AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **58(2)** (2014), Pages 333–346

The asymptotic existence of orthogonal designs

E. Ghaderpour H. Kharaghani*

Department of Mathematics and Computer Science
University of Lethbridge
Lethbridge, Alberta, T1K 3M4
Canada

Abstract

Given any ℓ-tuple $(s_1, s_2, \ldots, s_\ell)$ of positive integers, there is an integer $N = N(s_1, s_2, \ldots, s_\ell)$ such that an orthogonal design of order $2^n(s_1 + s_2 + \cdots + s_\ell)$ and type $(2^n s_1, 2^n s_2, \ldots, 2^n s_\ell)$ exists, for each $n \geq N$. This complements a result of Eades et al. which in turn implies that if the positive integers $s_1, s_2, \ldots, s_\ell$ are all highly divisible by 2, then there is a full orthogonal design of type $(s_1, s_2, \ldots, s_\ell)$.

1 Introduction

A *Hadamard matrix* of order n is a square $\{\pm 1\}$-matrix H of order n such that $HH^t = nI_n$, where H^t is the transpose of H. A *complex orthogonal design* of order n and type $(s_1, \ldots, s_\ell)$, denoted $COD(n;\ s_1, \ldots, s_\ell)$, is a square matrix X of order n with entries from $\{0, \epsilon_1 x_1, \ldots, \epsilon_\ell x_\ell\}$, where the x_j's are commuting variables and $\epsilon_j \in \{\pm 1, \pm i\}$ for each j, that satisfies

$$XX^* = \left(\sum_{j=1}^{\ell} s_j x_j^2\right) I_n,$$

where X^* denotes the conjugate transpose of X and I_n is the identity matrix of order n. A complex orthogonal design (COD) in which $\epsilon_j \in \{\pm 1\}$ for all j is called an *orthogonal design*, denoted $OD(n; s_1, \ldots, s_\ell)$. An orthogonal design (OD) in which there is no zero entry is called a *full* OD. Equating all variables to 1 in any full OD results in a Hadamard matrix.

It is shown (see [9]) that the number of variables in an OD of order $n = 2^a b$, b odd, cannot exceed the Radon number $\rho(n)$, where $\rho(n)$ is defined as follows:

$$\rho(n) := 8c + 2^d, \quad \text{where} \quad a = 4c + d,\ 0 \leq d < 4.$$

* Supported by an NSERC-Group Discovery Grant.
Corresponding author: `kharaghani@uleth.ca`

The credit for the consideration of *asymptotic existence* results should be given to Seberry [9, 15] for her fundamental approach in showing that for each positive integer p, there is a Hadamard matrix of order $2^n p$ for each $n \geq 2\log_2(p-3)$. Thus for each positive integer n, the existence of Hadamard matrices is in doubt for only a finite number of orders of the form $2^t n$. Two of Seberry's students, Robinson [13] and Eades [6], did extensive work on ODs in their Ph.D. theses and made significant advances towards showing the asymptotic existence of a number of ODs. The work of Wolfe [16] provided enough ammunition for other researchers to pursue a different approach to the asymptotic existence of ODs. There are now a number of asymptotic existence results for ODs and thus Hadamard matrices; see [1, 2, 3, 4, 5, 8, 12] for a sample.

Eades in his Ph.D. thesis [7] states that

> If the positive integers $s_1, s_2, \dots, s_u$, are all *highly* divisible by 2, then in many cases the existence of an OD of type $s_1, s_2, \dots, s_u$ and order n may be established.

He then proves the following general construction.

Theorem 1 *Suppose that r and n are positive integers, $b_1, b_2, \dots, b_\ell$ are powers of 2, and there is an OD of type $(b_1, b_2, \dots, b_\ell)$ and order $2^r n$. If $s_1, s_2, \dots, s_u$ are positive integers with sum $2^d(b_1 + b_2 + \cdots + b_\ell)$ for some $d \geq 0$, then there is an integer N such that for each $a \geq N$, there is an*

$$OD\big(2^{a+d+r}n;\ 2^a s_1, 2^a s_2, \dots, 2^a s_u\big).$$

One of the main results of the paper is an improvement of this result of Eades. We show that the existence of the ODs of type $(b_1, b_2, \dots, b_\ell)$ and order $2^r n$ can be removed from Theorem 1. More specifically, we prove in Section 2, Theorem 4, that for any ℓ-tuple $\big(s_1, s_2, \dots, s_\ell\big)$ of positive integers, there is an integer $N = N\big(s_1, s_2, \dots, s_\ell\big)$ such that for each $n \geq N$ there is an OD of order $2^n\big(s_1+s_2+\ldots+s_\ell\big)$ and type $\big(2^n s_1, 2^n s_2, \dots, 2^n s_\ell\big)$.

Let M be an $OD\big(n;\ c_1, \dots, c_k\big)$ on variables $\alpha_1, \dots, \alpha_k$, and N be an $OD\big(n;\ d_1, \dots, d_m\big)$ on variables $\beta_1, \dots, \beta_m$, where the two sets of variables are disjoint. Then the pair $(M; N)$ is said to form an *amicable orthogonal design*, denoted

$$AOD\big(n;\ c_1, \dots, c_k;\ d_1, \dots, d_m\big),$$

if $MN^t = NM^t$. The pair $(M; N)$ is called *anti-amicable* if $MN^t = -NM^t$.

