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A note on quasi-symmetric designs

Rajendra M. Pawale

Department of Mathematics
University of Mumbai
2nd Floor, Ranade Bhavan, Vidyanagari, Santacruz (East)
Mumbai-400 098
India
rmpawale@yahoo.co.in

Abstract

A quasi-symmetric design is a (v, k, λ) design with two intersection numbers x, y where $0 \leq x < y < k$. We show that for fixed x, y, λ with $x > 1, \lambda > 1, y \neq \lambda$ and $\lambda\,(4xy + ((y-x)^2 - 2x - 2y + 1)\,\lambda)$ a perfect square of a positive integer, there exist finitely many quasi-symmetric designs. We rule out the possibilities of quasi-symmetric designs corresponding to $y = x + 3$ and $(\lambda, x) = (9, 2), (8, 3)$ and prove that the only feasible parameters associated with $x = 2$, $y = 5, \lambda = 8$ and $x = 4$, $y = 7$, $\lambda = 9$ are $v = 1001$, $k = 65$ and $v = 4642$, $k = 154$ respectively. These are some of the open cases reported in the classification of quasi-symmetric designs with $y = x + 3$, $x > 0$ obtained by Mavron, McDonough and Shrikhande [*Des. Codes Crypto.* 63 (2012) no. 1, 73–86].

The technique used rules out the possibilities of quasi-symmetric designs corresponding to $x = 1$ and $\lambda = (y-1)^2$ for $y \geq 4$.

1 Introduction

Let X be a finite set of v elements called points, and β be a set of k-element subsets of X called blocks, such that each pair of points occurs in λ blocks. Then the pair $D = (X, \beta)$ is called a 2-(v, k, λ) design. For a 2-(v, k, λ) design D, the number of blocks containing α in X is r, which is independent of α. The number of blocks in D is denoted by b.

A number x, for $0 \leq x < k$, is called an intersection number of D if there exist $B, B' \in \beta$ such that $|B \cap B'| = x$. A 2-design with two intersection numbers is said to be a quasi-symmetric design. These intersection numbers are denoted by x and y. Assume $0 \leq x < y < k$. We consider a proper quasi-symmetric design, i.e. both the intersection numbers occur. The parameters $(v, b, r, k, \lambda; x, y)$ are called standard parameters of a quasi-symmetric design. The standard parameters are called *feasible* if they satisfy necessary conditions given in Lemma 1. The following are well-known results in quasi-symmetric designs.

Lemma 1 ([1, 2, 6, 8]). *Let D be a quasi-symmetric design, with the standard parameter set $(v, b, r, k, \lambda; x, y)$. Then the following relations hold:*

1. $vr = bk$ *and* $\lambda(v-1) = r(k-1)$.
2. $y - x$ *divides* $k - x$ *and* $r - \lambda$.
3. $$(k-1)xyr^2 + (xy - k^2(x+y-1))\,\lambda r + k\lambda\,(k^2(\lambda-1) + k(x+y-\lambda) - xy) = 0. \tag{1.1}$$
4. *If* $\lambda > 1$ *then* $x < \lambda$.

Lemma 2 ([8], Theorem 2.6). *For a fixed k, there are only a finite number of quasi-symmetric designs with $y \geq 2$.*

Theorem 3 ([6], Theorem 3.2). *Let D be a proper quasi-symmetric design with the standard parameter set $(v, b, r, k, \lambda; x, y)$ with $x \neq 0$ and $y = x + 1$. Then D is a design with parameters given in (1) or (2) as follows or D is a complement of one of the design in (1).*

(1) $v = (1+m)(2+m)/2$, $b = (2+m)(3+m)/2$, $r = m+3$, $k = m+1$, $\lambda = 2$, $x = 1$, $y = 2$ *and* $m = 2, 3, \ldots$.

(2) $v = 5$, $b = 10$, $r = 6$, $k = 3$, $\lambda = 3$ *and* $x = 1$, $y = 2$.

