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On the cyclic decomposition of circulant graphs into bipartite graphs

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Abstract

It is known that if a bipartite graph G with n edges possesses any of three types of ordered labelings, then the complete graph K_{2nx+1} admits a cyclic G-decomposition for every positive integer x. We introduce variations of the ordered labelings and show that whenever a bipartite graph G admits one of these labelings, then there exists a cyclic G-decomposition of an infinite family of circulant graphs. We also show that all 2-regular bipartite graphs admit one of these variant labelings.

1 Introduction

If a and b are integers we denote $\{a, a+1, \dots, b\}$ by $[a, b]$. (If $a > b$, then $[a, b] = \varnothing$.) If A and B are subsets of the integers and if $\max(A) \leq \min(B)$, we will write $A \leq B$. We define $A < B$, $A \geq B$, and $A > B$ analogously. If $\{a\} \leq B$, we will write $a \leq B$. Similarly, if $B \leq \{b\}$, then we will write $B \leq b$. Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{Z}_n$ the group of integers modulo n. For a graph G, let $V(G)$ and $E(G)$ denote the vertex set of G and the edge set of G, respectively. The *order* and the *size* of a graph G are $|V(G)|$ and $|E(G)|$, respectively. Unless otherwise noted, we will only consider graphs with no isolated vertices.

Let $V(K_m) = \mathbb{Z}_m$ and let G be a subgraph of K_m. By *clicking* G, we mean applying the isomorphism $i \mapsto i+1$ to $V(G)$. Let H and G be graphs such that G is a subgraph of H. A *G-decomposition* of H is a set $\Delta = \{G_1, G_2, \dots, G_t\}$ of pairwise edge-disjoint subgraphs of H each of which is isomorphic to G and such that $E(H) = \bigcup_{i=1}^{t} E(G_i)$. A G-decomposition of K_m is also known as a (K_m, G)-*design*. A (K_m, G)-design Δ is *cyclic* if clicking is a permutation of Δ. For recent surveys on G-designs, see [3] and [7].

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Unless noted otherwise, we will let $V(K_m) = [0, m-1]$. The *label* of an edge $\{i, j\}$ in K_m is $|i-j|$ while the *length* of $\{i, j\}$ is $\min\{|i-j|, m-|i-j|\}$. We shall refer to an edge $\{i, j\}$ whose length is not $|i-j|$ as a *wrap-around* edge. Note that if $\{i, j\}$ has length $|i-j|$ in K_m, then $\{i, j\}$ will have length $|i-j|$ in $K_{m'}$ for all $m' \geq m$. If m is odd, then K_m consists of m edges of length i for $i \in [1, \frac{m-1}{2}]$. If m is even, then K_m consists of m edges of length i for $i \in [1, \frac{m}{2}-1]$ and $\frac{m}{2}$ edges of length $\frac{m}{2}$; moreover, in this case the edges of length $\frac{m}{2}$ constitute a 1-factor in K_m.

Let $L \subseteq \{1, 2, \ldots, \lfloor m/2 \rfloor\}$. The subgraph of K_m induced by all the edges with lengths in L is called a *circulant graph* and is denoted by $\langle L \rangle_m$. Of course, circulant graphs are *Cayley graphs* on cyclic groups. As noted earlier, $\langle \{m/2\} \rangle_m$ is a 1-factor in K_m when m is even. Otherwise, for $1 \leq i < m/2$, it is easy to see that $\langle \{i\} \rangle_m$ consists of δ vertex disjoint cycles $C_{m/\delta}$, where $\delta = \gcd(i, m)$.

Let k and n be positive integers and let G be a graph of size n. It would be of interest to know whether there exists a G-decomposition of the circulant $\langle [k, n+k-1] \rangle_{2n+2k-1}$. When $k = 1$, the circulant $\langle [k, n+k-1] \rangle_{2n+2k-1}$ is the complete graph K_{2n+1}. A popular conjecture of Ringel [19] states that there exists a (K_{2n+1}, G)-design for every tree G of size n. It is very likely that every tree of size n will decompose the circulant $\langle [k, n+k-1] \rangle_{2n+2k-1}$ for every positive integer k. In fact, it would be of interest to know what graphs of size n do not decompose $\langle [k, n+k-1] \rangle_{2n+2k-1}$ for some positive k.

A popular approach to dealing with Ringel's Conjecture is the use of graph labelings. In fact, numerous conjectures in graph labelings are stronger than Ringel's Conjecture (see [14]). For example, Kotzig (see [20]) conjectures that every tree admits what is called a ρ-labeling. This would imply that there is a cyclic (K_{2n+1}, G)-design for every tree G of size n. It can be conjectured similarly that there is a cyclic G-decomposition of $\langle [k, n+k-1] \rangle_{2n+2k-1}$ for every tree G of size n.

1.1 Extensions of Rosa-type Labelings

For any graph G, a one-to-one function $f\colon V(G) \to \mathbb{N}$ is called a *labeling* (or *valuation*) of G. In [20], Rosa introduced a hierarchy of labelings. We generalize Rosa's labelings and add a few items to this hierarchy. Let G be a graph with n edges and no isolated vertices and let f be a labeling of G. Let $f(V(G)) = \{f(u)\colon u \in V(G)\}$. Define a function $\bar{f}\colon E(G) \to \mathbb{N}$ by $\bar{f}(e) = |f(u) - f(v)|$, where $e = \{u, v\} \in E(G)$. We will refer to $\bar{f}(e)$ as the *label* of e. If $F \subseteq E(G)$, let $\bar{f}(F) = \{\bar{f}(e)\colon e \in F\}$. Let k be a positive integer and consider the following conditions:

(ℓ1) $f(V(G)) \subseteq [0, 2(n+k-1)]$,

(ℓ2) $f(V(G)) \subseteq [0, n+k-1]$,

(ℓ3) $\bar{f}(E(G)) = \{x_k, x_{k+1}, \ldots, x_{n+k-1}\}$, where for each $i \in [k, n+k-1]$ either $x_i = i$ or $x_i = 2(n+k-1)+1-i = 2(n+k)-1-i$,

(ℓ4) $\bar{f}(E(G)) = [k, n+k-1]$.