Let X be a $COD\big(n;\ c_1, \dots, c_k\big)$ on variables $\alpha_1, \dots, \alpha_k$, and Y be a $COD\big(n;\ d_1, \dots, d_m\big)$ on variables $\beta_1, \dots, \beta_m$, where the two sets of variables are disjoint. Then $(X; Y)$ is called an *amicable complex orthogonal design*, denoted

$$ACOD\big(n;\ c_1, \dots, c_k;\ d_1, \dots, d_m\big),$$

if $XY^* = YX^*$.

We deal with the asymptotic existence of amicable orthogonal designs in Section 3. More specifically, we show in Theorem 5 that for any two sequences $(u_1, \ldots, u_s)$ and $(v_1, \ldots, v_t)$ of positive integers, there are integers h, h_1, h_2 and N such that there exists an

$$AOD\Big(2^n h;\ 2^{n+h_1}u_1, \ldots, 2^{n+h_1}u_s;\ 2^{n+h_2}v_1, \ldots, 2^{n+h_2}v_t\Big),$$

for each $n \geq N$.

Wolfe [16], continuing Shapiro's work [14], studied amicable and anti-amicable orthogonal designs in detail. The following result from his work will be used in Section 3. We give a construction which will be needed later.

Theorem 2 *Given an integer $n = 2^s d$, where d is odd and $s \geq 1$, there exist two sets $A = \{A_1, \ldots, A_{s+1}\}$ and $B = \{B_1, \ldots, B_{s+1}\}$ of signed permutation matrices of order n such that*

(i) A consists of pairwise disjoint anti-amicable matrices,

(ii) B consists of pairwise disjoint anti-amicable matrices,

(iii) for each i and j, $A_i B_j^t = B_j A_i^t$.

Proof. For each $2 \leq k \leq s+1$ let

$$A_1 = \Big(\otimes_{i=1}^{s} I\Big) \otimes I_d, \quad A_k = \Big(\otimes_{i=1}^{k-2} I\Big) \otimes R \otimes \Big(\otimes_{i=k}^{s} P\Big) \otimes I_d,$$

and

$$B_1 = \Big(\otimes_{i=1}^{s} P\Big) \otimes I_d, \quad B_k = \Big(\otimes_{i=1}^{k-2} I\Big) \otimes Q \otimes \Big(\otimes_{i=k}^{s} P\Big) \otimes I_d,$$

where $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and I and I_d are the identity matrices of orders 2 and d, respectively. Then the matrices A_i and B_i $(1 \leq i \leq s+1)$ satisfy the three properties (i), (ii) and (iii). □

The *nonperiodic autocorrelation function* [11] of a sequence $A = (x_1, \ldots, x_n)$ of type 1 square matrices of order m, is defined by

$$N_A(j) := \begin{cases} \displaystyle\sum_{i=1}^{n-j} x_{i+j} x_i^t & \text{if } \ j = 0, 1, 2, \ldots, n-1 \\ 0 & j \geq n \end{cases}$$

where x_i^t is the transpose of x_i.

Let $X = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ be a set of type 1 matrices. Then a pair of sequences $A = (x_1, \ldots, x_n)$ and $B = (y_1, \ldots, y_n)$ is called a *Golay pair of length n* *in type* 1 *matrices* x_i, y_i, $1 \le i \le n$, if $N_A(j) + N_B(j) = 0$ for all $j > 0$. Note that by our definition, the pair $A = (x, y)$ and $B = (y, -x)$ do not form a Golay pair of length 2 in type 1 matrices in general, because $N_A(1) + N_B(1) = 0$ only if $xy^t - yx^t = 0$. However, $A = (x, y)$ and $B = (x, -y)$ form a Golay pair of length 2 in type 1 matrices x and y. Note that the *directed sequences* terminology is used in [10, 11] for a similar concept.

Although the results of this note apply to more general settings, we would concentrate only on type 1 matrices of the form $\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$, where α and β are commuting variables.

We use the standard notation $a_{(k)}$ to show that the figure a is repeated k times and $\mathrm{circ}(a_1, \ldots, a_n)$ to denote a circulant matrix with the first row $(a_1, \ldots, a_n)$.

2 The asymptotic existence of orthogonal designs

We start with the following well-known result (see [10] Section 2).

Lemma 1 *For any positive integer n, there is a Golay pair of length 2^n in two type 1 matrices each appearing 2^{n-1} times in each of the sequences.*

Proof. Let A_{n-1} and B_{n-1} be a Golay pair of length 2^{n-1} in two type 1 matrices each appearing 2^{n-2} times in both A_{n-1} and B_{n-1}. Then $A_n = \big(A_{n-1}, B_{n-1}\big)$ and $B_n = \big(A_{n-1}, -B_{n-1}\big)$ form a Golay pair of length 2^n in two type 1 matrices as desired, where (A, B) means sequence A followed by sequence B. □

Theorem 3 *For any given sequence of positive integers $(b, a_1, a_2, \ldots, a_k)$, there exists a full COD of type $\Big(2^{N(m)} \cdot 1_{(b)}, 2^{N(m)} \cdot 2^{a_1}_{(4)}, \ldots, 2^{N(m)} \cdot 2^{a_k}_{(4)}\Big)$, where $m = 4k + b + 2$ if b is even, $m = 4k + b + 1$ if b is odd, and $N(m)$ is the smallest positive integer such that $m \le \rho\big(2^{N(m)-1}\big)$.*

Proof. Let $(b, a_1, a_2, \ldots, a_k)$ be a sequence of positive integers. We distinguish two cases:

