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Independent set and vertex covering in a proper monograph determined through a signature

L. L. Fontanil*   R. G. Panopio

Institute of Mathematical Sciences and Physics
College of Arts and Sciences
University of the Philippines Los Baños
College Laguna, 4031
Philippines
lauro_06@yahoo.com   rolygpanopio@yahoo.com

Abstract

A graph G is a *proper monograph* if its vertices can be labeled bijectively by a set S of positive real numbers, called a *signature* of G, such that two vertices are adjacent if and only if the absolute difference of the corresponding labels is also in S. In this paper, we adapt some concepts from directed graphs, sum graphs and mod difference digraphs to proper monographs to determine their independent sets and vertex coverings by means of their signatures.

1 Introduction and definitions

The concept of *autograph* was introduced by Bloom et al. [1] in 1979. A graph G is an autograph if its vertices can be labeled bijectively by a multiset S of real numbers such that two vertices are adjacent if and only if the absolute difference of the corresponding labels is in S. The multiset S of numbers is commonly known as a *signature* of the autograph G and each of its elements is called *signature value*. In this paper, an autograph G with a signature S is denoted by $G(S)$. An autograph is *proper* if it has a signature consisting of positive real numbers and it is called a *monograph* if it has a signature whose elements are distinct real numbers. In 1982, Gervacio [4] adapted the labeling principle of autograph to directed graphs, referring to the resulting graphs as *difference digraphs*. In 2009, Hegde and Vasudeva [8] introduced *mod difference digraphs* using modulo difference as the criteria for the adjacency of the vertices instead of the usual difference of numbers used in difference digraphs. In [9], they were able to show that structural properties of directed graphs can be studied in the context of mod difference digraphs.

* Corresponding author.

Since the introduction of autographs, most studies focused on determining which graphs are autographs [1, 5, 11], monographs [14], proper monographs [3], difference digraphs [2, 12, 13] or mod difference digraphs [6, 7, 8, 9]. The works of Panopio in [11] and Hegde and Vasudeva in [9], however, suggest that the study of these labeling methods can be done through a different approach, that is, to investigate some structural properties of the labeled graphs through the observation of their signatures.

In this study, we have adapted the idea of *working vertices* and *idle vertices* to obtain independent sets and vertex coverings in proper monographs. An *independent set I* of a graph is a set of pairwise non-adjacent vertices of this graph. If no other independent set contains I as a proper subset, then I is said to be *maximal*. On the other hand, a *vertex covering* of a graph is a set of vertices such that each edge of the graph is incident to at least one vertex of this set. The concept of working vertices originated from the works of Tuga et al. [15] and Miller et al. [10], while the concept of idle vertices was first introduced in [9] by Hegde and Vasudeva. Moreover, a concept similar to that of the working vertices was used in [9]. Lastly, we have defined and used a concept similar to the *out-degree* in directed graphs to determine more independent sets from proper monographs.

2 Working and idle vertices

Let G be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{[v_i, v_j] |$ for some $v_i, v_j \in V\}$. Suppose G is an autograph with $S = \{s_1, s_2, \ldots, s_n\}$ as its signature where s_i is the label of v_i, for $i = 1, 2, \ldots, n$. Note that in the succeeding discussions, we often identify the vertices with their corresponding signature values, i.e., a signature value refers to the vertex it represents. For instance, for $s_i, s_j \in S$, when s_i is said to be adjacent to s_j, it means that not only $|s_i - s_j| \in S$ but also corresponding vertices v_i and v_j are adjacent. Moreover, an edge $[v_i, v_j]$ will sometimes be referred to as the edge $[s_i, s_j]$.

Definition 2.1 *A vertex s in the autograph $G(S)$ is called a working vertex if there exist $s_i, s_j \in S$ such that $|s_i - s_j| = s$. Otherwise, s is called an idle vertex.*

From this definition, the remarks below immediately follow.