We fix x, y, λ and observe that k satisfies the Diophantine equation $T^2 = g(k)$, related to the discriminant of the quadratic equation (1.1) in r, where $g(k)$ is a polynomial in $\mathbb{Z}[k]$ of degree at most four. There are celebrated results by Baker, Faltings and Siegel (see Cohen [3]), which suggest that the Diophantine equation $T^2 = g(k)$ has only a finite number of integer solutions when degree of $g(k) > 1$. Finding these solutions is a difficult problem in number theory, the exception being the case when the leading coefficient of g is a perfect square of an integer. We explore this idea to obtain finiteness results and also prove non-existence of some quasi-symmetric designs.

The coefficient of highest power of k in $g(k)$ is $\lambda\,(4xy + ((y-x)^2 - 2x - 2y + 1)\,\lambda)$.

The set of feasible parameters of quasi-symmetric designs corresponding to $x = 1$, $y = 2$ and $\lambda = 2$ in Theorem 3 (1) is a well-known series. In [7], the author obtains the following two infinite series of feasible parameters of quasi-symmetric designs corresponding to $x = 1$, $y = 3$ and $\lambda = 4$ which are

$$r = 2\left(t^2 - t\right),\ b = t\left(t^3 - 2t^2 + 2\right),\ v = \frac{1}{2}(t+1)\left(t^3 - 2t^2 + 2\right)$$

and

$$r = 2\left(t^2 + t\right),\ b = t\left(t^3 + 2t^2 - 2\right),\ v = \frac{1}{2}(t-1)\left(t^3 + 2t^2 - 2\right)$$

where $k = t^2 - 1$ and t is a non-negative even integer.

In view of this, it is interesting to know the values of x, y and λ with $x \geq 1, \lambda > 1$, which associates to finitely many quasi-symmetric designs. Note that $\lambda\left(4xy + \left((y-x)^2 - 2x - 2y + 1\right)\lambda\right) = 0$ for the above mentioned series.

The Computer Algebra System MAXIMA [5] is used to simplify symbolic calculations given in Section 2, which contains our main results.

2 Main Results

Theorem 4. *There are only a finite number of quasi-symmetric designs with fixed x, y, λ with $x > 1$, $\lambda > 1$, $y \neq \lambda$ and $t = \sqrt{\lambda\left(4xy + \left((y-x)^2 - 2x - 2y + 1\right)\lambda\right)}$ a positive integer.*

Proof. In view of Theorem 3 we may assume $y \geq x + 2$. Consider $\Delta_1 = t^6\Delta$, where Δ is the discriminant of the quadratic (1.1). Take

$$\begin{aligned}\theta &= k^2t^4 - 2kxy(x+y-2\lambda+1)\lambda t^2 \\ &\quad + xy\lambda\left(t^2(2yx + 2x + 2y - (x+y+1)\lambda) - 2xy(x+y-2\lambda+1)^2\lambda\right)\end{aligned}$$

and observe the following:

1. $\dfrac{\Delta_1 - \theta^2}{4x^2y^2\lambda^3} = (x-\lambda)(y-\lambda)(\lambda-1)\left(2kt^2(x-y-1)(x-y+1)(x+y-1) + \zeta\right)$, where ζ is an expression in x, y and λ.

2. $\Delta_1 - (\theta+1)^2$ is a polynomial in k of degree two, with the coefficient of k^2 being $-2\lambda^2\left(4xy + \left((y-x)^2 - 2x - 2y + 1\right)\lambda\right)^2$.

3. $\Delta_1 - (\theta-1)^2$ is a polynomial in k of degree two, with the coefficient of k^2 being $2\lambda^2\left(4xy + \left((y-x)^2 - 2x - 2y + 1\right)\lambda\right)^2$.

If $\Delta_1 - \theta^2 = 0$, then either $\lambda = x$ or $\lambda = y$ or $\lambda = 1$ or k is a function of x, y and λ, as $y \geq x + 2$. Our assumption and (4) of Lemma 1 rules out the possibilities $\lambda = x$, $\lambda = y$ and $\lambda = 1$. Hence, k is a function of x, y and λ.