If in addition G is bipartite with bipartition $\{A, B\}$ of $V(G)$ (with every edge in G having one end vertex in A and the other in B), consider also

(ℓ5) for each $\{a, b\} \in E(G)$ with $a \in A$ and $b \in B$, we have $f(a) < f(b)$,

(ℓ6) there exists an integer λ (called a *boundary value* of f) such that $f(a) \leq \lambda$ for all $a \in A$ and $f(b) > \lambda$ for all $b \in B$.

Then a labeling satisfying the conditions:

(ℓ1) and (ℓ3) is called a ρ_k*-labeling*;

(ℓ1) and (ℓ4) is called a σ_k*-labeling*;

(ℓ2) and (ℓ4) is called a β_k*-labeling.*

A β_k-labeling is necessarily a σ_k-labeling which in turn is a ρ_k-labeling. When $k = 1$, these labelings correspond, respectively, to the β, σ, and ρ-labelings that were introduced by Rosa [20]. We shall refer to the labelings introduced above simply as k*-labelings.*

If G is bipartite and a ρ_k, σ_k, or β_k-labeling of G also satisfies (ℓ5), then the labeling is *ordered* and is denoted by ρ_k^+, σ_k^+, or β_k^+, respectively. If in addition (ℓ6) is satisfied, the labeling is *uniformly-ordered* and is denoted by ρ_k^{++}, σ_k^{++}, or β_k^{++}, respectively.

A β-labeling is better known as a *graceful* labeling and a uniformly-ordered β-labeling is an α*-labeling* as introduced in [20]. Because the concept of an α-labeling is well known, we will call a β^{++}-labeling an α-labeling, and we will use the notation α_k in place of β_k^{++}. Moreover, what we are calling a β_k-labeling was previously independently introduced as a k*-graceful labeling* by Slater [21] and by Mahéo and Thuillier [18].

The following lemma shows that if a bipartite graph G admits an ordered k-labeling, then G admits a uniformly-ordered $(k+m)$-labeling for all but a finite number of positive integers m.

Lemma 1. *Let G be a bipartite graph with no isolated vertices and vertex bipartition $\{A, B\}$ and let k be a positive integer. Let f be an ordered k-labeling of G with $f(a) < f(b)$ for every $\{a, b\} \in E(G)$ with $a \in A$ and $b \in B$. Let $D = \{f(a) - f(b)\colon\ a \in A, b \in B \text{ with } f(a) > f(b)\}$. If f is a β_k^+, σ_k^+, or ρ_k^+-labeling, then G admits a β_{k+m}^+, σ_{k+m}^+, or ρ_{k+m}^+-labeling, respectively for all $m \in \mathbb{N} \setminus D$. Moreover, if $m > D$, then the $(k+m)$-labeling of G is uniformly-ordered.*

Proof. Suppose G has n edges and vertex bipartition $\{A, B\}$. Let k be a positive integer and let f be a β_k^+, σ_k^+, or ρ_k^+-labeling of G such that $f(a) < f(b)$ for all $\{a, b\} \in E(G)$ with $a \in A$ and $b \in B$. Also, let $D = \{f(a) - f(b)\colon\ a \in A, b \in B \text{ with } f(a) > f(b)\}$ and let $m \in \mathbb{N} \setminus D$. Consider the labeling $f'\colon\ V(G) \to [0, 2n + 2k + 2m - 1]$ defined by $f'(u) = f(u)$ if $u \in A$ and $f'(v) = f(v) + m$ if $v \in B$. Since $m \neq f(a) - f(b)$ for any $a \in A$ and $b \in B$, we have $f'(v) = f(v) + m \neq f(u) = f'(u)$ for any $u \in A$

and $v \in B$. Thus f' is one-to-one. Depending on which type of ordered k-labeling f is, it is simple to verify that f' is the corresponding ordered $(k+m)$-labeling.

Now, suppose m exceeds all elements in D. Then $m > f(a) - f(b)$ for any $a \in A$ and $b \in B$, and we have $f'(v) = f(v) + m > f(u) = f'(u)$ for any $u \in A$ and $v \in B$, i.e. $f'(B) > f'(A)$. Thus f' is uniformly ordered. ∎

If the k-labeling f in Lemma 1 is uniformly-ordered, then D is empty and and the resulting $(k+m)$-labeling is also uniformly-ordered.

Corollary 2. *Let G be a bipartite graph and let k and m be positive integers. If G admits an α_k, σ_k^{++}, or ρ_k^{++}-labeling, then G also admits an α_{k+m}, σ_{k+m}^{++}, or ρ_{k+m}^{++}-labeling, respectively.*

It is well known (see [20]) that if a graph G of size n has all even degrees and if G admits a σ-labeling, then we must have $n \equiv 0$ or $3 \pmod 4$. Moreover, if G is bipartite, then G has an even number of edges, so $n \equiv 0 \pmod 4$. This condition is known as the *parity condition* and has a k-labelings counterpart.