Case 1. b is even. Consider the type 1 matrices x_i, $0 \le i \le \dfrac{b}{2}$, y_j and z_j, $1 \le j \le k$ of order 2. For each j, $1 \le j \le k$, let G_{j1} and G_{j2} be a Golay pair of length 2^{a_j} in two type 1 matrices y_j and z_j. Let

$$s_1 = 0 \quad \text{and} \quad s_j = 2\sum_{r=1}^{j-1} 2^{a_r},\ 2 \le j \le k+1. \tag{1}$$

Let $d = \frac{b}{2} + s_{k+1}$ and define

$$M_0 := \operatorname{circ}\big(0_{(d)}, x_0, 0_{(d-1)}\big), \qquad M_1 := \operatorname{circ}\big(x_1, 0_{(2d-1)}\big), \tag{2}$$
$$M_h := \operatorname{circ}\big(0_{(h-1)}, x_h, 0_{(2d-h)}\big), \qquad 2 \le h \le \frac{b}{2}.$$

For each j, $1 \le j \le k$, define

$$N_{2j-1} := \operatorname{circ}\Big(0_{(\frac{b}{2}+s_j)}, G_{j1}, 0_{(2d-\frac{b}{2}-s_j-2^{a_j})}\Big),\ N_{2j} := \operatorname{circ}\Big(0_{(\frac{b}{2}+s_j+2^{a_j})}, G_{j2}, 0_{(2d-\frac{b}{2}-s_{j+1})}\Big).$$

Considering that all the variables in these matrices are assumed to be type 1 matrices of order 2, these matrices are in fact commuting block-circulant matrices (see [9, 11]), and the 0 entries denote the zero matrix of order 2. Let $m = 4k+b+2$ and let $N(m)$ be the smallest positive integer such that $m \le \rho\big(2^{N(m)-1}\big)$. So there is a set

$$A' = \big\{A_1, \dots, A_m\big\} \tag{3}$$

of mutually disjoint anti-amicable signed permutation matrices of order $2^{N(m)-1}$. These matrices are known as Hurwitz-Radon matrices (see [9] chapter 1). Suppose H is a Hadamard matrix of order $2^{N(m)-1}$. Let

$$\begin{aligned} C = {} & \frac{1}{2}\big(M_0 + M_0^t\big) \otimes A_1 H + \frac{i}{2}\big(M_0 - M_0^t\big) \otimes A_2 H \qquad (4)\\ & + \frac{1}{2}\big(M_1 + M_1^t\big) \otimes A_3 H + \frac{i}{2}\big(M_1 - M_1^t\big) \otimes A_4 H \\ & + \sum_{h=2}^{\frac{b}{2}} \Big(\big(M_h + M_h^t\big) \otimes \frac{1}{2}\big(A_{2h+1} + A_{2h+2}\big)H + i\big(M_h - M_h^t\big) \otimes \frac{1}{2}\big(A_{2h+1} - A_{2h+2}\big)H\Big) \\ & + \sum_{j=1}^{2k} \Big(\big(N_j + N_j^t\big) \otimes \frac{1}{2}\big(A_{2j+b+1} + A_{2j+b+2}\big)H \qquad (5)\\ & \qquad + i\big(N_j - N_j^t\big) \otimes \frac{1}{2}\big(A_{2j+b+1} - A_{2j+b+2}\big)H\Big). \end{aligned}$$

We show that

$$CC^* = 2^{N(m)}\omega I_{2^{N(m)}d}, \tag{6}$$

where $\omega = \frac{1}{2}x_0x_0^t + \frac{1}{2}x_1x_1^t + x_2x_2^t + \cdots + x_{\frac{b}{2}}x_{\frac{b}{2}}^t + 2^{a_1}y_1y_1^t + 2^{a_1}z_1z_1^t + \cdots + 2^{a_k}y_ky_k^t + 2^{a_k}z_kz_k^t$. To this end, we first note that each of the sets

$$\Big\{\frac{1}{2}\big(M_0 + M_0^t\big),\ \ \frac{i}{2}\big(M_0 - M_0^t\big),\ \ \frac{1}{2}\big(M_1 + M_1^t\big),\ \ \frac{i}{2}\big(M_1 - M_1^t\big)\Big\},$$

$$\Big\{\big(M_h + M_h^t\big),\ \ \big(N_j + N_j^t\big);\ \ 2 \le h \le \frac{b}{2},\ \ 1 \le j \le 2k\Big\}$$

and

$$\Big\{i\big(M_h - M_h^t\big),\ i\big(N_j - N_j^t\big);\ \ 2 \le h \le \frac{b}{2},\ \ 1 \le j \le 2k\Big\}$$

consist of mutually disjoint Hermitian circulant matrices. Moreover, for $u = 0, 1$, we have

$$\frac{1}{4}\big(M_u + M_u^t\big)\big(M_u + M_u^t\big)^t + \frac{1}{4}\big(M_u - M_u^t\big)\big(M_u - M_u^t\big)^t = x_u x_u^t I_{2d}$$

and for each h, $2 \le h \le \dfrac{b}{2}$,

$$\big(M_h + M_h^t\big)\big(M_h + M_h^t\big)^t + \big(M_h - M_h^t\big)\big(M_h - M_h^t\big)^t = 4x_h x_h^t I_{2d}.$$

