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Palettes in block colourings of designs

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Abstract

For any proper block colouring of a Steiner system, a palette of an element is the set of colours on blocks incident with it. We obtain bounds on the minimum possible number of distinct palettes in proper block colourings of Steiner triple systems and Steiner systems $\mathrm{S}(2,4,v)$.

1 Introduction

A block colouring of a design $(V,\mathcal{B})$ is a mapping $f:\mathcal{B}\to C$ where C is the set of colours. A block colouring is proper if for $B,B'\in\mathcal{B}$, $f(B)\neq f(B')$ whenever $B\cap B'\neq\emptyset$.

Recently, Horňák et al. [8] have introduced an interesting concept of a palette in edge-colourings of graphs and an associated parameter called the palette index of a graph.

By analogy, given a design $(V, \mathcal{B})$ and its proper colouring f, we define the *palette* of an element $v \in V$ with respect to f to be the set $S_f(v)$ of colours of blocks containing v. The minimum number $p(V, \mathcal{B})$ of distinct palettes taken over all possible block colourings of a design $(V, \mathcal{B})$ is called the *palette index* of $(V, \mathcal{B})$.

The only types of designs that will be considered in this paper are Steiner systems $\mathrm{S}(2,3,v)$ (Steiner triple systems of order v, STS(v)), and Steiner systems $\mathrm{S}(2,4,v)$. An STS(v) is a pair $(V, \mathcal{B})$ where V is a finite set, and $\mathcal{B}$ is a collection of 3-element subsets of V (called blocks or triples) such that every 2-subset of V is contained in exactly one triple. Similarly, a Steiner system $\mathrm{S}(2,4,v)$ is a pair $(V, \mathcal{B})$ where V is a finite set, and $\mathcal{B}$ is a collection of 4-elements subsets of V (called blocks) such that every 2-subset of V is contained in exactly one block. It is well known that an STS(v) exists if and only if $v \equiv 1$ or $3 \pmod 6$), and an $\mathrm{S}(2,4,v)$ exists if and only if $v \equiv 1$ or $4 \pmod{12}$.

2 Palette indices of Steiner triple systems

Define $p(v) = \min\ p(V, \mathcal{B})$ where the minimum is taken over all Steiner triple systems on V.

Example 1. For the unique STS(7), a proper colouring must colour each of the seven blocks with a different colour. Clearly, any two palettes are distinct, and thus $p(7) = 7$. On the other hand, the unique STS(9) is resolvable, and a proper colouring (with 4 colours) is obtained by colouring blocks in the same parallel class by the same colour. Clearly, for such a colouring all palettes are the same, and thus $p(9) = 1$.

The following is now immediate.

Lemma 2.1 *For a design $(V, \mathcal{B})$, we have $p(V, \mathcal{B}) = 1$ if and only if $(V, \mathcal{B})$ is resolvable.*

Corollary 2.2 *$p(v) = 1$ if and only if $v \equiv 3 \pmod 6$.*

Thus it remains to consider STSs of order $v \equiv 1 \pmod 6$. By Corollary 2.2, $p(v)$ can never equal 1. We show first that in this case the palette index cannot equal 2 or 3 either.

Lemma 2.3 *For every STS(v) $(V, \mathcal{B})$, $p(V, \mathcal{B}) \neq 2$.*

Proof: Suppose that $p(V, \mathcal{B}) = 2$ and f is a coloring yielding two distinct palettes S_f^1 and S_f^2. Then V can be partitioned into two subsets V_1 and V_2 induced by palettes S_f^1 and S_f^2, respectively, i.e. $S_f^1 = S_f(v)$ for each $v \in V_1$, and $S_f^2 = S_f(u)$ for each $u \in V_2$. W.l.o.g., we may assume that a coloring f is chosen in such way that the set