Lemma 3. *Let G be a graph of size n and suppose every vertex of G has even degree. If G admits a σ_k-labeling, then either (a) $n \equiv 0 \pmod 4$, (b) k is even and $n \equiv 1 \pmod 4$, or (c) k is odd and $n \equiv 3 \pmod 4$. Moreover, if G is bipartite, then $n \equiv 0 \pmod 4$.*

Proof. Let f be a σ_k-labeling of G. Then we have the sum of the edge labels in G is $\sum_{\{u,v\} \in E(G)} |f(u) - f(v)|$ which is necessarily even (since every vertex has even degree) and equals $\sum_{i=1}^{n} (k+i-1) = nk + (n-1)n/2$. Thus the conclusions follow. ∎

Lemma 4. *Let G be a bipartite graph of size n and let $d = \gcd(\{\deg(v) \colon v \in V(G)\})$. If G admits a σ_k^+-labeling for some positive integer k, then d divides $n(2k+n-1)/2$.*

Proof. Let G have vertex bipartition $\{A, B\}$, where $A = \{a_1, a_2, \ldots, a_r\}$ and $B = \{b_1, b_2, \ldots, b_s\}$. Let f be a σ_k^+-labeling of G such that $f(a) < f(b)$ for each $\{a, b\} \in E(G)$ with $a \in A$ and $b \in B$. Then the sum of the edge labels in G can be computed with $\sum_{i=1}^{n} (k+i-1) = nk + (n-1)n/2 = n(2k+n-1)/2$ or with $\sum_{j=1}^{s} \deg(b_j) f(b_j) - \sum_{i=1}^{r} \deg(a_i) f(a_i)$. Since $d \mid \sum_{j=1}^{s} \deg(b_j) f(b_j) - \sum_{i=1}^{r} \deg(a_i) f(a_i)$, we have $d \mid n(2k+n-1)/2$. ∎

We note that Lemma 4 has no application if n is odd, since d divides n. However, if n is even and $n \equiv d \pmod{2d}$, then G cannot admit a σ_k^+-labeling. For instance, $K_{5,5} - I$, where I is a 1-factor, does not admit a σ_k^+-labeling for any positive integer k.

We turn our attention briefly to disconnected graphs. It was shown in [13] that the vertex-disjoint union of graphs that admit α-labelings has a σ^+-labeling (called a θ-labeling in [13]). However, for graph decomposition purposes, we need the resulting σ^+-labeling to satisfy additional conditions.

Theorem 5. *Let $G_1, G_2, \ldots, G_t$ be vertex-disjoint bipartite graphs that admit α-labelings and let $H = \bigcup_{i=1}^{t} G_i$. Let H have size n and let $\{A, B\}$ be a bipartition of $V(H)$. Then H admits a σ^+-labeling h that satisfies $h(a) < h(b)$ for every edge $\{a, b\}$ with $a \in A$ and $b \in B$ and satisfies $h(u) - h(v) < n$ for any $u \in A$ and $v \in B$.*

Proof. For $1 \le i \le t$, let bipartite graph G_i have n_i edges (with $n_i \ge 1$), α-labeling g_i with boundary value λ_i, and vertex bipartition $\{A_i, B_i\}$ where $g_i(a) \le \lambda_i < g_i(b)$ for all $a \in A_i$ and $b \in B_i$. Without loss of generality, we can assume that $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_t$. Then H is bipartite with vertex bipartition $\{A, B\}$ where $A = \bigcup_{i=1}^t A_i$ and $B = \bigcup_{i=1}^t B_i$. Let $n = |E(H)| = \sum_{i=1}^t n_i$.

If $t = 1$, then g_1 is an α-labeling of H (which necessarily satisfies the conclusion), so assume $t \ge 2$. We define a labeling h on $V(H)$ by

$$h(v) = \begin{cases} g_i(v) + \frac{i-1}{2} + \sum_{j=1}^{\frac{i-1}{2}} \lambda_{2j-1} & \text{for } i \text{ odd, } v \in A_i, \\ g_i(v) + \frac{i-1}{2} + \sum_{j=1}^{\frac{i-1}{2}} \lambda_{2j-1} + \sum_{j=i+1}^{t} n_j & \text{for } i \text{ odd, } v \in B_i, \\ g_i(v) + n + \frac{i}{2} + \sum_{j=1}^{\frac{i-2}{2}} \lambda_{2j} & \text{for } i \text{ even, } v \in A_i, \\ g_i(v) + n + \frac{i}{2} + \sum_{j=1}^{\frac{i-2}{2}} \lambda_{2j} + \sum_{j=i+1}^{t} n_j & \text{for } i \text{ even, } v \in B_i. \end{cases}$$

To show that h is a σ^+-labeling, we note that, if t_{o} and t_{e} are respectively the greatest odd integer and greatest even integer less than or equal to t, then we have

$$0 \le h(A_1) < h(A_3) < \cdots < h(A_{t_{\mathrm{o}}}) < h(B_{t_{\mathrm{o}}}) < h(B_{t_{\mathrm{o}}-2}) < \cdots < h(B_1) \le n$$

and

$$n + 1 \le h(A_2) < h(A_4) < \cdots < h(A_{t_{\mathrm{e}}}) < h(B_{t_{\mathrm{e}}}) < h(B_{t_{\mathrm{e}}-2}) < \cdots < h(B_2) \le 2n.$$

Hence h is one-to-one and no edge label will exceed n. Furthermore, $\bar{h}(E(G_i)) = [1, n_i] + \sum_{j=i+1}^t n_j = [1 + \sum_{j=i+1}^t n_j, \sum_{j=i}^t n_j]$, and we have

$$1 \le \bar{h}(E(G_t)) < \bar{h}(E(G_{t-1})) < \cdots < \bar{h}(E(G_1)) \le n.$$

Hence $\bar{h}(E(H)) = [1, n]$, and the conditions for h being a σ^+-labeling are satisfied.

Now, we consider the difference of $\max(h(A))$ and $\min(h(B))$. Note that since g_i is an α-labeling, the boundary value λ_i is unique and $\min G_i(B_i) = \lambda_i + 1$.