Also, for each j, $1 \le j \le k$, we have

$$\begin{aligned}\sum_{r=2j-1}^{2j}\Big(\big(N_r + N_r^t\big)\big(N_r + N_r^t\big)^t + \big(N_r - N_r^t\big)\big(N_r - N_r^t\big)^t\Big) &= 2\sum_{r=2j-1}^{2j}\big(N_r N_r^t + N_r^t N_r\big)\\ &= 2^{a_j+2}\big(y_j y_j^t + z_j z_j^t\big) I_{2d}.\end{aligned}$$

Note that for each j, $3 \le j \le \dfrac{b}{2} + 2k + 1$, the matrices $\dfrac{1}{2}\big(A_{2j-1} + A_{2j}\big)H$ and $\dfrac{1}{2}\big(A_{2j-1} - A_{2j}\big)H$ are disjoint with $0, \pm 1$ entries. Furthermore, since the set A' consists of mutually anti-amicable matrices, the set

$$\Big\{A_1H,\ A_2H,\ A_3H,\ A_4H,\ \frac{1}{2}\big(A_{2j-1} \pm A_{2j}\big)H\ \ \big(3 \le j \le \frac{b}{2} + 2k + 1\big)\Big\}$$

consists of mutually anti-amicable matrices. Since for each j, $3 \le j \le \dfrac{b}{2} + 2k + 1$,

$$\begin{aligned}\Big(\frac{1}{2}\big(A_{2j-1} \pm A_{2j}\big)H\Big)\Big(\frac{1}{2}\big(A_{2j-1} \pm A_{2j}\big)H\Big)^t &= \frac{2^{N(m)-1}}{4}\big(A_{2j-1} \pm A_{2j}\big)\big(A_{2j-1} \pm A_{2j}\big)^t I_{2^{N(m)-1}}\\ &= 2^{N(m)-2} I_{2^{N(m)-1}},\end{aligned}$$

the validity of equation (6) follows.

In the equation (6), we now replace x_0 by $\begin{bmatrix} \alpha & \alpha \\ -\alpha & \alpha \end{bmatrix}$, x_1 by $\begin{bmatrix} \beta & \beta \\ -\beta & \beta \end{bmatrix}$, x_h by $\begin{bmatrix} \alpha_h & \beta_h \\ -\beta_h & \alpha_h \end{bmatrix}$, $2 \le h \le \dfrac{b}{2}$, y_j by $\begin{bmatrix} \alpha_j' & \beta_j' \\ -\beta_j' & \alpha_j' \end{bmatrix}$, and z_j by $\begin{bmatrix} \alpha_j'' & \beta_j'' \\ -\beta_j'' & \alpha_j'' \end{bmatrix}$, $1 \le j \le k$. The resulted matrix will be a full COD of type $\big(2^{N(m)} \cdot 1_{(b)}, 2^{N(m)} \cdot 2^{a_1}_{(4)}, \ldots, 2^{N(m)} \cdot 2^{a_k}_{(4)}\big)$, where the α, β, α_h's, β_h's, α_j''s, β_j''s, α_j'''s and β_j'''s are commuting variables.

Case 2. b is odd. Consider the following circulant matrices of order $2d+1$, where $d = \dfrac{b-1}{2} + s_{k+1}$ with the same s_j's as in equation (1),

$$\begin{aligned} M_1 &= \mathrm{circ}\big(x_1, 0_{(2d)}\big),\\ M_h &= \mathrm{circ}\big(0_{(h-1)}, x_h, 0_{(2d-h+1)}\big), \qquad 2 \le h \le \frac{b+1}{2}. \end{aligned}$$

For each j, $1 \le j \le k$, assume

$$N_{2j-1} = \text{circ}\Big(0_{\left(\frac{b+1}{2}+s_j\right)}, G_{j1}, 0_{\left(2d-\frac{b-1}{2}-s_j-2^{a_j}\right)}\Big),$$
$$N_{2j} = \text{circ}\Big(0_{\left(\frac{b+1}{2}+s_j+2^{a_j}\right)}, G_{j2}, 0_{\left(2d-\frac{b-1}{2}-s_{j+1}\right)}\Big).$$

The rest of proof is similar to Case 1, and so $m = 4k + b + 1$. □

Remark 1 The choice of $N(m)$ in Theorem 3 and the next few asymptotic results is crucial; the smaller $N(m)$, the better asymptotic result. All $N(m)$'s appearing in this note are either equal to or 1 less than the ceiling of $(m+2)/2$, depending on the value of m.

Let $(u_1, \ldots, u_\ell)$ be an ℓ-tuple of positive integers and suppose 2^t is the largest power of 2 appearing in the binary expansions of u_i, $i = 1, 2, \ldots, \ell$. Using the binary expansion of each u_i, one can write

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_\ell \end{bmatrix} = E \begin{bmatrix} 1 \\ 2 \\ \vdots \\ 2^t \end{bmatrix}, \tag{7}$$

where $E = [e_{ij}]$ is the unique $\ell \times (t+1)$ matrix with 0 and 1 entries. We call E the *binary matrix* corresponding to the ℓ-tuple $(u_1, \ldots, u_\ell)$.

For convenience and in order to make the first column of the binary matrix E nonzero, in the following lemma, we assume that the ℓ-tuples of positive integers have at least one odd element.