$S_f^1 \setminus S_f^2$ has the minimum possible cardinality. Since $|S_f^1| = |S_f^2|$ then there exists a color $c_1 \in S_f^1 \setminus S_f^2$ and a color $c_2 \in S_f^2 \setminus S_f^1$. Triples colored with c_1 form a parallel class on V_1; similarly triples colored with c_2 form a parallel class on V_2. Thus $|V_1| \equiv 0 \pmod 3$ and $|V_2| \equiv 0 \pmod 3$. To reduce the number of colors in $(S_f^1 \setminus S_f^2) \cup (S_f^2 \setminus S_f^1)$ we recolor triples colored with c_2 by replacing c_2 with c_1. Thus we get a new coloring f' such that $|S_{f'}^1 \setminus S_{f'}^2| < |S_f^1 \setminus S_f^2|$, a contradiction. □

Lemma 2.4 *If $v \equiv 1 \pmod 6$ then every STS(v) $(V, \mathcal{B})$ has $p(V, \mathcal{B}) \geq 4$.*

PROOF: By Corollary 2.2 and Lemma 2.3, $p(V, \mathcal{B}) \geq 3$. Suppose that f is a coloring yielding exactly three palettes S_f^1, S_f^2 and S_f^3 . Then V can be partitioned into three subsets V_1, V_2 and V_3, induced by palettes S_f^1, S_f^2 and S_f^3, respectively. Obviously, no color is used in all three different types of palettes since otherwise triples colored with such a color would form a parallel class on V but then $|V| \equiv 0 \pmod 3$, a contradiction. To color triples with vertices in two different sets V_i and V_j, $i \neq j$, we use colors which are present in two different palettes. Suppose that $c \in (S_f^1 \cup S_f^2) \setminus S_f^3$. The triples colored with c form a parallel class on $V_1 \cup V_2$ implying $|V_1| + |V_2| \equiv 0 \pmod 3$. By the same argument we get $|V_1| \cup |V_3| \equiv 0 \pmod 3$ and $|V_2| \cup |V_3| \equiv 0 \pmod 3$. Thus $2(|V_1| + |V_2| + |V_3|) \equiv 0 \pmod 3$, a contradiction. □

The *chromatic index* χ' of an STS(v) is the minimum number of colours in any proper block colouring. It is well known that $\chi'(V, \mathcal{B}) \geq \frac{v+1}{2}$ whenever $v \equiv 1 \pmod 6$, and equality can be attained if and only if $v \geq 19$ [4].

Our first result gives an upper bound on $p(v)$.

Lemma 2.5 $p(v) \leq \frac{v+1}{2}$ *whenever $v \geq 19$.*

PROOF: Let $(V, \mathcal{B}, \mathcal{R})$ be a Hanani triple system (cf. [4]) of order v which is known to exist if and only if $v \equiv 1 \pmod 6), v \geq 19$. Here $\mathcal{R} = \{R_i, \ldots, R_{\frac{v+1}{2}}\}$ where $R_i, i = 1, \ldots, \frac{v-1}{2}$ are almost parallel classes, and $R_{\frac{v+1}{2}}$ is the half-parallel class. Colour the blocks of each R_i in $\mathcal{R}$ with one colour. In such a colouring, the palettes of the $\frac{v+1}{2}$ elements *not* in the half-parallel class are all the same, while the palettes of the $\frac{v-1}{2}$ elements of the half-parallel class are all distinct. □

Alternatively, we could have used the existence of nearly Kirkman triple systems of order $v \equiv 1 \pmod 6$, $v \geq 19$, to obtain the same bound. But such a block colouring would use $v - 1$ colours in total, while the colouring given in the proof above uses $\frac{v+1}{2}$ colours, the minimum possible number of colours.

Example 2. Let S_1 be the cyclic STS(13) (with full automorphism group of order 39), and let S_2 be the non-cyclic STS(13) (with automorphism group of order 6). Using a computer we established that $p(S_1) = 8$, $p(S_2) = 7$.

