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Cycles through a given arc and certain partite sets in strong multipartite tournaments

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Abstract

Moon [*J. Combin. Inform. System Sci.* 19 (1994), 207–214] showed that every strong tournament contains a Hamiltonian cycle through at least three pancyclic arcs. In this paper, we extend the result of Moon and prove that if D is a strong c-partite tournament with $c \geq 3$, then D contains a cycle C containing vertices from exactly c partite sets such that C contains at least three arcs, each of which belongs to a cycle containing vertices from exactly l partite sets for each $l \in \{3, 4, \ldots, c\}$. In addition, this bound is best possible.

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1 Introduction and preliminaries

A *multipartite tournament* or *c-partite tournament* is an orientation of a complete c-partite graph. This paper will follow the notation and terminology defined in [2]. All digraphs are finite without loops or multiple arcs. The vertex set of a digraph D is denoted by $V(D)$ and the cardinality of $V(D)$ is denoted by $|V(D)|$. If xy is an arc of a digraph D, then we say that x *dominates* y and write $x \rightarrow y$. The arc set of a digraph D is denoted by $E(D)$. For disjoint subsets X and Y of $V(D)$, if every vertex of X dominates every vertex of Y, we say that X *dominates* Y, denoted by $X \rightarrow Y$. A digraph D is *strong* or *strongly connected* if for each ordered pair of vertices u and v, there is a path in D from u to v. A *strong component* of a digraph D is a maximal induced subdigraph of D that is strong.

An *m-cycle* is a cycle of length m and C_m denotes an m-cycle. A cycle in a digraph D is *Hamiltonian* if it includes all vertices of D. An arc is *pancyclic* in D if it belongs to an m-cycle for all $3 \leq m \leq |V(D)|$.

Tournaments are the best studied class of digraphs. In particular, some results on properties of cycles and paths in tournaments are beautiful and fascinating. In 1994, Moon [5] proved the following result on the number of pancyclic arcs for arbitrary strong tournaments.

Theorem 1.1. *Every non-trivial strong tournament contains a Hamiltonian cycle through at least three pancyclic arcs.*

In the last decade, more attention has been paid to multipartite tournaments. It is interesting to extend the most basic and important results on tournaments to multipartite tournaments. Since an arbitrary strong multipartite tournament does not necessarily have a Hamiltonian cycle, Theorem 1.1 is not valid for strong multipartite tournaments. Recently, an arc that belongs to an m-cycle for each $m \in \{3, 4, \ldots, c\}$ in c-partite tournaments has been introduced. In 2011, Li, Li, Guo and Guo [4] proved the following result for arbitrary strong multipartite tournaments.

Theorem 1.2. *If D is a strong c-partite tournament with $c \geq 3$, then D contains at least three arcs that belong to an m-cycle for each $m \in \{3, 4, \ldots, c\}$.*

As a generalization of Theorem 1.1, Theorem 1.2 also confirmed a conjecture posed by Volkmann [6]. In addition, the study of an arc that belongs to a cycle containing vertices from exactly l partite sets for each $l \in \{3, 4, \ldots, c\}$ in c-partite tournaments has attracted some attention. In 2002, Guo and Kwak [3] obtained the following result.

Theorem 1.3. *Let D be a regular c-partite tournament with $c \geq 3$, if the cardinality common to all the partite sets of D is odd, then every arc of D belongs to a cycle that contains vertices from exactly l partite sets for each $l \in \{3, 4, \ldots, c\}$.*

Theorem 1.3 implies Alspach′s Theorem [1] that every regular tournament is arc pancyclic. For arbitrary strong multipartite tournaments, our result is another generalization of Theorem 1.1.

Let D be a c-partite tournament where $c \geq 3$. If D' is a subdigraph in D, let $p(D')$ denotes the number of partite sets of D represented by the vertices of D'. Suppose that e is an arc of subdigraph D' in D and that $p(D') = k$, where $3 \leq k \leq c$. Let us say that e is a k-*special* arc (of D) in D' if for each $3 \leq i \leq k$, there exists a cycle $C(i)$ in the induced subdigraph $\langle V(D') \rangle$ such that $p(C(i)) = i$ and $C(i)$ contains the arc e. The following Lemma is useful for the proof of our main result.

Lemma 1.4. *Let e be an arc of a c-partite tournament D where $c \geq 3$. Suppose there exists a cycle C_i of length i for $3 \leq i \leq m$ such that*

(i) $V(C_3) \subset V(C_4) \subset \cdots \subset V(C_m)$;

(ii) $p(C_m) = k$, *for some k such that* $3 \leq k \leq min\{c, m\}$; *and*

(iii) $e \in E(C_i)$, *for* $3 \leq i \leq m$.