1. In an autograph $G(S)$ with $s_i > 0$, for each i, and at least two of which are distinct, a vertex is idle if it has the largest signature value. On the other hand, the vertex with the smallest signature value s' is working if there is a positive $s_i \in S$ such that $s_i - s' \in S$.

2. Vertices with negative signature values are idle (i.e., all working vertices have non-negative signature values).

3. In a proper autograph, a vertex s_α is a working vertex if and only if there is an $s_i \in S$ such that $s_i - s_\alpha \in S$. Hence, a vertex s_ω is an idle vertex in a proper autograph if and only if there is no $s_i \in S$ such that $s_i - s_\omega \in S$. That is, a vertex s_ω is idle in a proper monograph if for every vertex s_i adjacent to s_ω, $s_\omega > s_i$.

4. In an autograph $G(S)$ with $0 \in S$, all vertices with positive signature values are working.

5. Working and idle vertices are highly dependent on the given signature of an autograph. Thus, these vertices may vary from one signature of an autograph to another.

6. In a complete proper monograph with s_n as its largest signature value, all vertices, except the one with the signature value s_n, are working. Since for every signature value $s_i \neq s_n$ there is an $s_t \in S$ such that $s_n - s_i = s_t$. With this remark, the next result immediately follows.

Proposition 2.2 *If a proper monograph contains at least two idle vertices, then it is not complete.*

Let T be a tree. According to Bloom et al. in [1], T is a proper monograph.

Theorem 2.3 *In a proper monograph $T(S)$, there are no pairwise distinct vertices $s_i, s_j, s_k \in S$ such that $|s_i - s_j| = |s_j - s_k| \in S$.*

Proof. Suppose that in a proper monograph $T(S)$ there are pairwise distinct vertices $s_i, s_j, s_k \in S$ such that $|s_i - s_j| = |s_j - s_k| = s$, for some $s \in S$.

Case 1. Let $s_i < s_j < s_k$. Then, $s_j - s_i = s$ and $s_k - s_j = s$. So, $s_j - s = s_i$ and $s_k - s = s_j$. This means that s is adjacent to both s_k and s_j. Clearly, s cannot be equal to either s_k or s_j. Hence, s is either the vertex s_i or a vertex different from s_i. In either case, (s_j, s_k, s) is a triangle in T.

Case 2. Suppose $s_j < s_i < s_k$. Thus, $s_k - s_j = s$ and $s_i - s_j = s$. This implies that $s_k - s = s_j$ and $s_i - s = s_j$, therefore, (s_i, s, s_k) is a triangle in T, a contradiction.

Case 3. Now, assume that $s_i, s_k < s_j$. Then, $s_j - s_i = s$ and $s_j - s_k = s$. Thus, $s_i = s_k$, contradicting the assumption that T is a proper monograph.

The proof for the cases where $s_j, s_k < s_i$ is similar to the combined proofs of *Case 1* and *Case 2*. Hence, the result follows. □

The next set of results are some consequences of Theorem 2.3 and the fact that stars, paths and cycles are known to be proper monographs [1, 11, 14].

Let $K_{1,n}$ be the star with $n + 1$ vertices.

Corollary 2.4 *The proper monograph $K_{1,n}$ has n working vertices.*

Corollary 2.5 *Let the center of the proper monograph $K_{1,n}(S)$ be labeled by the signature value c. Then, either c is the largest signature value or there exists an $s_r \in S$ such that $s_r = 2c$.*

Proof. Because of Corollary 2.4, $K_{1,n}(S)$ has n working vertices and 1 idle vertex. Suppose the center c is idle and c is not the largest signature value in S. Then, there exists an $s_i \in S$ such that $s_i > c$. Hence, $s_i - c = s_t$, for some $s_t \in S$. Thus, $s_i - s_t = c$, implying that c is working. Therefore, $s_i < c$, for each i.

Now, suppose c is working. Then, there is an $s_r \in S$ such that $|c - s_r| = c$. Hence, either $c - s_r = c$ or $s_r - c = c$. Since $s_r = 0$ is impossible, it follows that $s_r = 2c$. □

The following concept will be vital in the succeeding results.