Steiner triple systems of order 13 need to be considered separately, as the bound given in Lemma 2.5 does not apply: neither Hanani triple systems nor nearly Kirkman triple systems exist for order 13. Nevertheless, Example 2 above seems to suggest that the bound given in Lemma 2.5 may be close to the minimal value of the actual palette index. But, actually, we are able to show that $p(v)$ never exceeds a constant.

Our main tool in this is the following variant of a well-known recursive construction (cf. [4]).

Construction A. Let $(V, \mathcal{B})$ be an STS(v) with a sub-STS(w) $(W, \mathcal{C})$. Let $Y = (V \setminus W) \times \{1, 2, 3\}$, $X = Y \cup W$, and let $(Y, \mathcal{D})$ be a 3-GDD (with groups $(V \setminus W) \times \{i\}, i = 1, 2, 3$ and blocks $\mathcal{D}$). Let $\mathcal{E} = ((\mathcal{B} \setminus \mathcal{C}) \times \{1, 2, 3\}) \cup \mathcal{C} \cup \mathcal{D}$. Then $(X, \mathcal{E})$ is an STS($3v - 2w$) with a sub-STS(w).

Theorem 2.6 *Suppose that in the above Construction A, both the STS*(v) $(V, \mathcal{B})$ *and the* 3*-GDD* $(Y, \mathcal{D})$ *are resolvable. Then* $p(Y, \mathcal{E}) \leq w + 3$.

Proof: Let $R_1^i, \ldots, R_{\frac{v-1}{2}}^i$ be the parallel classes in (any) resolution of $(V \setminus W) \times \{i\} \cup W, ((\mathcal{B} \setminus \mathcal{C}) \times \{i\}) \cup \mathcal{C})$, and let $S_1, \ldots, S_{v-w}$ be the parallel classes in (any) resolution of $(Y, \mathcal{D})$. Colour now the blocks in the parallel class R_j^1 with colour j, the blocks in the parallel class R_j^2 (except for blocks contained entirely in W) with colour $j + \frac{v-1}{2}$, and the blocks in the parallel class R_j^3 (except the blocks contained entirely in W) with colour $j + v - 1$. Finally, colour the blocks of the parallel class S_j with the colour $j + 3\frac{v-1}{2}$. In such a colouring, for each $i \in \{1, 2, 3\}$, the palettes of each element in $(V \setminus W) \times \{i\}$ are the same while the palettes of the elements of W may or may not be all distinct. Thus the total number of distinct palettes of $(Y, \mathcal{E})$ cannot exceed $w + 3$. □

Lemma 2.7 *Let* $v \equiv 7 \pmod{18}$, $v \geq 25$. *Then* $p(v) = 4$.

Proof: A classical result of Ray-Chaudhuri and Wilson (cf. [4]) states that for every $v \equiv 3 \pmod 6$, there exists a resolvable STS(v). The existence of a resolvable 3-GDD with three groups of size $v - w$ each is equivalent to the existence of a pair of orthogonal latin squares of order $v - w$. Now apply Theorem 2.6 with $|W| = 1$ (that is, with a trivial subsystem of order 1); this shows that $p(v) \leq 4$. Together with Lemma 2.4 we get in this case $p(v) = 4$. □

In order to proceed to the case of $v \equiv 13 \pmod{18}$, we need first to establish the following result.

Lemma 2.8 *A resolvable STS*(v) *containing a sub-STS*(7) *exists for all* $v \equiv 3 \pmod 6$, $v \geq 15$.

Proof: A resolvable STS(15) with a sub-STS(7) is well-known to exist (cf. [4]). It can be embedded in a resolvable STS(w) for all $w \equiv 3 \pmod 6$, $w \geq 45$ [11]. This proves the statement of the lemma for all v except for $v \in \{21, 27, 33, 39\}$. The existence of a resolvable STS(21) with a sub-STS(7) is established, e.g., in [9].

