + 2k + 1$. Clearly, the labeling ϕ_2 is an optimal vertex irregular total k-labeling and we have arrived at the desired result. ■

3 Irregular total labelings for disjoint union of cycles

For a cycle C_n with $n \ge 3$ edges, it was proved in [6] that $\mathrm{tes}(C_n) = \mathrm{tvs}(C_n) = \lceil \frac{n+2}{3} \rceil$. In this section, we determine the exact value of the total edge (vertex) irregularity strength of the disjoint union of cycles denoted by $\bigcup_{j=1}^{m} C_{n_j}$.

The disconnected graph $\bigcup_{j=1}^{m} C_{n_j}$ consists of the vertex set and edge set as follows:

$$V\left(\bigcup_{j=1}^{m} C_{n_j}\right) = \bigcup_{j=1}^{m} \bigcup_{i=1}^{n_j} \{v_i^j\},$$

$$E\left(\bigcup_{j=1}^{m} C_{n_j}\right) = \bigcup_{j=1}^{m} \bigcup_{i=1}^{n_j} \{v_i^j v_{i+1}^j\},$$

where the subscript $n_j + 1$ is replaced by 1, for $1 \le j \le m$.

Theorem 5. *Let $m \ge 2$ and $n_j \ge 5$ for every $1 \le j \le m$. Then*

$$\mathrm{tes}\left(\bigcup_{j=1}^{m} C_{n_j}\right) = \mathrm{tvs}\left(\bigcup_{j=1}^{m} C_{n_j}\right) = \left\lceil \frac{\sum_{j=1}^{m} n_j + 2}{3} \right\rceil.$$

Proof. First, let us consider the total edge irregularity strength. Suppose that $m \geq 2$, $5 \leq n_1 \leq n_2 \leq \cdots \leq n_m$ and $k = \left\lceil \frac{\sum_{j=1}^{m} n_j + 2}{3} \right\rceil$. According to Theorem 1 it is sufficient to prove the existence of an edge irregular total k-labeling for the graph $\bigcup_{j=1}^{m} C_{n_j}$.

We define a labeling $\phi_3 : V(\bigcup_{j=1}^{m} C_{n_j}) \cup E(\bigcup_{j=1}^{m} C_{n_j}) \to \{1, 2, \ldots, k\}$ as follows: for $1 \leq j \leq m$

$$\phi_3(v_i^j) = \begin{cases} 1, & \text{if } 1 \leq i \leq \lceil \frac{n_j}{3} \rceil \\ \lfloor \frac{k}{2} \rfloor, & \text{if } \lceil \frac{n_j}{3} \rceil + 1 \leq i \leq \lfloor \frac{2n_j}{3} \rfloor \\ k, & \text{if } \lfloor \frac{2n_j}{3} \rfloor + 1 \leq i \leq n_j \end{cases}$$

$$\phi_3(v_i^j v_{i+1}^j) = \begin{cases} \sum_{s=2}^{j} (\lceil \frac{n_{s-1}}{3} \rceil - 1) + i, & \text{if } 1 \leq i \leq \lceil \frac{n_j}{3} \rceil - 1 \\ \sum_{r=1}^{m} \lceil \frac{n_r}{3} \rceil - m - \lfloor \frac{k}{2} \rfloor + j + 1, & \text{if } i = \lceil \frac{n_j}{3} \rceil \\ \sum_{r=1}^{m} \lceil \frac{n_r}{3} \rceil + 2(1 - \lfloor \frac{k}{2} \rfloor) - \lceil \frac{n_j}{3} \rceil + i + & \\ \sum_{s=2}^{j} (n_{s-1} - 2\lceil \frac{n_{s-1}}{3} \rceil - 1), & \text{if } \lceil \frac{n_j}{3} \rceil + 1 \leq i \leq \lfloor \frac{2n_j}{3} \rfloor - 1 \\ \sum_{r=1}^{m} \lfloor \frac{2n_r}{3} \rfloor + 2 - \lfloor \frac{3k}{2} \rfloor + j, & \text{if } i = \lfloor \frac{2n_j}{3} \rfloor \\ \sum_{r=1}^{m} \lfloor \frac{2n_r}{3} \rfloor + \sum_{s=2}^{j} (\lceil \frac{n_{s-1}}{3} \rceil - 1) + 2 + & \\ m + i - \lfloor \frac{2n_j}{3} \rfloor - 2k, & \text{if } \lfloor \frac{2n_j}{3} \rfloor + 1 \leq i \leq n_j - 1 \\ \sum_{r=1}^{m} \lfloor \frac{2n_r}{3} \rfloor - m - k + j + 1, & \text{if } i = n_j. \end{cases}$$