Definition 2.6 *The vertex s is said to correspond to the edge $[s_i, s_j]$ if and only if $|s_i - s_j| = s$. The role of a vertex s denoted by $role(s)$, is the number of the edges in a graph that correspond to s.*

Theorem 2.7 *The proper monograph P_n has η working vertices, where $\frac{n-1}{2} \leq \eta \leq n-1$.*

Proof. Clearly, the vertex of P_n with the largest signature value is idle. This means that P_n can only have at most $n-1$ working vertices. Now, to show that the number η of working vertices of P_n is at least $\frac{n-1}{2}$, we prove that for every working vertex s, $role(s) \leq 2$. That is, we prove that there are no pairwise distinct vertices $s_i, s_j, s_k \in S$ such that $|s_i - s_{i-1}| = |s_j - s_{j-1}| = |s_k - s_{k-1}|$, for some $s_{i-1}, s_{j-1}, s_{k-1} \in S$. On the contrary, suppose that in S there are $s_i > s_j > s_k$ and $s_i > s_{i-1}, s_j > s_{j-1}, s_k > s_{k-1}$ such that $|s_i - s_{i-1}| = |s_j - s_{j-1}| = |s_k - s_{k-1}| = s$, for some $s \in S$. Then, $s_i - s = s_{i-1}$, $s_j - s = s_{j-1}$ and $s_k - s = s_{k-1}$, implying that s is adjacent to s_i, s_j, s_k and s has at least the degree 3, in contradiction to the fact that P_n is a path. It follows that a working vertex has a role of at most two in P_n, therefore, $\frac{n-1}{2} \leq \frac{n}{2} \leq \eta$ if n is even and $\frac{n-1}{2} \leq \eta$ when n is odd. □

Using the same technique presented in the proof of Theorem 2.7, the following theorem can be easily shown.

Theorem 2.8 *If s is a working vertex in a k-regular proper monograph, then the number of distinct $s_i \in S$ such that $|s_i - s_j| = s$, for some $s_j \in S$, does not exceed k.*

Theorem 2.9 *In a proper monograph $C_n(S)$, $n \geq 4$, there are no pairwise distinct vertices $s_i, s_j, s_k \in S$ such that $|s_i - s_j| = |s_j - s_k| \in S$.*

Proof. Assuming that the conclusion is not true, then as in the proof of Theorem 2.3, either (s_i, s_j, s_k) will form a triangle in C_n or two of s_i, s_j, s_k will have the same signature values. □

Theorem 2.10 *The proper monograph C_n has η working vertices, where $\frac{n-1}{2} < \eta \leq n-1$.*

Proof. First, consider the triangle $C_3(\{s_1, s_2, s_3\})$ with $s_3 > s_1, s_2$. Suppose, $|s_1 - s_2| = |s_2 - s_3| = |s_3 - s_1| = s$, for some $s \in \{s_1, s_2, s_3\}$. Assuming $s_2 > s_1$, then $s_2 - s_1 = s$, $s_3 - s_2 = s$ and $s_3 - s_1 = s$. This implies that $s_2 = 2s$, $s_3 = 3s$ and $s_3 = 2s$. A similar contradiction will be obtained when $s_1 > s_2$. Thus, C_3 must have more than one working vertices.

By Theorem 2.9 and following the idea of the proof of Theorem 2.7, it can be shown that in C_n with $n \geq 4$, a working vertex s has $role(s) \leq 2$. So, the result follows. □

3 Independent sets and vertex coverings

A graph might have a lot of distinct proper monograph labelings, but the results above show that in several cases proper monographs can only accommodate a particular number of working vertices, i.e., the number of the working vertices is bounded. Obviously, this also implies bounds for the number of the idle vertices in these proper monographs. In the following, we investigate the relations between idle vertices and independent sets.