Consider now the following resolvable STS(27) $(V, \mathcal{B})$:
$V = Z_{13} \times \{1,2\} \cup \{\infty\}$; $\mathcal{B} = \{R + i : i \in Z_{13}\}$;
$R = \{\{0_1, 1_1, 4_1\}, \{3_1, 5_1, 10_1\}, \{\infty, 6_1, 6_2\}, \{8_1, 9_2, 12_2\}, \{7_1, 0_2, 2_2\},$
$\{2_1, 1_2, 5_2\}, \{9_1, 3_2, 11_2\}, \{12_1, 4_2, 10_2\}, \{11_1, 7_2, 8_2\}\}$.

This STS(27) contains a sub-STS(7) on the set $\{\infty, 0_1, 1_1, 4_1, 0_2, 1_2, 4_2\}$.

Similarly, consider a resolvable STS(39) $(V, \mathcal{B})$:
$V = Z_{19} \times \{1,2\} \cup \{\infty\}$; $\mathcal{B} = \{R + i : i \in Z_{19}\}$;
$R = \{\{0_1, 1_1, 4_1\}, \{3_1, 11_1, 16_1\}, \{8_1, 15_1, 17_1\}, \{\infty, 2_1, 2_2\}, \{14_1, 0_2, 5_2\},$
$\{7_1, 1_2, 9_2\}, \{5_1, 3_2, 13_2\}, \{9_1, 4_2, 16_2\}, \{10_1, 6_2, 7_2\}, \{18_1, 8_2, 10_2\},$
$\{12_1, 11_2, 15_2\}, \{6_1, 12_2, 18_2\}, \{13_1, 14_2, 17_2\}\}$.

This STS(39) contains a sub-STS(7) on the set $\{\infty, 0_1, 1_1, 4_1, 0_2, 1_2, 4_2\}$.

Finally, consider a resolvable STS(33) $(V, \mathcal{B})$:
$V = Z_{15} \times \{1,2\} \cup \{\infty_1, \infty_2, \infty_3\}$; $\mathcal{B} = R_0 \cup \{R + i : i \in Z_{15}\}$;
$R_0 = \{\{i_1, (i+5)_1, (i+10)_1, \{i_2, (i+5)_2, (i+10)_2\} : i \in Z_{15}\}$,
$R = \{\{0_1, 1_1, 1_4\}, \{3_1, 9_1, 11_1\}, \{\infty_1, 2_1, 2_2\}, \{\infty_2, 5_1, 10_2\}\{\infty_3, 12_1, 3_2\},$
$\{13_1, 0_2, 7_2\}, \{6_1, 1_2, 14_2\}, \{8_1, 4_2, 5_2\}, \{14_1, 6_2, 12_2\}, \{7_1, 8_2, 11_2\},$
$\{10_1, 9_2, 13_2\}\}$.

This STS(33)(contains a sub-STS(7) on $\{\infty_1, 0_1, 1_1, 4_1, 0_2, 1_2, 4_2\}$. □

Lemma 2.9 *Let* $v \equiv 13 \pmod{18}$. *Then* $p(v) \le 10$.

PROOF: By Example 2, we have $p(13) = 7$. To prove the statement for $v \ge 31$, all we have to do is to invoke Theorem 2.6 with $|W| = 7$. □

Actually, we can improve the upper bound of Lemma 2.9 for "one-half" of the cases in question. But first, we need a modification of Construction A.

Construction A'. Let $(V, \mathcal{B})$ be an STS(v) with a sub-STS(w) $(W, \mathcal{C})$. Let $Y = (V \setminus W) \times \{1, \dots, m\}$, $X = Y \cup W$, and let $(Y, \mathcal{D})$ be a 3-GDD (with m groups $(V \setminus W) \times \{i\}, i = 1, \dots, m$ and blocks $\mathcal{D}$. Let $\mathcal{E} = ((\mathcal{B} \setminus \mathcal{C}) \times \{1, \dots, m\}) \cup \mathcal{C} \cup \mathcal{D})$. Then $(X, \mathcal{E})$ is an STS$(m(v-w)+w)$ with a sub-STS(w).