One can check that all vertex and edge labels are at most k. Moreover, under the labeling ϕ_3 the weights of the edges are as follows.

$$\mathrm{wt}(v_i^j v_{i+1}^j) = \begin{cases} i+2+\sum\limits_{s=2}^{j}(\lceil\frac{n_{s-1}}{3}\rceil-1), & \text{for } 1\le i\le\lceil\frac{n_j}{3}\rceil-1\\ \sum\limits_{r=1}^{m}\lceil\frac{n_r}{3}\rceil-m+2+j, & \text{for } i=\lceil\frac{n_j}{3}\rceil\\ \sum\limits_{r=1}^{m}\lceil\frac{n_r}{3}\rceil+i-\lceil\frac{n_j}{3}\rceil+2+ & \\ \sum\limits_{s=2}^{j}(n_{s-1}-2\lceil\frac{n_{s-1}}{3}\rceil-1), & \text{for } \lceil\frac{n_j}{3}\rceil+1\le i\le\lfloor\frac{2n_j}{3}\rfloor-1\\ \sum\limits_{r=1}^{m}\lfloor\frac{2n_r}{3}\rfloor+2+j, & \text{for } i=\lfloor\frac{2n_j}{3}\rfloor\\ \sum\limits_{r=1}^{m}\lfloor\frac{2n_r}{3}\rfloor+\sum\limits_{s=2}^{j}(\lceil\frac{n_{s-1}}{3}\rceil-1)+ & \\ m+2+i-\lfloor\frac{2n_j}{3}\rfloor, & \text{for } \lfloor\frac{2n_j}{3}\rfloor+1\le i\le n_j-1\\ \sum\limits_{r=1}^{m}\lfloor\frac{2n_r}{3}\rfloor-m+j+2, & \text{for } i=n_j. \end{cases}$$

It is easy to verify that the edge-weights are distinct for all pairs of distinct edges and constitute the progression of consecutive integers $3,4,\ldots,\sum\limits_{j=1}^{m} n_j+2$. Thus the labeling ϕ_3 is the desired edge irregular total k-labeling.

Now we orient the cycles C_{n_j}, $1\le j\le m$, such that $\overrightarrow{v_i^j v_{i+1}^j}$ is the outgoing arc from the vertex v_i^j, for every $1\le i\le n_j$ and $1\le j\le m$.

Let us define a labeling $\phi_4 : V(\bigcup\limits_{j=1}^{m} C_{n_j})\cup E(\bigcup\limits_{j=1}^{m} C_{n_j})\to\{1,2,\ldots,k\}$ in the following way: for $1\le j\le m$

$$\begin{aligned}
\phi_4(\overrightarrow{v_i^j v_{i+1}^j}) &= \phi_3(v_i^j) \text{ for } 1\le i\le n_j-1,\\
\phi_4(\overrightarrow{v_{n_j}^j v_1^j}) &= \phi_3(v_{n_j}^j),\\
\phi_4(v_{i+1}^j) &= \phi_3(\overrightarrow{v_i^j v_{i+1}^j}) \text{ for } 1\le i\le n_j-1,\\
\phi_4(v_1^j) &= \phi_3(\overrightarrow{v_{n_j}^j v_1^j}).
\end{aligned}$$

One can easily see that the labeling ϕ_4 is the desired vertex irregular total k-labeling. This concludes the proof. ■

4 Conclusion

In the foregoing sections we studied the existence of the edge (vertex) irregular total k-labeling for the disjoint union of prisms and disjoint union of cycles. We determined the exact value of the total edge (vertex) irregularity strength for the disjoint union of prisms $\bigcup\limits_{j=1}^{m} D_{n_j}$ for $m\ge 2$ and arbitrary $n_j\ge 3$.

Also we determined the exact value of the total edge (vertex) irregularity strength for the disjoint union of cycles $\bigcup_{j=1}^{m} C_{n_j}$ for $m \geq 2$ and arbitrary $n_j \geq 5$. We are not able to describe any optimal edge (vertex) irregular total $\left\lceil \frac{\sum_{j=1}^{m} n_j + 2}{3} \right\rceil$-labeling for the disjoint union of cycles for $m \geq 2$ and arbitrary $n_j \geq 3$. However, we suggest the following.

Conjecture 1. $\mathrm{tes}\left(\bigcup_{j=1}^{m} C_{n_j}\right) = \mathrm{tvs}\left(\bigcup_{j=1}^{m} C_{n_j}\right) = \left\lceil \frac{\sum_{j=1}^{m} n_j + 2}{3} \right\rceil$ *for any* $m \geq 2$ *and* $n_j \geq 3$.

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