Then e is a k-special arc in C_m.

Proof. We may assume that $k > 3$. Since $|V(C_i)| = i$, for $3 \leq i \leq m$, and $V(C_i) \subset V(C_{i+1})$, for $3 \leq i \leq m-1$, it follows that $p(C_i) \leq p(C_{i+1}) \leq p(C_i) + 1$, for $3 \leq i \leq m-1$. Now $p(C_3) = 3$ and $p(C_m) = k$; consequently, the non-decreasing sequence $\{p(C_i) : 3 \leq i \leq m\}$ includes all the numbers in $\{3, 4, \ldots, k\}$. Let i_j denote the least integer i such that $p(C_i) = j$ for $3 \leq j \leq k$. Then the cycles $\{C_{i_j} : 3 \leq j \leq k\}$ satisfy the conditions for e to be a k-special arc in C_m. $\square$

2 Main results

To prove our result requires more terminology and notation. Let D be a strong c-partite tournament where $c \geq 3$. If C is a cycle in D, the *residual* set of C is the set of vertices of D that belong to partite sets not represented by vertices of C. An arc yz of the cycle C with residual set S has an S-*bypass*, if there is a vertex $w \in S$ such that $y \rightarrow w$ and $w \rightarrow z$. If $p(C) = k$, where $3 \leq k \leq c$, let $N(C)$ denote the number of k-special arcs in C. Finally, let $r = r(D)$ denote the largest integer such that there exists a strong subtournament T of D of order r. Then it follows from Theorem 1.1 that T contains a Hamiltonian cycle through at least three pancyclic arcs and let C_r denote the Hamiltonian cycle of T.

Theorem 2.1. *Let D be a strong c-partite tournament with $c \geq 3$. Then there exists a sequence of cycles $C(k)$ in D, $r \leq k \leq c$, such that $C(r) = C_r$ and*

(i) $V(C(r)) \subset V(C(r+1)) \subset \cdots \subset V(C(c))$;

(ii) $p(C(k)) = k$, *for* $r \leq k \leq c$; *and*

(iii) $N(C(k)) \geq 3$, *for* $r \leq k \leq c$.

Proof. By Theorem 1.1, C_r contains at least three pancyclic arcs. So, $N(C(r)) \geq 3$ and $C(r) = C_r$ satisfies conclusions *(i)–(iii)*. We may assume that $r < c$ and that the required conclusions have been proved for all k such that $r \leq k \leq h$ for some h such that $h < c$. Let $C = C(h) = v_1v_2 \cdots v_mv_1$ denote the cycle that satisfies the conclusions when $k = h$ where, clearly, $h \leq m$. We will extend $C(h)$ to a longer cycle $C^* = C(h+1)$ that satisfies the conclusions when $k = h+1$. Let S be the residual set of C. In the following, our argument falls into two cases.

Case 1. No arc of C has an S-bypass.

Obviously, $S \neq \emptyset$ and S has no vertex that has an out-neighbor and an in-neighbor on C. Then S can be decomposed into two sets S_1 and S_2 such that $S_2 \to V(C) \to S_1$. We may assume, without loss of generality, that $S_1 \neq \emptyset$. Since D is strongly connected, there is a path from S_1 to C. Let $P = y_1y_2 \cdots y_q$ be any shortest such path, where necessarily $q \geq 3$.

Subcase 1.1. $V(P) \cap S_2 \neq \emptyset$.

It is easy to see that $V(P) \cap S_2 = \{y_{q-1}\}$. If $q = 3$, then the induced subdigraph $\langle \{y_1, y_2\} \cup V(C_r) \rangle$ is a strong sub-tournament of order $r+2$, a contradiction. Therefore, we have $q \geq 4$. Without loss of generality, we assume that y_{q-2} and v_m are in distinct partite sets, which implies that $v_m \to y_{q-2}$. Now, $C_{m-i+3} = v_my_{q-2}y_{q-1}v_iv_{i+1} \cdots v_m$ is an $(m-i+3)$-cycle through arc v_my_{q-2} for every $1 \leq i \leq m$. Obviously, $V(C_3) \subset V(C_4) \subset \cdots \subset V(C_{m+2})$ and $p(C_{m+2}) = h+1$. Let $e = v_my_{q-2}$ and $C^* = C(h+1) = C_{m+2} = v_my_{q-2}y_{q-1}v_1v_2 \cdots v_m$. By Lemma 1.4, e is an $(h+1)$-special arc in C^*. Since $E(C)\backslash\{v_mv_1\} \subset E(C^*)$ and $e \notin E(C)$, it follows that $N(C^*) \geq N(C)$. Then the cycle $C^* = C(h+1)$ is the desired cycle.