Lemma 2.10 *Let* $v \equiv 13 \pmod{36}$. *Then* $p(v) \le 7$.

PROOF: Use Construction A' with $m = 6$. Let $V = Z_{6t+2} \times \{1,2,3,4,5,6\} \cup \{\infty\}$ so that $|V| = 36t + 13$. Let $(\mathcal{G}, \mathcal{B}, \mathcal{R})$ be a resolvable 3-GDD with groups $g_i = Z_{6t+2} \times \{i\}, i \in \{1,2,3,4,5,6\}$, and parallel classes $\mathcal{R} = \{R_1, \dots, R_{6t+2}\}$; such a resolvable 3-GDD is known to exist (see, e.g., [2], IV.5). For each $i \in \{1,2,3,4,5,6\}$, let $(g_i \cup \{\infty\}, \mathcal{B}_i)$ be a resolvable STS$(6t+3)$, Proceeding now similarly as in Construction A and the proof of Theorem 2.6, namely colouring the blocks within each parallel class in $\mathcal{B}_i, i \in \{1,2,3,4,5,6\}$ with the same colour (but with distinct colours for distinct parallel classes), and using yet different colours for the parallel classes $R_i, i \in \{1, \dots, 6t+2\}$, we see that the palettes of any two points in $Z_{6t+2} \times \{i\}$ are the same, for a total of 7 distinct palettes (the seventh palette being that of the point $\{\infty\}$).
□

It remains for us to deal with the case of $v \equiv 1 \pmod{18}$. We start with an embedding lemma.

Lemma 2.11 *There exists a resolvable STS(39) with a sub-STS(13).*

Proof: Consider the following resolvable STS(39) $(V, \mathcal{B})$:
$V = Z_{13} \times \{1,2,3\}$; $\mathcal{B} = \{R + i : i \in Z_{13}\} \cup \{S_i : i = 1,2,3,4,5,6\}$;
$R = \{\{0_1, 1_1, 4_1\}, \{5_1, 7_1, 12_1\}, \{0_2, 2_2, 8_2\}, \{1_2, 10_2, 11_2\}, \{0_3, 3_3, 12_3\},$
$\{2_3, 8_3, 10_3\}, \{2_1, 4_2, 6_3\}, \{3_1, 6_2, 9_3\}, \{6_1, 12_2, 5_3\}, \{8_1, 3_2, 11_3\}, \{9_1, 5_2, 1_3\},$
$\{10_1, 7_2, 4_3\}, \{11_1, 9_2, 7_3\}$; R is a base parallel class for a total of 13 parallel classes: further six parallel classes are given by
$S_1 = \{\{i_1, i_2, i_3\}; i \in Z_{13}\}$; $S_2 = \{\{1_1, 2_2, 3_3\} : i \in Z_{13}\}$;
$S_3 = \{\{4_1, 8_2, 12_3\} : i \in Z_{13}\}$; $S_4 = \{\{5_1, 10_2, 2_3\} : i \in Z_{13}\}$;
$S_5 = \{\{7_1, 1_2, 8_3\} : i \in Z_{13}\}$; $S_6 = \{\{12_1, 11_2, 10_3\} : i \in Z_{13}\}$.
This STS(39) contains a sub-STS(13) on $Z_{13} \times \{i\}$ for each $i \in \{1,2,3\}$. □

Lemma 2.12 *Let* $v \equiv 1 \pmod{18}$*,* $19 \leq v \leq 199$*,* $v \notin \{37, 109, 145\}$*. Then* $p(v) \leq 16$.