Theorem 3.1 *Let G be an autograph with signature S, which contains some positive elements. Then, the set of the positive idle vertices of $G(S)$ is an independent set in $G(S)$.*

Proof. Let $G(S)$ be an autograph and $I = \{\overline{s_i} \in S | \overline{s_i}$ is a positive idle vertex$\}$. Since some elements of S are positive and the vertex in $G(S)$ with the largest signature value is idle, I is non-empty. If I is a singleton, the result follows. Suppose there are $\overline{s_j}, \overline{s_k} \in I$ such that $|\overline{s_j} - \overline{s_k}| \in S$. So, there exists an $s_m \in S$ such that $|\overline{s_j} - \overline{s_k}| = s_m$. Hence, $\overline{s_j} - \overline{s_k} = s_m$ or $\overline{s_k} - \overline{s_j} = s_m$. Thus, $\overline{s_j} - s_m = |\overline{s_j} - s_m| = \overline{s_k}$ or $\overline{s_k} - s_m = |\overline{s_k} - s_m| = \overline{s_j}$. Therefore, $\overline{s_j}$ or $\overline{s_k}$ is a working vertex. In any case, a contradiction to the assumption can be observed. □

Theorem 3.2 *Let $G(S)$ be a proper monograph of order n. If the elements of S can be arranged in a sequence $s_1, s_2, \ldots, s_n$ such that $|s_i - s_{i-1}| \in S$ for $i = 2, 3, \ldots, n$ and s_i is an idle vertex for every odd i, then $I = \{s_i | i$ is odd$\}$ is a maximal independent set in $G(S)$.*

Proof. Let $G(S)$ be a proper monograph of order n. Also, suppose that the elements of S can be arranged in a sequence $s_1, s_2, \ldots, s_n$ such that $|s_i - s_{i-1}| \in S$ for $i = 2, 3, \ldots, n$ and s_i is an idle vertex for every odd i. From Theorem 3.1, I is an independent set of $G(S)$. Suppose I is not maximal. Then, there is an independent set K such that $I \subset K$. Hence, K contains some $s_i \in S$ where i is even. Obviously,

$i-1$ is odd, $s_{i-1} \in I \subset K$ and $|s_i - s_{i-1}| \in S$. So, s_i and s_{i-1} are adjacent, a contradiction to the assumption that K is independent. □

Theorem 3.2 tells that if a proper monograph has a Hamiltonian path whose vertices are alternately working and idle, then a maximal independent set can be obtained by collecting all of its idle vertices. The following results can be shown similarly.

Theorem 3.3 *Let $G(S)$ be a proper monograph of order n. If the elements of S can be arranged in a sequence $s_1, s_2, \ldots, s_n$ such that $|s_i - s_{i-1}| \in S$ for $i = 2, 3, \ldots, n$ and s_i is an idle vertex for every even i, then $\{s_i | i \text{ is even}\}$ is a maximal independent set in $G(S)$.*

Theorem 3.4 *Let $G(S)$ be a proper monograph of order $3n-2$. If the elements of S can be arranged in a sequence $s_1, s_2, \ldots, s_{3n-2}$ such that $|s_i - s_{i-1}| \in S$ for $i = 2, 3, \ldots, 3n-2$ and $s_1, s_4, s_7, s_{10}, \ldots, s_{3n-2}$ are idle, then $\{s_i | i = 1, 4, 7, 10, \ldots, 3n-2\}$ is a maximal independent set in $G(S)$.*

The following results are about vertex coverings of proper monographs obtained by collecting all of its working vertices. These results make use of the fact that a set of vertices is a vertex covering if and only if its complement is an independent set.

Corollary 3.5 *The set of all working vertices of a proper monograph $G(S)$ is a vertex covering of $G(S)$.*

Proposition 3.6 *Let W be the set of all working vertices of a proper monograph $G(S)$ and $w \in W$. $W - \{w\}$ is a vertex covering if w is not adjacent to any idle vertex.*

Proof. Since w is a working vertex, it is not an isolated vertex. Also, since w is not adjacent to any idle vertex all vertices adjacent to w must be working. It follows that $W - \{w\}$ is a vertex covering. □

The following can be proven similarly.