Subcase 1.2. $V(P) \cap S_2 = \emptyset$.

It follows readily from the definitions of S_1 and P that $y_i \to y_1$ for $3 \leq i \leq q$. Without loss of generality, let $y_q = v_m$. We obtain an i-cycle $C_i = y_1y_2 \cdots y_iy_1$ through the arc y_1y_2 for every $3 \leq i \leq q$, and a $(q+j)$-cycle $C_{q+j} = y_1y_2 \cdots y_qv_1v_2 \cdots v_jy_1$ through the arc y_1y_2 for every $1 \leq j \leq m-1$. Obviously, $V(C_3) \subset V(C_4) \subset \cdots \subset V(C_{m+q-1})$ and $p(C_{m+q-1}) = h+1$. Let $e = y_1y_2$ and $C^* = C(h+1) = C_{m+q-1} = y_1y_2 \cdots y_qv_1v_2 \cdots v_{m-1}y_1$. Then $E(C)\backslash\{v_{m-1}v_m\} \subset E(C^*)$ and $e \notin E(C)$. The remaining details are the same as in Subcase 1.1.

Case 2. At least one arc yz of C has an S-bypass.

Suppose that w is a vertex of S such that $y \to w$ and $w \to z$. Let C^* denote the cycle obtained from C by replacing the arc yz by arcs yw and wz. Then $p(C^*) = h+1$ since C^* contains all the vertices of C plus the vertex w from S. Let pq be any h-special arc in C. If $pq \neq yz$, then pq belongs to the cycle C^* and, hence, since $V(C) \subset C(C^*)$, pq is an $(h+1)$-special arc in C^*; so all the $N(C)$ h-special arcs in C are $(h+1)$-special arcs in C^* with the possible exception of yz.

Let us suppose that yz is an h-special arc in C. Then there are $h-2$ cycles C' in $\langle V(C)\rangle$ that contain the arc yz and are such that the number $P(C')$ take on the values $3, 4, \ldots, h$. These cycles can be transformed into cycles C'' in $\langle V(C^*)\rangle$ that contain the arcs yw and wz, instead of yz, and are such that the number $P(C'')$ take on the values $4, 5, \ldots, h+1$. Now consider the cycle C' containing yz such that $P(C') = 3$. It follows from Theorem 1.1 and the proof of Lemma 1.4 that we may assume that C' is a 3-cycle, i.e.,that $C' = yziy$ for some vertex i of C, where $i \neq y$ or z. Since w is a vertex of S, vertices w and i are in distinct partite sets. If $w \to i$, the arc yw belongs to the 3-cycle $ywiy$; and if $i \to w$, the arc wz belongs to the 3-cycle $wziw$. So at least one of the arcs yw and wz belongs to a 3-cycle C'' in $\langle V(C^*)\rangle$ in addition to the longer transformed cycles C'' already mentioned. Hence, if yz is one of the $N(C(h))$ h-special arcs in C, then either yw or wz is an $(h+1)$-special arc in C^*. Combining these observations, we see that $N(C^*) \geq N(C)$ whether or not yz is an h-special arc in C. Consequently, the cycle C^* satisfies the required conclusions when $k = h+1$. This sufficies to complete the proof of the theorem by induction. □

From the proof of Theorem 2.1, we obtain the following result.

Corollary 2.2. *If D is a strong c-partite tournament with $c \geq 3$, then D contains a cycle C such that $p(C) = c$ and C contains at least three c-special arcs in D.*

Corollary 2.2 is a generalization of Theorem 1.1. In addition, the bound is best possible in the sense that there exists a strong c-partite tournament D such that each cycle C with $p(C) = c$ in D contains exactly three c-special arcs in D .

Example 2.3. *Let $A_1, A_2, \ldots, A_c$ be the partite sets of a c-partite tournament H with $c \geq 5$ such that $|A_1| = |A_2| = |A_3| = |A_c| = 1$, say $A_i = \{a_i\}$ for $i \in \{1, 2, 3, c\}$. If $A_1 \to A_2$, $A_i \to A_1$ for $3 \leq i \leq c$, and $A_i \to A_j$ for $2 \leq i < j \leq c$, then H is strong and each cycle C with $p(C) = c$ contains only three c-special arcs (a_1a_2, a_2a_3, a_ca_1) in H.*

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