Case 1: t is even.
Then we have

$$\begin{aligned} \max(h(A)) &= \max(g_t(A_t)) + n + \tfrac{t}{2} + \textstyle\sum_{j=1}^{\frac{t-2}{2}} \lambda_{2j} \\ &= \lambda_t + n + \tfrac{t}{2} + \textstyle\sum_{j=1}^{\frac{t-2}{2}} \lambda_{2j}, \end{aligned}$$

and, since g_{t-1} is an α-labeling,

$$\begin{aligned} \min(h(B)) &= \min(g_{t-1}(B_{t-1})) + \tfrac{t-2}{2} + \textstyle\sum_{j=1}^{\frac{t-2}{2}} \lambda_{2j-1} + n_t \\ &= (\lambda_{t-1} + 1) + \tfrac{t-2}{2} + \textstyle\sum_{j=1}^{\frac{t-2}{2}} \lambda_{2j-1} + n_t. \end{aligned}$$

Hence

$$\begin{aligned} \max(h(A)) - \min(h(B)) &= \lambda_t + n + \tfrac{t}{2} + \sum_{j=1}^{\frac{t-2}{2}} \lambda_{2j} - \left(\lambda_{t-1} + 1 + \tfrac{t-2}{2} + \sum_{j=1}^{\frac{t-2}{2}} \lambda_{2j-1} + n_t \right) \\ &= n - n_t + \textstyle\sum_{j=1}^{\frac{t}{2}} (\lambda_{2j} - \lambda_{2j-1}). \end{aligned}$$

By the assumption that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t$, we have $\sum_{j=1}^{\frac{t}{2}}(\lambda_{2j} - \lambda_{2j-1}) \leq 0$, and thus $\max(h(A)) - \min(h(B)) \leq n - n_t < n$. Therefore $h(u) - h(v) < n$ for any $u \in A$ and $v \in B$.

Case 2: t is odd.
Then we have

$$\begin{aligned}\max(h(A)) &= \max(g_{t-1}(A_{t-1})) + n + \tfrac{t-1}{2} + \textstyle\sum_{j=1}^{\frac{t-3}{2}} \lambda_{2j} \\ &= \lambda_{t-1} + n + \tfrac{t-1}{2} + \textstyle\sum_{j=1}^{\frac{t-3}{2}} \lambda_{2j},\end{aligned}$$

and, since g_t is an α-labeling,

$$\begin{aligned}\min(h(B)) &= \min(g_t(B_t)) + \tfrac{t-1}{2} + \textstyle\sum_{j=1}^{\frac{t-1}{2}} \lambda_{2j-1} + 0 \\ &= (\lambda_t + 1) + \tfrac{t-1}{2} + \textstyle\sum_{j=1}^{\frac{t-1}{2}} \lambda_{2j-1}.\end{aligned}$$

Hence

$$\begin{aligned}\max(h(A)) - \min(h(B)) &= \lambda_{t-1} + n + \tfrac{t-1}{2} + \sum_{j=1}^{\frac{t-3}{2}} \lambda_{2j} - \left(\lambda_t + 1 + \tfrac{t-1}{2} + \sum_{j=1}^{\frac{t-1}{2}} \lambda_{2j-1}\right) \\ &= n - (\lambda_t + 1) + \textstyle\sum_{j=1}^{\frac{t-1}{2}}(\lambda_{2j} - \lambda_{2j-1}).\end{aligned}$$

By the assumption that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t$, we have $\sum_{j=1}^{\frac{t-1}{2}}(\lambda_{2j} - \lambda_{2j-1}) \leq 0$, and thus $\max(h(A)) - \min(h(B)) \leq n - (\lambda_t + 1) < n$. Therefore, $h(u) - h(v) < n$ for any $u \in A$ and $v \in B$. ∎

The following was given in [6].

Theorem 6. *Let $G_1, G_2, \ldots, G_t$ be vertex-disjoint bipartite graphs with $n_1, n_2, \ldots, n_t$ edges, respectively. If G_1 admits a ρ^{++}-labeling in which no vertex is labeled $2n_1$ and $G_2, G_3, \ldots, G_t$ admit α-labelings, then the vertex-disjoint union $G_1 \cup G_2 \cup G_3 \cup \cdots \cup G_t$ admits a ρ^{++}-labeling.*

We discovered recently that the proof of Theorem 6 is incomplete as published in [6]. We prove a stronger result here that subsumes the results from Theorem 6. We first give two lemmas.

Lemma 7. *Let G_1 be a bipartite graph with n_1 edges that admits a ρ^{++}-labeling g_1 with boundary value λ_1. Let G_2 be a bipartite graph with n_2 edges that admits an α-labeling g_2 with boundary value λ_2. If $\lambda_1 + 1 \notin g_1(V(G_1))$, then the vertex-disjoint union $G_1 \cup G_2$ admits a ρ^{++}-labeling h such that $2(n_1 + n_2) \notin h(V(G_1 \cup G_2))$.*

Proof. For $i \in \{1, 2\}$, let G_i have vertex bipartition $\{A_i, B_i\}$ such that $g_i(A_i) \leq \lambda_i < g_i(B_i)$. Furthermore, let $\lambda_1 + 1 < g_1(B_1)$. Let H denote the vertex-disjoint union $G_1 \cup G_2$ and let $n = n_1 + n_2$.

We define a labeling h on $V(H)$ by

$$h(v) = \begin{cases} g_1(v) & v \in A_1, \\ g_1(v) + n_2 & v \in B_1, \\ g_2(v) + \lambda_1 + 1 & v \in V(G_2). \end{cases}$$

Clearly, h is one-to-one on each of the sets A_1, B_1, and $V(G_2)$. Since $\lambda_1 + 1 < g_1(B_1)$, we have

$$0 \leq h(A_1) < \lambda_1 + 1 \leq h(V(A_2)) < h(V(B_2)) \leq \lambda_1 + 1 + n_2 < h(B_1) \leq 2n_1 + n_2 < 2n. \quad (1)$$

Hence h is one-to-one and $h(V(H)) \subseteq [0, 2n]$.