Proposition 3.7 *Let W be the set of all working vertices of a proper monograph $G(S)$ and $\{w_1, w_2, \ldots, w_k\} \subset W$. $W - \{w_1, w_2, \ldots, w_k\}$ is a vertex covering of $G(S)$ if, for every $i \in \{1, 2, \ldots, k\}$, w_i is neither adjacent to an idle vertex of $G(S)$ nor to a vertex w_j, where $j \in \{1, 2, \ldots, k\} - \{i\}$.*

Some working vertices in a proper monograph can also form independent sets as the following results include. A modified concept of the out-degree in digraphs will be utilized here to show these results.

Definition 3.8 *Let $s \in S$ be the signature value of a vertex v in the autograph $G(S)$. The low-degree of the vertex v denoted by $ld(s)$ is the number of vertices adjacent to v with signature values lower than s.*

Proposition 3.9 *Let $G(S)$ be a proper monograph. Then, $A = \{s \in S | ld(s) = 0\}$ is an independent set in $G(S)$.*

Proof. Assume, $s_i, s_j \in A$ such that $|s_i - s_j| \in S$. Thus, $|s_i - s_j| = s_k$, for some $s_k \in S$. This implies that either $s_i - s_j = s_k$ or $s_j - s_i = s_k$. Hence, $ld(s_i) \neq 0$ or $ld(s_j) \neq 0$. Either case implies a contradiction to the definition of A. Therefore, $|s_i - s_j| \notin S$ and it follows that A is an independent set. $\square$

Proposition 3.10 *Let $G(S)$ be a proper monograph. Then, $A = \{s \in S | ld(s) = 1\}$ is an independent set in $G(S)$ if for every $s_i \in A$, $s_i \neq 2s_j$, for all $s_j \in S$.*

Proof. Assume, $s_k, s_l \in A$ such that $|s_k - s_l| \in S$. Then, there exists an $s_m \in S$ such that $s_k - s_l = s_m$ or $s_l - s_k = s_m$. But since $ld(s_k) = 1$ and $ld(s_l) = 1$, it follows that $s_l = s_m$ or $s_k = s_m$. In either case, there is a contradiction to the assumption that for every $s_i \in A$, $s_i \neq 2s_j$, for all $s_j \in S$. $\square$

4 Final remarks

The above results can be useful not only to determine maximal but also *maximum independent sets* of graphs (a maximum independent set is an independent set with the most number of elements). Therefore, it is an interesting problem to investigate the relations between certain signatures of proper monographs (especially the set of the idle vertices) and their maximum independent sets. However, it is important to note that not all proper monographs have the property that their maximum independent sets are bijectively mapped to the sets of their idle vertices. As seen in Corollary 2.4, it is impossible for $K_{1,n}$ to have a proper monograph labeling having the said property. In the case of the path P_n, however, we can give the following labeling. Let $v_1, v_2, \ldots, v_n$ be the consecutive vertices of P_n and the define the labeling L as follows. For $i = n$, $L(v_n) = 5^{\frac{n}{2}} + 1$ if n is even (or $L(v_n) = 2(5^{\frac{n-1}{2}} + 1)$ if n is odd) and for $i < n$,

$$L(v_i) = \begin{cases} 12 & \text{if } i = 1 \\ 5^{i/2} + 1 & \text{if } i = 2k,\ k \in \left\{ x \in \mathbb{N} \middle| 1 \leq x \leq \frac{n-1}{2} \right\} \\ L(v_{i-1}) + L(v_{i+1}) & \text{if } i = 2k - 1,\ k \in \left\{ x \in \mathbb{N} \middle| 2 \leq x \leq \frac{n}{2} \right\} \end{cases}$$

Finally, aside from independent sets and vertex coverings, it is also interesting to determine which other properties of graphs can be easily described or investigated by means of proper monograph labelings.

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