If $\Delta_1 - \theta^2 > 0$ then $\theta^2 < \Delta_1 < (\theta+1)^2$ and if $\Delta_1 - \theta^2 < 0$ then $(\theta-1)^2 < \Delta_1 < \theta^2$, for k bigger than some function of x, y and λ.

Since Δ is a perfect square, Δ_1 must be a perfect square. Hence k is bounded by some function of x, y and λ. Use Lemma 2 to complete the proof. ☐

Corollary 5. *The number of quasi-symmetric designs is finite under each of the following conditions:*

1. *for integers $u \geq 3$ such that $x = u^2 - 4, y = u^2 - 1, \lambda = u^2$;*
2. *for positive integers u, s with $s > u + 1$, and $x = u^2$, $y = (u+1)^2, \lambda = s^2$;*
3. *for x, y such that $x > 1$ and $\lambda = (y-x)^2$.*

Proof. In all cases observe that the value of t defined in Theorem 4 is a positive integer. $\square$

Theorem 6. *There does not exist a quasi-symmetric designs with $x = 1$ and $\lambda = (y-1)^2$ for $y \geq 4$.*

Proof. By Lemma 1 (2), there exists a positive integer m such that $k = m(y-1)+1$. Substitute $x = 1$, $k = m(y-1)+1$ and $\lambda = (y-1)^2$ in the equation (1.1) and observe that

$$(ym-2m+1)(ym-m+1)(y-1)^3 - r(ym-m+2)(y-1)^2 + r^2 = 0.$$

The discriminant Δ of this quadratic in r is $(y-1)^2\Delta_1$, where

$$\Delta_1 = 4m(y-1)(y-2)^2 + 4(y-1)(y-2) + m^2(y-3)^2(y-1)^2.$$

For $m = 2$, observe that $(2y^2-6y+4)^2 < \Delta_1 < (2y^2-6y+5)^2$; Δ_1 is not a perfect square. Hence $m \geq 3$. Observe that for $\theta = m(y-1)(y-3)^2 + 2(y-2)^2$,

$$\Delta_1(y-3)^2 - \theta^2 = -4(y-2)\left(y^2-3y+1\right)$$

whereas

$$\Delta_1(y-3)^2 - (\theta-1)^2 = 2m(y-1)(y-3)^2 - 4y^3 + 24y^2 - 44y + 23.$$

Hence, $(\theta-1)^2 < \Delta_1(y-3)^2 < \theta^2$ for $y \geq 6$ and $m \geq 3$.

Now for $y = 4$, $\Delta_1 = 24 + 48m + 9m^2$. Observe that $(3m+7)^2 < \Delta_1 < (3m+8)^2$ for $m \geq 5$. Note that for $y = 4$ and $m = 3, 4$, Δ_1 is not a perfect square.

Also for $y = 5$, $\Delta_1 = 16(3 + 9m + 4m^2)$. Observe that $(8m+8)^2 < \Delta_1 < (8m+9)^2$.

Hence $y \neq 4, 5$. $\square$

In [4], Mavron, McDonough and Shrikhande have given a parametric classification of quasi-symmetric designs with $y = x + 3$, $x > 0$, where (λ, x) associated with pairs $(7,2)$, $(8,2)$, $(9,2)$, $(10,2)$, $(8,3)$, $(9,3)$, $(9,4)$, $(10,5)$ is one of the open cases reported. Out of these, quasi-symmetric designs associated with pairs $(8,2)$, $(9,2)$, $(9,4)$, $(8,3)$ are settled in the following theorems.