Next, we examine the set of edge labels $\bar{h}(E(H))$. For each $\ell \in [1, n_2]$, there exists an edge $e \in E(G_2)$ such that $\bar{g_2}(e) = \ell$. Hence

$$\bar{h}(e) = \bar{g_2}(e) = \ell.$$

Moreover, for each $\ell \in [n_2 + 1, n]$, there exists an edge $e \in E(G_1)$ such that either $\bar{g_1}(e) + n_2 = \ell$ or $(2n_1 + 1 - \bar{g_1}(e)) + n_2 = \ell$. Hence either

$$\bar{h}(e) = \bar{g_1}(e) + n_2 = \ell$$

or

$$2n + 1 - \bar{h}(e) = 2(n_1 + n_2) + 1 - (\bar{g_1}(e) + n_2) = (2n_1 + 1 - \bar{g_1}(e)) + n_2 = \ell.$$

Therefore, $\bar{h}(E(H)) \supseteq \{x_1, x_2, \ldots, x_n\}$, where for each $\ell \in [1, n]$ either $x_\ell = \ell$ or $2n + 1 - x_\ell = \ell$. Since $|\bar{h}(E(H))| = n$, we have $\bar{h}(E(H)) = \{x_1, x_2, \ldots, x_n\}$. Thus h is a ρ-labeling.

Finally, let $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$. Then $\{A, B\}$ is a bipartition of $V(H)$. It is clear from (1) that $h(A) < h(B)$ and thus h is a ρ^{++}-labeling. Moreover, it is clear from (1) that $2n \notin h(V(H))$. ∎

Lemma 8. *Let G_1 be a bipartite graph with n_1 edges that admits a ρ^{++}-labeling g_1 with boundary value λ_1. Let G_2 be a bipartite graph with n_2 edges that admits an α-labeling g_2 with boundary value λ_2. If $2n_1 \notin g_1(V(G_1))$, then the vertex-disjoint union $G_1 \cup G_2$ admits a ρ^{++}-labeling h with boundary value λ such that $\lambda + 1 \notin h(V(G_1 \cup G_2))$.*

Proof. For $i \in \{1, 2\}$, let G_i have vertex bipartition $\{A_i, B_i\}$ such that $g_i(A_i) \leq \lambda_i < g_i(B_i)$. Furthermore, let $g_1(B_1) < 2n_1$. Let H denote the vertex-disjoint union $G_1 \cup G_2$ and let $n = n_1 + n_2$.

We define a labeling h on $V(H)$ by

$$h(v) = \begin{cases} g_1(v) + \lambda_2 + 1 & v \in A_1, \\ g_1(v) + \lambda_2 + 1 + n_2 & v \in B_1, \\ g_2(v) & v \in A_2, \\ g_2(v) + 2n_1 + n_2 & v \in B_2. \end{cases}$$

Clearly, h is one-to-one on each of the sets A_1, B_1, A_2, and B_2. Since $g_1(B_1) < 2n_1$, we have

$$0 \leq h(A_2) < \lambda_2+1 \leq h(V(A_1)) \leq \lambda_1+\lambda_2+1 < h(V(B_1)) < 2n_1+\lambda_2+1+n_2 \leq h(B_2) \leq 2n. \quad (2)$$

Hence h is one-to-one and $h(V(H)) \subseteq [0, 2n]$.

Next, we examine the set of edge labels $\bar{h}(E(H))$. For each $\ell \in [1, n_2]$, there exists an edge $e \in E(G_2)$ such that $\bar{g_2}(e) = n_2 + 1 - \ell$. Hence

$$2n + 1 - \bar{h}(e) = 2(n_1 + n_2) + 1 - (\bar{g_2}(e) + 2n_1 + n_2) = n_2 + 1 - \bar{g_2}(e) = \ell.$$

Moreover, for each $\ell \in [n_2 + 1, n]$, there exists an edge $e \in E(G_1)$ such that either $\bar{g_1}(e) + n_2 = \ell$ or $(2n_1 + 1 - \bar{g_1}(e)) + n_2 = \ell$. Hence either

$$\bar{h}(e) = \bar{g_1}(e) + n_2 = \ell$$

or

$$2n + 1 - \bar{h}(e) = 2(n_1 + n_2) + 1 - (\bar{g_1}(e) + n_2) = (2n_1 + 1 - \bar{g_1}(e)) + n_2 = \ell.$$

Therefore, $\bar{h}(E(H)) \supseteq \{x_1, x_2, \ldots, x_n\}$, where for each $\ell \in [1, n]$ either $x_\ell = \ell$ or $2n + 1 - x_\ell = \ell$. Since $|\bar{h}(E(H))| = n$, we have $\bar{h}(E(H)) = \{x_1, x_2, \ldots, x_n\}$. Thus h is a ρ-labeling.

Finally, let $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$. Then $\{A, B\}$ is a bipartition of $V(H)$. It is clear from (2) that $h(A) \leq \lambda_1 + \lambda_2 + 1 < h(B)$ and thus h is a ρ^{++}-labeling with boundary value $\lambda_1 + \lambda_2 + 1$. Moreover, since $\min(h(B)) = \min(h(B_1)) \geq \lambda_1 + \lambda_2 + 2 + n_2$, we have $\lambda_1 + \lambda_2 + 2 \notin h(V(H))$. ∎

By combining the results from Lemmas 7 and 8, we obtain the following theorem which subsumes Theorem 6.

Theorem 9. *Let G be a bipartite graph with n edges that admits ρ^{++}-labeling g with boundary value λ. Let H_1, H_2, $\ldots$, H_k be bipartite graphs that admit α-labelings. If $\{\lambda + 1, 2n\} \not\subseteq g(V(G))$, then the vertex-disjoint union $G \cup H_1 \cup H_2 \cup \cdots \cup H_k$ admits a ρ^{++}-labeling.*

Labelings that are used in graph decompositions are called *Rosa-type* because of Rosa's original article [20] on the topic. For a survey of Rosa-type labelings and their graph decomposition applications, see [14]. A comprehensive dynamic survey on general graph labelings is maintained by Gallian [16].