Lemma 2 *Suppose that $(u_1, \ldots, u_\ell)$ is an ℓ-tuple of positive integers such that at least one of the u_i's is odd. Then there exists an integer $m = m(u_1, \ldots, u_\ell)$ such that there is a*

$$COD\Big(2^m(u_1 + \cdots + u_\ell);\ 2^m u_1, \ldots, 2^m u_\ell\Big).$$

Proof. Let $(u_1, \ldots, u_\ell)$ be an ℓ-tuple of positive integers such that at least one of u_i's is odd, and let $d = u_1 + \cdots + u_\ell$.

By applying Theorem 3 all we need is to equate variables appropriately. We do this by applying the following procedure.

We form the $\ell \times (t+1)$ binary matrix $E = [e_{ij}]$ corresponding to the ℓ-tuple $(u_1, \ldots, u_\ell)$, where t is the largest exponent appearing in the binary expansions of u_i, $i = 1, 2, \ldots, \ell$. Let

$$\gamma_{j-1} := \sum_{i=1}^{\ell} e_{ij}, \qquad 1 \le j \le t+1. \tag{8}$$

$$
\begin{aligned}
& k := t;\ \gamma'_t := \left\lfloor \frac{\gamma_t}{4} \right\rfloor; \quad (\lfloor x \rfloor \text{ is floor of } x) \\
& \text{while } \ k > 0 \ \text{ do} \\
& \qquad \{\beta_k := \gamma_k \pmod 4; \\
& \qquad\ k := k - 1; \\
& \qquad\ \gamma_k := \gamma_k + 2\beta_{k+1}; \\
& \qquad\ \text{if } \ k \neq 0 \ \text{ then} \\
& \qquad\qquad \gamma'_k := \left\lfloor \frac{\gamma_k}{4} \right\rfloor; \\
& \qquad\ \text{else} \\
& \qquad\qquad \gamma'_k := \gamma_k; \}
\end{aligned}
\tag{9}
$$

Now we apply Theorem 3 to the sequence $\left(\gamma'_0, 1_{(\gamma'_1)}, 2_{(\gamma'_2)}, \ldots, t_{(\gamma'_t)}\right)$. Thus, there is an integer m such that there is a

$$
COD\left(2^m d;\ 2^m \cdot 1_{(\gamma'_0)}, 2^m \cdot 2_{(4\gamma'_1)}, 2^m \cdot 2^2_{(4\gamma'_2)}, \ldots, 2^m \cdot 2^t_{(4\gamma'_t)}\right), \tag{10}
$$

where $m = N\left(4\sum_{j=1}^{t} \gamma'_j + \gamma'_0 + 2\right)$ if γ'_0 is even, and $m = N\left(4\sum_{j=1}^{t} \gamma'_j + \gamma'_0 + 1\right)$ if γ'_0 is odd.

Equating variables in (10) in an appropriate way, we obtain a

$$
COD\left(2^m d;\ 2^m u_1, \ldots, 2^m u_\ell\right).
$$

□

Lemma 3 *For any ℓ-tuple $(s_1, \ldots, s_\ell)$ of positive integers, there is an integer $r = r(s_1, \ldots, s_\ell)$ such that there is a*

$$
COD\left(2^r(s_1 + \cdots + s_\ell);\ 2^r s_1, \ldots, 2^r s_\ell\right).
$$

Proof. Suppose that $(s_1, \ldots, s_\ell)$ is an ℓ-tuple of positive integers and let

$$
(s_1, \ldots, s_\ell) = 2^q(u_1, \ldots, u_\ell),
$$

where q is the unique integer such that one of u_i's is odd. By Lemma 2, there exists an integer $m = m(u_1, \ldots, u_\ell)$ such that there is a

$$
COD\left(2^m(u_1 + \cdots + u_\ell);\ 2^m u_1, \ldots, 2^m u_\ell\right).
$$

Choose $r = m - q$, if $m \geq q$, and if $m < q$, then $A \otimes H$ is a

$$
COD\left(2^q(u_1 + \cdots + u_\ell);\ 2^q u_1, \ldots, 2^q u_\ell\right) = COD\left(s_1 + \cdots + s_\ell;\ s_1, \ldots, s_\ell\right),
$$

where H is a Hadamard matrix of order 2^{q-m}, and therefore we may choose $r = 0$ to complete the proof. □

Theorem 4 *For any ℓ-tuple $(s_1, \ldots, s_\ell)$ of positive integers, there is an integer $N = N(s_1, \ldots, s_\ell)$ such that for each $n \geq N$ there is an*

$$OD\Big(2^n(s_1 + \cdots + s_\ell);\ 2^n s_1, \ldots, 2^n s_\ell\Big).$$

Proof. Let $(s_1, \ldots, s_\ell)$ be a ℓ-tuple of positive integers. From Lemma 3, there is an integer $r = r(s_1, \ldots, s_\ell)$ such that there is a

$$COD\Big(2^r(s_1 + \cdots + s_\ell);\ 2^r s_1, \ldots, 2^r s_\ell\Big),$$

call it A. We may write $A = X + iY$, where X and Y are disjoint and amicable matrices such that $XX^t + YY^t = AA^*$. It can be seen that the matrix B,

$$B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes X + \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \otimes Y$$

is an

$$OD\Big(2^{r+1}(s_1 + \cdots + s_\ell);\ 2^{r+1}s_1, 2^{r+1}s_2, \ldots, 2^{r+1}s_\ell\Big).$$