Proof: Lemma 2.5 shows $p(19) \leq 10$. A resolvable STS(27) with a sub-STS(13) exists by [10]; applying Construction A with $v = 27$ and $w = 13$ shows the validity of the statement of the lemma for $v = 55$. For $v \in \{73, 127, 181\}$, use Construction A' with $m = 9$, $w = 1$, and $v = 9, 15$ and 21, respectively; this shows $p(v) \leq 10$ for $v \in \{73, 127, 181\}$. For $v \in \{91, 163, 199\}$, use Construction A' with $m = 6, w = 7$ and $v = 21, 33$ and 39, respectively; the existence of an embedding of an STS(7) in STS(v) follows from Lemma 2.8; thus we have $p(v) \leq 13$ for $v \in \{91, 163, 199\}$. This completes the proof. □

Lemma 2.13 *Let* $v \equiv 1 \pmod{18}$*,* $v \notin \{37, 109, 145\}$*. Then* $p(v) \leq 16$.

Proof: Let $(V, \mathcal{B})$ be a resolvable STS(27) with a sub-STS(13); such an STS exists by [10]. By [11], the latter can be embedded into a resolvable STS(u) for all $v \geq 217$. Applying Theorem 2.6 with $|W| = 13$ shows the validity of the statement of the lemma for all $v \leq 217$. Values of v for $19 \leq v \leq 199$, $v \neq 37, 109, 145$, are treated in Lemma 2.12. □

Finally, for the last three orders not handled above, an upper bound on $p(v)$ is given in the following lemma.

Lemma 2.14 *For* $v \in \{37, 109, 145\}$*, the palette index* $p(v) \leq 19$.

Proof: From Lemma 2.5, we get $p(37) \leq 19$. Using Construction A' with $m = 18$, $w = 1$ and $v = 9$ yields $p(145) \leq 19$. Finally, for $v = 109$, we must use a result of [3] on the existence of a Hanani triple system of order 19 with a sub-STS(9); we observe that the sub-STS(9) is necessarily formed by the elements in its (unique) half-parallel class. We then use a construction very similar to Construction A' with $m = 10, w = 9$ and $v = 19$ where instead of a resolvable 3-GDD of type 10^{10} we use a 3-frame of type 10^{10} (cf. [2]), placing such a Hanani triple system on each set $((V \setminus W) \times \{i\}) \cup W$, $i = 1, \ldots, 10$. The palettes of any two elements of $(V \setminus W) \times \{i\}$ for fixed i are the same, and thus the total number of distinct palettes is $10 + 9 = 19$, yielding $p(109) \leq 19$. □

Similarly as in Lemma 2.9, we can improve the bound of Lemma 2.13 for a subset of admissible values in question.

Lemma 2.15 *a) If* $v \equiv 19 \pmod{36}$*, then* $p(v) \leq 13$.
b) If $v \equiv 19 \pmod{54}$*, then* $p(v) \leq 10$.

Proof: For a), use Construction A' with $m = 6, w = 7$ and $v \equiv 3 \pmod 6$, $v \geq 15$, taking into account Lemma 2.8. For b), use Construction A' with $m = 9, w = 1$ and $v \equiv 3 \pmod 6, v \geq 9$. □

3 Palette indices of Steiner systems S$(2, 4, v)$

Now let $(V, \mathcal{B})$ be a Steiner system $S(2, 4, v)$ and let $p_4(V, \mathcal{B})$ be the palette index of $(V, \mathcal{B})$. Similarly as for STSs, define $p_4(v) = \min p_4(V, \mathcal{B})$ where the minimum is taken over all Steiner systems $S(2, 4, v)$ on V.

Similarly, as for Steiner triple systems, we have the following corollary to Lemma 2.1.

Corollary 3.1 $p_4(v) = 1$ *if and only if* $v \equiv 4 \pmod{12}$.

Proof: A result of Hanani, Ray-Chaudhuri and Wilson [6] guarantees the existence of a resolvable $S(2, 4, v)$ for all $v \equiv 4 \pmod{12}$. □

When $v \equiv 1 \pmod{12}$, we have partial results concerning the palette indices. The main tool for proving our results is an analogue of Construction A above.