Theorem 7. *The only feasible parameters of quasi-symmetric designs associated with $x = 2$, $y = 5$, $\lambda = 8$ are $v = 1001$, $k = 65$, $r = 125$, $b = 1925$ and with $x = 4$, $y = 7$, $\lambda = 9$ are $v = 4642$, $k = 154$, $r = 273$, $b = 8229$.*

Proof. Substituting $x = 2$, $y = 5$, $\lambda = 8$ in the equation (1.1), we get $\Delta = 64\Delta_1$, where $\Delta_1 = k^4 + 40k^3 - 75k^2 - 50k + 100$. Observe that $(k^2+20k-238)^2 < \Delta_1 < (k^2+20k-237)^2$ for $k \geq 9430$. Hence $k < 9430$. Simple calculations show that Δ_1 is a perfect square only for $k = 15$ and $k = 65$. For $k = 15$ we get $r = 50$, $v = 177/2$, and for $k = 65$, $r = 125$, $v = 1001$, $b = 1925$.

When $x = 4$, $y = 7$, $\lambda = 9$ where $\Delta = 36\Delta_2$, $\Delta_2 = k^4 + 168k^3 - 420k^2 - 784k + 1764$. In this case $(k^2 + 84k - 3738)^2 < \Delta_2 < (k^2 + 84k - 3737)^2$ for $k \geq 313516$. Hence $k < 313516$. Feasible value of k is 154. The remaining parameters are obtained using equation (1.1) and Lemma 1. □

Theorem 8. *There do not exist quasi-symmetric designs for the following cases:*

1. $x = 2$, $y = 5$ *and* $\lambda = 9$.
2. $x = 3$, $y = 6$ *and* $\lambda = 8$.

Proof. Substitute $x = 2$, $y = 5$, $\lambda = 9$ in the equation (1.1) and find the discriminant $\Delta = 36\Delta_2$, where $\Delta_2 = k^4 + 100k^3 - 190k^2 - 100k + 225$. Observe that, for $k \geq 67150$, $(k^2 + 50k - 1345)^2 < \Delta_2 < (k^2 + 50k - 1344)^2$. The discriminant must be a perfect square, and hence $k < 67150$. Simple calculations show that Δ_2 is not a perfect square for k with $5 < k < 67150$.

When $x = 3$, $y = 6$, $\lambda = 8$, the discriminant $\Delta = 64\Delta_2$, where $\Delta_2 = k^4 + 54k^3 - 117k^2 - 162k + 324$. Here $(k^2 + 27k - 423)^2 < \Delta_2 < (k^2 + 27k - 422)^2$ for $k \geq 11313$. As before we see that Δ_2 is not a perfect square for k with $6 < k < 11313$, except for $k = 51$. Using equation (1.1) and Lemma 1 we get $r = 76, v = 476$ but $b = 2128/3$. This excludes the possibility of quasi-symmetric designs with $x = 3$, $y = 6$, $\lambda = 8$. □

Remark 9. On substituting $x = 1$ in the equation (1.1) we obtain $f_1(r) = 0$, where $f_1(r) = yr^2 - (k+1)y\lambda r + \lambda^2 k^2 - \lambda k^2 + y\lambda k$.

Now $\Delta = y\lambda(y\lambda - 4\lambda + 4)k^2 + 2y^2(\lambda - 2)\lambda k + y^2\lambda^2$, the discriminant of the quadratic in r, $f_1(r) = 0$. Clearly Δ is a perfect square of an integer. Take $\Delta = T^2$, for some positive integer T, and $\alpha = \lambda^2 y^2 - 2\lambda y^2 + k\lambda(y\lambda - 4\lambda + 4)y$; we find that $y\lambda(y\lambda - 4\lambda + 4)T^2 - \alpha^2 = 4y^3(y - \lambda)(\lambda - 1)\lambda^2$. This is the *Pell-Fermat* type equation, which can be solved for integer solutions (see Section 6.3.5 of Cohen [3]).

Further, if $t = \sqrt{y\lambda(y\lambda - 4\lambda + 4)}$ is a positive integer then by following a similar procedure, as given in the proof of Theorem 4, we see k is bounded by a function of y and λ.

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