Let $N = r + 1$, and H is a Hadamard matrix of order 2^{n-N}. Then $B \otimes H$ is an

$$OD\Big(2^n(s_1 + \cdots + s_\ell);\ 2^n s_1, \ldots, 2^n s_\ell\Big).$$

$\square$

Example 1 Consider the 5-tuple $(8, 12, 20, 68, 136)$. We may write this as $2^2(2, 3, 5, 17, 34)$. We apply the equation (7) to $(2, 3, 5, 17, 34)$ as follows:

$$\begin{bmatrix} 2 \\ 3 \\ 5 \\ 17 \\ 34 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2^2 \\ 2^3 \\ 2^4 \\ 2^5 \end{bmatrix}.$$

From the equation (8), we have $\gamma_0 = 3, \gamma_1 = 3, \gamma_2 = 1, \gamma_3 = 0, \gamma_4 = 1$ and $\gamma_5 = 1$. By applying the procedure (9), we find $\gamma'_0 = 5, \gamma'_1 = 1, \gamma'_2 = 1, \gamma'_3 = 1, \gamma'_4 = 0$ and $\gamma'_5 = 0$. So, we apply Theorem 3 to the sequence $(b, a_1, a_2, a_3) = (5, 1, 2, 3)$. Since b is odd, we use Case 2 of the theorem, and so $m = 4 \times 3 + 5 + 1 = 18$. $N(18) = 10$ as 10 is the smallest positive integer such that $18 \leq \rho\big(2^{10-1}\big)$. Thus there is a

$$COD\Big(2^{10} \cdot 61;\ 2^{10} \cdot 1_{(5)}, 2^{10} \cdot 2_{(4)}, 2^{10} \cdot 2^2_{(4)}, 2^{10} \cdot 2^3_{(4)}\Big).$$

By equating variables, we obtain a

$$COD\Big(2^8 \cdot 244;\ \ 2^8 \cdot 8,\ 2^8 \cdot 12,\ 2^8 \cdot 20,\ 2^8 \cdot 68,\ 2^8 \cdot 136\Big).$$

Example 2 We apply the equation (7) to the 4-tuple $(1, 5, 7, 17)$. Thus,

$$\begin{bmatrix} 1 \\ 5 \\ 7 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2^2 \\ 2^3 \\ 2^4 \end{bmatrix}.$$

From (8), we have $\gamma_0 = 4, \gamma_1 = 1, \gamma_2 = 2, \gamma_3 = 0, \gamma_4 = 1$. By applying the procedure (9), we find $\gamma'_0 = 6, \gamma'_1 = 1, \gamma'_2 = 1, \gamma'_3 = 0, \gamma'_4 = 0$. Now we apply Theorem 3 to the sequence $(b, a_1, a_2) = (6, 1, 2)$. Since b is even, we use Case 1 of Theorem 3, and so $m = 4 \times 2 + 6 + 2 = 16$. $N(16) = 8$ as 8 is the smallest positive integer such that $16 \leq \rho(2^{8-1})$. Thus there is a

$$COD\Big(2^8 \cdot 30;\;\; 2^8 \cdot 1_{(6)}, 2^8 \cdot 2_{(4)}, 2^8 \cdot 2^2_{(4)}\Big).$$

By equating variables, we obtain a

$$COD\Big(2^8 \cdot 30;\;\; 2^8 \cdot 1, 2^8 \cdot 5, 2^8 \cdot 7, 2^8 \cdot 17\Big).$$

3 The asymptotic existence of amicable orthogonal designs

We now include an asymptotic result related to the amicable orthogonal designs.

Lemma 4 *If there exists an* $ACOD(n;\; u_1, \ldots, u_s;\; v_1, \ldots, v_t)$*, then there exists an*

$$AOD(2n;\; 2u_1, \ldots, 2u_s;\; 2v_1, \ldots, 2v_t).$$

Proof. Suppose that $(X; Y)$ is a complex amicable orthogonal design. We write $X = A + iB$ and $Y = C + iD$, where A and B (C and D) are disjoint and amicable matrices such that $AA^t + BB^t = XX^*$ and $CC^t + DD^t = YY^*$. Let $R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Since $(X; Y)$ is a complex amicable orthogonal design,

$$AC^t + BD^t = CA^t + DB^t, \qquad AD^t - BC^t = CB^t - DA^t.$$

Let $X' = A \otimes RH + B \otimes H$ and $Y' = C \otimes RH + D \otimes H$. Then

$$X'Y'^t = 2\big(AC^t + BD^t\big) \otimes I + 2\big(AD^t - BC^t\big) \otimes R$$
$$Y'^tX' = 2\big(CA^t + DB^t\big) \otimes I + 2\big(CB^t - DA^t\big) \otimes R.$$

Therefore $(X'; Y')$ is an amicable orthogonal design as desired. □

We are now ready for the main result of this section.

Theorem 5 *For any two sequences $(u_1, \dots, u_s)$ and $(v_1, \dots, v_t)$ of positive integers, there are integers h, h_1, h_2 and N such that there exists an*

$$AOD\Big(2^n h;\ 2^{n+h_1}u_1, \dots, 2^{n+h_1}u_s;\ 2^{n+h_2}v_1, \dots, 2^{n+h_2}v_t\Big),$$

for each $n \geq N$.

Proof. Suppose that $(u_1, \dots, u_s)$ and $(v_1, \dots, v_t)$ are two sequences of positive integers. Let $(u_1, \dots, u_s) = 2^{q_1}(u'_1, \dots, u'_s)$ and $(v_1, \dots, v_t) = 2^{q_2}(v'_1, \dots, v'_t)$, where q_1 and q_2 are the unique integers such that at least one of u_i's and one of v_j's is odd.