Construction B. Let $(V, \mathcal{B})$ be an $S(2, 4, v)$ with a sub-$S(2, 4, w)$ $(W, \mathcal{C})$. Let $Y = (V \setminus W) \times \{1, 2, 3, 4\}$, $X = Y \cup W$, and let $(Y, \mathcal{D})$ be a 4-GDD (with groups $(V \setminus W) \times \{i\}$, $i = 1, 2, 3$ and blocks $\mathcal{D}$; existence of such a 4-GDD is equivalent to the existence of a pair of orthogonal latin squares of order $v - w$). Let $\mathcal{E} = ((\mathcal{B} \setminus \mathcal{C}) \times \{1, 2, 3, 4\}) \cup \mathcal{C} \cup \mathcal{D}$. Then $(X, \mathcal{E})$ is an $S(2, 4, 4v - 3w)$ with a sub-$S(2, 4, w)$.

Theorem 3.2 *Suppose that in the above Construction B, both the* $S(2, 4, v)$ $(V, \mathcal{B})$ *and the* 4-*GDD* $(Y, \mathcal{D})$ *are resolvable. Then* $p(Y, \mathcal{E}) \leq w + 4$.

Proof: The proof is completely analogous to that of Theorem 2.6. □

The existence of a resolvable 4-GDD $(Y, \mathcal{D})$ is equivalent to the existence of three pairwise orthogonal latin squares of order $v - w$. The latter are well known to exist for all orders exceeding ten.

Corollary 3.3 *Let* $v \equiv 13 \pmod{48}$, $v \geq 61$. *Then* $p_4(v) \leq 5$.

PROOF: As mentioned already, for every $v \equiv 4 \pmod{12}$ there exists a resolvable $S(2, 4, v)$ and a resolvable 4-GDD of order $v - 1$. Now apply Theorem 3.2 with $w = 1$. □

Lemma 3.4 *Let* $v \equiv 25 \pmod{48}$, $v = 121$ *or* $v \geq 601$. *Then* $p_4(v) \leq 17$.

PROOF: The unique $S(2, 4, 13)$ can be embedded in a resolvable $S(2, 4, 40)$ [7]. This in turn can be embedded into an $S(2, 4, u)$ for all $u \geq 160$ [1, 5]. We can apply now Theorem 3.2 with $w = 13$. □

In order to show for the remaining two classes modulo 48 that the palette index $p_4(v)$ is constant, we need to postulate the existence of two specific designs: D_1, a resolvable $S(2, 4, 76)$ with a sub-$S(2, 4, 25)$, and D_2, a resolvable $S(2, 4, 112)$ with a sub-$S(2, 4, 37)$.

Lemma 3.5 *Suppose that there exists an embedding of an* $S(2, 4, 25)$ *into a resolvable* $S(2, 4, 76)$. *Let* $v \equiv 37 \pmod{48}$, $v = 229$ *or* $v \geq 1141$. *Then* $p_4(v) \leq 29$.

PROOF: It is shown in [5] that a resolvable $S(2, 4, 76)$ can be embedded in a resolvable $S(2, 4, u)$ for all $u \equiv 4 \pmod{12}$, $u \geq 304$. Applying Theorem 3.2 with $w = 25$ yields the result. □

Lemma 3.6 *Suppose there exists an embedding of an* $S(2, 4, 37)$ *into a resolvable* $S(2, 4, 112)$. *Let* $v \equiv 1 \pmod{48}$, $v = 337$ *or* $v \geq 2129$. *Then* $p_4(v) \leq 41$.

PROOF: It is shown in [5] that a resolvable $S(2, 4, 112)$ can be embedded in a resolvable $S(2, 4, u)$ for all $u \equiv 4 \pmod{12}$, $v \geq 560$. Applying Theorem 3.2 with $w = 37$ proves the result. □

Unfortunately, we have not yet been able to show the existence of the embeddings needed in Lemmas 3.5 and 3.6.

4 Conclusion

In Section 2 we concerned ourselves only with the question of a minimum possible palette index of an STS of given order. A more general and clearly also much more difficult question would be that about the *largest* possible palette index of an STS(v), or indeed about the spectrum of possible values for palette indices of Steiner triple systems of order v.

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