Rosa-type labelings are critical to the study of cyclic graph decompositions as seen in the following results from Rosa [20].

Theorem 10. *Let G be a graph with n edges. There exists a cyclic G-decomposition of K_{2n+1} if and only if G has a ρ-labeling.*

Theorem 11. *Let G be a graph with n edges that has a σ-labeling. Then there exists a cyclic G-decomposition of $K_{2n+2} - I$, where I is a 1-factor in K_{2n+2}.*

Theorem 12. *Let G be a bipartite graph with n edges that has an α-labeling. Then there exists a cyclic G-decomposition of K_{2nx+1} for all positive integers x.*

From a graph decompositions perspective, Theorem 12 offers a great advantage over the other two theorems. However, many bipartite graphs, including infinite classes of trees, fail to admit α-labelings. In [13] it is shown that ρ^+-labelings yield similar results to Theorem 12.

Theorem 13. *Let G be a bipartite graph with n edges that has a ρ^+-labeling. Then there exists a cyclic G-decomposition of K_{2nx+1} for all positive integers x.*

Unlike with α-labelings, it is not currently known if there is a bipartite graph (without too many isolated vertices) that fails to admit a ρ^+-labeling. In fact, El-Zanati and Vanden Eynden conjecture that every bipartite graph admits a ρ^{++}-labeling (see [14]).

In this manuscript, we show how to use k-labelings to get extensions of the above theorems to cyclic G-decompositions of the corresponding circulant graphs. We also investigate which bipartite 2-regular graphs admit the various k-labelings.

2 Main Results

We note that a ρ_k-labeling of G of size n induces an embedding of G in $K_{2n+2k-1}$ (with $V(K_{2n+2k-1}) = [0, 2n+2k-2]$) so that for each ℓ, such that $k \leq \ell \leq n+k+\ell$, there an edge of the length ℓ in $E(G)$. Moreover, $\langle [k, n+k-1] \rangle_{2n+2k-1} = K_{2n+2k-1} - \langle [1, k-1] \rangle_{2n+2k-1}$. Thus we have the following result corresponding to Theorem 10.

Theorem 14. *Let G be a graph with n edges and let k be a positive integer. There exists a cyclic G-decomposition of $\langle [k, n+k-1] \rangle_{2n+2k-1}$ if and only if G has a ρ_k-labeling.*

Similarly, a σ_k-labeling of G can be viewed as inducing an embedding of G in K_{2n+2k} (with $V(K_{2n+2k}) = [0, 2n+2k-1]$) so that there is one edge in $E(G)$ with each label ℓ for $k \leq \ell \leq n+k-1$. (Recall that σ-labelings do not allow wrap-around edges.) Moreover, $\langle [k, n+k-1] \rangle_{2n+2k} = K_{2n+2k} - \langle [1, k-1] \cup \{k+n\} \rangle_{2n+2k}$. Thus we have the following result corresponding to Theorem 11.

Theorem 15. *Let G be a graph with n edges and let k be a positive integer. If G admits a σ_k-labeling, then there exists a cyclic G-decomposition of $\langle [k, n+k] \rangle_{2n+2k} - I$, where I is a 1-factor.*

Because σ_k-labelings do not allow wrap-around edges, Theorem 14 can be broadened greatly in terms of decompositions of circulant graphs.

Theorem 16. *Let G be a graph with n edges and let $k \geq 1$ be an integer. If G admits a σ_k-labeling, then there exists a cyclic G-decomposition of $\langle [k, n+k-1] \rangle_{2n+2k-1+t}$ for each nonnegative integer t.*

As would be expected, Theorem 13 has a k-labelings counterpart.

Theorem 17. *Let G be a bipartite graph with n edges and let k be a positive integer. If G admits a ρ_k^+-labeling, then there exists a cyclic G-decomposition of $\langle[k, nx+k-1]\rangle_{2nx+2k-1}$ for each positive integer x.*

Proof. Let $\{A, B\}$ be a bipartition of $V(G)$. Let h be a ρ_k^+-labeling of G, so that $h(u) < h(v)$ for every $\{u, v\} \in E(G)$ with $u \in A$ and $v \in B$. We define a multigraph G' with

$$V(G') = \{h(a)\colon\ a \in A\} \cup \{h(b) + 2n(i-1)\colon\ b \in B, i \in [1, x]\} \subseteq [0, 2(nx + k - 1)]$$

and

$$E(G') = \big\{\{h(a), h(b) + 2n(i-1)\}\colon\ \{a, b\} \in E(G), a \in A, b \in B, i \in [1, x]\big\}.$$

We will show that the lengths in $K_{2nx+2k-1}$ of the nx edges of G' are exactly the nx integers in $[k, nx+k-1]$, and so G' is actually a graph. Then if we define $h'(v) = v$ for $v \in V(G')$, then h' is a ρ_k-labeling of G' and there is a cyclic G'-decomposition of $\langle[k, nx+k-1]\rangle_{2nx+2k-1}$ by Theorem 14. But for fixed $i \in [1, x]$ the corresponding edges of G' induce a graph isomorphic to G, so G' has a G-decomposition and the theorem follows.