Let $u'_1 + \cdots + u'_s = c_1$ and $v'_1 + \cdots + v'_t = c_2$. We may use the procedure (9) in the proof of Lemma 2 for sequences $(u'_1, \dots, u'_s)$ and $(v'_1, \dots, v'_t)$ to get sequences $(b, a_1, a_2, \dots, a_k)$ and $(\beta, \alpha_1, \alpha_2, \dots, \alpha_\ell)$ of positive integers, respectively. We have $c_1 = b + 4\sum_{i=1}^{k} 2^{a_i}$ and $c_2 = \beta + 4\sum_{i=1}^{\ell} 2^{\alpha_i}$. Without loss of generality we may assume that $c_1 \geq c_2$, and b and β are both even. Let $m = max\big\{4k+b+2, 4\ell+\beta+2\big\}$.

Suppose that $A = \big\{A_1, \dots, A_m\big\}$ and $B = \big\{B_1, \dots, B_m\big\}$ are the same set of matrices of order 2^{m-1} as in Theorem 2.

Apply Theorem 3 to the sequence $\big(b, a_1, a_2, \dots, a_k\big)$ by using the set A which contains matrices of order 2^{m-1} instead of the set A' in (3) which contains matrices of order $2^{N(m)-1}$. It can be seen that there is a COD, say C, of order $2^m c_1$ and type $\Big(2^m \cdot 1_{(b)}, 2^m \cdot 2^{a_1}_{(4)}, \dots, 2^m \cdot 2^{a_k}_{(4)}\Big)$.

Again apply Theorem 3 to the sequence $\big(\beta + c_1 - c_2, \alpha_1, \alpha_2, \dots, \alpha_\ell\big)$ by using the set B instead of the set A' in (3). It can be seen that there is a COD, say D, of order $2^m c_1$ and type $\Big(2^m \cdot 1_{(\beta)}, 2^m \cdot 2^{\alpha_1}_{(4)}, \dots, 2^m \cdot 2^{\alpha_\ell}_{(4)}\Big)$. Note that there is no need to use circulant matrices M_i's corresponding to the $c_1 - c_2$ variables to construct matrix D, and we do not necessarily need to use all matrices in sets A and B.

Since the circulant matrices used to construct C and D in (4) are Hermitian of order c_1 and $A_i B_j^t = B_j A_i^t$ for $1 \leq i, j \leq m$, $(C; D)$ is an

$$ACOD\Big(2^m c_1;\ 2^m \cdot 1_{(b)}, 2^m \cdot 2^{a_1}_{(4)}, \dots, 2^m \cdot 2^{a_k}_{(4)};\ 2^m \cdot 1_{(\beta)}, 2^m \cdot 2^{\alpha_1}_{(4)}, \dots, 2^m \cdot 2^{\alpha_\ell}_{(4)}\Big).$$

Equating variables in C and D in an appropriate way, we obtain an

$$ACOD\Big(2^m c_1;\ 2^m u'_1, \dots, 2^m u'_s;\ 2^m v'_1, \dots, 2^m v'_t\Big),$$

and so by Lemma 4, there exists an

$$AOD\Big(2^{m'} c_1;\ 2^{m'} u'_1, \dots, 2^{m'} u'_s;\ 2^{m'} v'_1, \dots, 2^{m'} v'_t\Big), \tag{11}$$

where $m' = m + 1$.

Now if $q_1 = q_2 = 0$, then we choose $h = c_1$, $h_1 = h_2 = 0$ and $N = m'$. If $q_1 \le q_2 \le m'$, then we choose $h = c_1$, $h_1 = -q_1$, $h_2 = -q_2$ and $N = m'$. For cases $q_1 \le m' \le q_2$ and $m' \le q_1 \le q_2$, the Kronecker product of a Hadamard matrix of order $2^{q_2 - m'}$ with the amicable orthogonal design (11) implies $h = 2^{q_2}c_1$, $h_1 = q_2 - q_1$ and $h_2 = N = 0$. Therefore, there exists an

$$AOD\Big(2^n h;\ 2^{n+h_1}u_1, \ldots, 2^{n+h_1}u_s;\ 2^{n+h_2}v_1, \ldots, 2^{n+h_2}v_t\Big),$$

for each $n \ge N$.

If β and b are not both even, then we may use Case 2 in Theorem 3 with a similar argument.

□

Example 3 Let $\big(u_1, u_2, u_3, u_4, u_5\big) = (8, 12, 20, 68, 136)$ and $\big(v_1, v_2, v_3, v_4\big) = (1, 5, 7, 17)$. We use the same notation as in the proof of Theorem 5. Thus, we have $\big(u'_1, u'_2, u'_3, u'_4, u'_5\big) = (2, 3, 5, 17, 34)$, $\big(v'_1, v'_2, v'_3, v'_4\big) = (1, 5, 7, 17)$, $q_1 = 2$, $q_2 = 0$, $c_1 = \sum_{i=1}^{5} u'_i = 61$, $c_2 = \sum_{i=1}^{4} v'_i = 30$ and $c_1 \ge c_2$.