Let $\ell \in [k, nx + k - 1]$. We will show that some edge of G' has label ℓ or $2nx + 2k - 1 - \ell$. First, we show that there exist integers q and r, with $0 \le q < x$ and $k \le r < n + k$, such that either $\ell = 2nq + r$ or $2nx + 2k - 1 - \ell = 2nq + r$. By the division algorithm there exist integers q_i and r_i, for $i \in \{1, 2\}$, such that

$$\ell - k = 2nq_1 + r_1, \quad \text{where } 0 \le r_1 < 2n,$$

and

$$2nx + k - 1 - \ell = 2nq_2 + r_2, \quad \text{where } 0 \le r_2 < 2n.$$

Note that since $\ell \in [k, nx + k - 1]$, $q_1 \ge 0$ and $q_2 \ge 0$. Also,

$$q_1 = \tfrac{\ell-k-r_1}{2n} \le \tfrac{nx+k-1-k-r_1}{2n} = \tfrac{nx-1-r_1}{2n} < \tfrac{nx}{2n} < x,$$

while also

$$q_2 = \tfrac{2nx+k-1-\ell-r_2}{2n} \le \tfrac{2nx+k-1-k-r_2}{2n} = \tfrac{2nx-1-r_2}{2n} < x.$$

We claim that $r_i < n$ for either $i = 1$ or $i = 2$. For if not, then $r_1 + r_2 \ge 2n$. Now

$$q_1 + q_2 = \tfrac{\ell-k-r_1}{2n} + \tfrac{2nx+k-1-\ell-r_2}{2n} = x - \tfrac{r_1+r_2+1}{2n},$$

and so $2n$ divides $r_1 + r_2 + 1 > 2n$. Thus, $r_1 + r_2 + 1 \ge 4n$, but this contradicts the fact that neither r_1 nor r_2 exceeds $2n - 1$. Therefore, $r_I < n$ for some $I \in \{1, 2\}$. Set $q = q_I$ and $r = r_I + k$. Then $k \le r < n + k$, and we noted already that $0 \le q < x$. If $I = 1$, then

$$\ell = 2nq_1 + r_1 + k = 2nq + r$$

while if $I = 2$, then

$$2nx + 2k - 1 - \ell = 2nx + k - 1 - \ell + k = 2nq_2 + r_2 + k = 2nq + r.$$

Since h is a ρ_k^+-labeling of G, there exists an edge $\{a, b\}$, where $a \in A$ and $b \in B$, with label either r or $2n + 2k - 1 - r$. In what follows, if $b \in B$, we denote $h(b) + 2n(i-1)$ by b_i.

Case 1: The label of $\{a, b\}$ is r.
Since $h(b) - h(a) = r$, we have

$$\begin{aligned} h'(b_{q+1}) - h'(a) &= h(b) + 2nq - h(a) \\ &= 2nq + r. \end{aligned}$$

Thus, $h'(b_{q+1}) - h'(a) = \ell$ if $\ell = 2nq + r$, and $h'(b_{q+1}) - h'(a) = 2nx + 2k - 1 - \ell$ if $2nx + 2k - 1 - \ell = 2nq + r$.

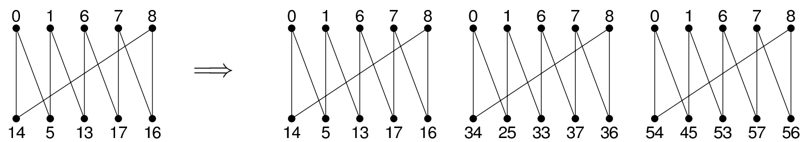
Case 2: The label of $\{a, b\}$ is $2n + 2k - 1 - r$.
Since $h(b) - h(a) = 2n + 2k - 1 - r$, we have

$$\begin{aligned} h'(b_{x-q}) - h'(a) &= h(b) + 2n(x - q - 1) - h(a) \\ &= h(b) - h(a) + 2nx - 2nq - 2n \\ &= 2n + 2k - 1 - r + 2nx - 2nq - 2n \\ &= 2nx + 2k - 1 - (2nq + r). \end{aligned}$$

Thus, $h'(b_{x-q}) - h'(a) = 2nx + 2k - 1 - \ell$ if $\ell = 2nq + r$, and $h'(b_{x-q}) - h'(a) = \ell$ if $2nx + 2k - 1 - \ell = 2nq + r$.

Since G' has size nx and each of the nx edge lengths $k, k+1, \ldots, nx + k - 1$ is the length of an edge, h' is a ρ_k-labeling of G', and we have a cyclic G-decomposition of $\langle [k, nx + k - 1] \rangle_{2nx+2k-1}$. ∎

In Figure 1, we show an example of a ρ_4^+-labeling of C_{10} and the three starters for a cyclic C_{10}-decomposition of $\langle [4, 33] \rangle_{67}$.



(a) A ρ_4^+-labeling of C_{10}.

(b) The three starters for a cyclic C_{10}-decomposition of $\langle [4, 33] \rangle_{67}$.

Figure 1: An ordered k-labeling yielding decompositions of more than one circulant graph.

If the ordered labeling in Theorem 17 is a σ_k^+-labeling with a slight restriction, then a more general result can be obtained.

Theorem 18. *Let G be a bipartite graph with n edges and let k be a positive integer. Let $\{A, B\}$ be a bipartition of $V(G)$ and let h be a σ_k^+-labeling of G with the property that $h(u) < h(v)$ for every $\{u, v\} \in E(G)$ with $u \in A$ and $v \in B$. Suppose moreover that $h(a) - h(b) \neq n$ for any $a \in A$ and $b \in B$. Then for all integers $x \geq 1$ and $t \geq 0$, there exists a cyclic G-decomposition of both $\langle [k, nx + k - 1] \rangle_{2nx+2k-1+t}$ and $\langle [k, nx + k] \rangle_{2nx+2k} - I$, where I is a 1-factor.*

Proof. We define a multigraph G' with

$$V(G') = \{h(a)\colon\ a \in A\} \cup \{h(b) + n(i-1)\colon\ b \in B, i \in [1,x]\} \subseteq [0, 2(nx+k-1)]$$

and

$$E(G') = \big\{\{h(a), h(b) + n(i-1)\}\colon\ \{a,b\} \in E(G), a \in A, b \in B, i \in [1,x]\big\}.$$

Because $h(a) - h(b) \neq n$ for all $a \in A$ and $b \in B$, the two sets whose union comprises $V(G')$ are disjoint. We will show that the labels of the nx edges of G' are exactly the nx integers in $[k, nx+k-1]$, and so G' is actually a graph. Thus if we define $h'(v) = v$ for $v \in V(G')$, then h' is a σ_k-labeling of G' and by Theorem 16, there is a cyclic G'-decomposition of $\langle [k, nx+k-1] \rangle_{2nx+2k-1+t}$. Also, by Theorem 15, there is a cyclic G'-decomposition of $\langle [k, nx+k] \rangle_{2nx+2k} - I$, where I is the 1-factor induced by the edges of length $nx+k$. But for fixed $i \in [1,x]$ the corresponding edges of G' induce a graph isomorphic to G, so G' has a G-decomposition and the theorem follows.