In Examples 1 and 2, we applied the procedure (9) to the sequences

$$\big(u'_1, u'_2, u'_3, u'_4, u'_5\big) = (2, 3, 5, 17, 34) \quad \text{and} \quad \big(v'_1, v'_2, v'_3, v'_4\big) = (1, 5, 7, 17),$$

and we obtained the two sequences

$$\big(b, a_1, a_2, a_3\big) = (5, 1, 2, 3) \quad \text{and} \quad \big(\beta, \alpha_1, \alpha_2\big) = (6, 1, 2),$$

respectively. We may choose $m = \max\big\{4{\cdot}3{+}b{+}1, 4{\cdot}2{+}\beta{+}2\big\} = \max\big\{18, 16\big\} = 18$. Note that b is odd, and β is even. From the proof of Theorem 5, there is an

$$ACOD\Big(2^{18} \cdot 61;\ 2^{18} \cdot 1_{(5)}, 2^{18} \cdot 2_{(4)}, 2^{18} \cdot 2^2_{(4)}, 2^{18} \cdot 2^3_{(4)};\ 2^{18} \cdot 1_{(6)}, 2^{18} \cdot 2_{(4)}, 2^{18} \cdot 2^2_{(4)}\Big),$$

and so there is an

$$AOD\Big(2^{19} \cdot 61;\ 2^{19} \cdot 1_{(5)}, 2^{19} \cdot 2_{(4)}, 2^{19} \cdot 2^2_{(4)}, 2^{19} \cdot 2^3_{(4)};\ 2^{19} \cdot 1_{(6)}, 2^{19} \cdot 2_{(4)}, 2^{19} \cdot 2^2_{(4)}\Big).$$

Equating variables, we obtain an

$$AOD\Big(2^{19} \cdot 61;\ \ 2^{19} \cdot 2, 2^{19} \cdot 3, 2^{19} \cdot 5, 2^{19} \cdot 17, 2^{19} \cdot 34;\ \ 2^{19} \cdot 1, 2^{19} \cdot 5, 2^{19} \cdot 7, 2^{19} \cdot 17\Big).$$

Since $q_2 \le q_1 \le 19$, we choose $N = 19$, $h = 61$, $h_1 = -2$, $h_2 = 0$, and therefore for each $n \ge 19$, there exists an

$$AOD\Big(2^n{\cdot}61;\ \ 2^{n-2}{\cdot}8, 2^{n-2}{\cdot}12, 2^{n-2}{\cdot}20, 2^{n-2}{\cdot}68, 2^{n-2}{\cdot}136;\ \ 2^n{\cdot}1, 2^n{\cdot}5, 2^n{\cdot}7, 2^n{\cdot}17\Big).$$

Acknowledgments

The authors wish to thank the referees for their detailed comments which have considerably improved the presentation of the paper. Thanks to Professor Rob Craigen for his very useful comments.

References

[1] R. Craigen, Signed groups, sequences, and the asymptotic existence of Hadamard matrices, *J. Combin. Theory Ser. A* 71(2) (1995), 241–254.

[2] R. Craigen, W. H. Holzmann and H. Kharaghani, On the asymptotic existence of complex Hadamard matrices, *J. Combin. Des.* 5(5) (1997), 319–327.

[3] W. de Launey, On the asymptotic existence of Hadamard matrices, *J. Combin. Theory Ser. A* 116(4) (2009), 1002–1008.

[4] W. de Launey and D. Flannery, *Algebraic design theory*, vol. 175 of *Mathematical Surveys and Monographs*, Amer. Math. Soc., Providence, RI, 2011.

[5] W. de Launey and H. Kharaghani, On the asymptotic existence of cocyclic Hadamard matrices, *J. Combin. Theory Ser. A* 116(6) (2009), 1140–1153.

[6] P. Eades, Some asymptotic existence results for orthogonal designs, *Ars Combin.* 1 (1976), 109–118.

[7] P. Eades, *On the existence of orthogonal designs*, 1977, Ph.D. Thesis, The Australian National University, Canberra.

[8] P. Eades, P. J. Robinson, J. Seberry Wallis and I. S. Williams, An algorithm for orthogonal designs, In *Proc. Fifth Manitoba Conf. Numerical Math, Univ. Manitoba, Winnipeg, Man.*, 1975.

[9] A. V. Geramita and J. Seberry, *Orthogonal designs*, vol. 45 of *Lec. Notes Pure Appl. Math.*, Marcel Dekker Inc., New York, 1979. Quadratic forms and Hadamard matrices.

[10] W. H. Holzmann and H. Kharaghanir, Sequences and arrays involving free variables, *J. Combin. Des.* 9 (2001), 17–27.

[11] W. H. Holzmann and H. Kharaghani, On the amicability of orthogonal designs, *J. Combin. Des.* 17(3) (2009), 240–252.

[12] K. J. Horadam, *Hadamard matrices and their applications*, Princeton University Press, Princeton, NJ, 2007.

[13] P. J. Robinson, Using product designs to construct orthogonal designs, *Bull. Austral. Math. Soc.* 16(2) (1977), 297–305.

[14] D. B. Shapiro, *Similarities, quadratic forms, and clifford algebras*, ProQuest LLC, Ann Arbor, MI, 1974, Ph.D. Thesis, University of California, Berkeley.

[15] J. Seberry Wallis, On the existence of Hadamard matrices, *J. Combin. Theory Ser. A* 21(2) (1976), 188–195.

[16] W. W. Wolfe, *Orthogonal designs—amicable orthogonal designs, some algebraic and combinatorial techniques*, ProQuest LLC, Ann Arbor, MI, 1975, Ph.D. Thesis, Queen's University, Canada.

(Received 26 May 2013; revised 16 Oct 2013)