Let $\ell \in [k, nx+k-1]$. We will show that some edge of G' has label ℓ. By the division algorithm, there exist integers q and r, with $0 \leq q < x$ and $0 \leq r < n$, such that $\ell - k = nq + r$. Thus, $\ell = nq + k + r$. Since h is a σ_k^+-labeling of G, there exists an edge $\{a,b\}$, where $a \in A$ and $b \in B$, with label $k+r$. Denote the vertex $h(b) + nq$ by b_{q+1}.

Since $h(b) - h(a) = k + r$, we have

$$\begin{aligned} h'(b_{q+1}) - h'(a) &= h(b) + nq - h(a) \\ &= nq + k + r. \end{aligned}$$

Thus, $h'(b_{q+1}) - h'(a) = \ell$.

Since G' has size nx and each of the nx edge lengths $k, k+1, \ldots, nx+k-1$ is the label of an edge, h' is a σ_k-labeling of G', and the result follows. ∎

3 Decompositions of circulant graphs into 2-regular bipartite graphs

Let G be a bipartite graph with n edges and let k be a positive integer. From the perspective of decomposing circulant graphs, and in light of Theorems 17 and 18, the most desirable k-labelings of G would be uniformly-ordered k-labelings. Moreover, σ_k^{++}-labelings would be preferable to ρ_k^{++}-labelings. El-Zanati and Vanden Eynden (see [14]) conjecture that every bipartite graph admits a ρ^{++}-labeling (and thus, by Lemma 1, a ρ_k^{++}-labeling for each $k \geq 1$). Lemmas 3 and 4 rule out the existence of certain variations of σ_k-labelings.

It is known that bipartite 2-regular graphs that satisfy the parity condition (i.e., have size a multiple of 4) and have at most 3 components admit α-labelings, except for the graph $3C_4$ (see [15]). In fact, $3C_4$ is currently the only known example of a 2-regular bipartite graph that satisfies the parity condition and fails to have an

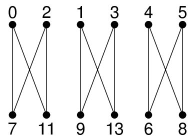


Figure 2: A σ^{++}-labeling of $3C_4$.

α-labeling. A σ^{++}-labeling of $3C_4$ is given in Figure 2. We conjecture that every bipartite 2-regular graph that satisfies the parity condition admits a σ^{++}-labeling. It is also likely that, with the sole exception of $3C_4$, all such graphs admit α-labelings. In [1], it is shown that rC_4 admits an α-labeling for all positive integers $r \neq 3$.

Because both C_{4m} (see [20]) and $C_{4m_1+2} \cup C_{4m_2+2}$ (see [2]) admit α-labelings, we have the following consequence of Theorem 5.

Theorem 19. *Let G be a 2-regular bipartite graph of size $n \equiv 0 \pmod 4$ and let $\{A, B\}$ be a bipartition of $V(G)$. Then G admits a σ^+-labeling f that satisfies $f(a) < f(b)$ for every edge $\{a, b\}$ with $a \in A$ and $b \in B$ and satisfies $f(u) - f(v) < n$ for all $u \in A$ and $v \in B$.*

Moreover, in light of Lemma 1, we have the following.

Corollary 20. *Let G be a 2-regular bipartite graph of size $n \equiv 0 \pmod 4$. Then G admits a σ_k^{++}-labeling for every integer $k \geq n$.*

Because C_{4m+2} admits a ρ^{++}-labeling that does not use $8m + 4$ as a vertex label (see [13]), we can use Theorem 9 to show that bipartite 2-regular graphs that do not satisfy the parity conditions admit ρ^{++}-labelings and hence ρ_k^{++}-labelings for every positive integer k. Moreover, since an α-labeling is a ρ^{++}-labeling, the following holds for all 2-regular bipartite graphs.

Theorem 21. *Every 2-regular bipartite graph admits a ρ_k^{++}-labeling for every positive integer k.*

In light of the above theorems and Theorems 17 and 18, the following hold for 2-regular bipartite graphs.

Corollary 22. *Let G be a 2-regular bipartite graph of size $n \equiv 0 \pmod 4$ and let $k \geq n$ be an integer. Then for all integers $x \geq 1$ and $t \geq 0$, there exists a cyclic G-decomposition of both $\langle [k, nx + k - 1] \rangle_{2nx+2k-1+t}$ and $\langle [k, nx + k] \rangle_{2nx+2k} - I$, where I is a 1-factor.*

Corollary 23. *Let G be a 2-regular bipartite graph of size n. Then there exists a cyclic G-decomposition of $\langle [k, nx + k - 1] \rangle_{2nx+2k-1}$ for all positive integers k and x.*

Little is known about cyclic decompositions of circulant graphs into non-bipartite 2-regular graphs. By using k-labelings of odd cycles, the following is shown in [12].

Corollary 24. *Let $n \geq 3$ be odd and $k \in [1, n]$ with $(n, k) \notin \{(3, 3), (5, 3)\}$. Then there exists a cyclic C_n-decomposition of $\langle [k, nx + k - 1] \rangle_{2nx+2k-1}$ for every positive integer x.*

3.1 Decompositions of K_m into Hamilton cycles and 2-regular bipartite graphs

Several authors have considered the problem of decomposing the complete graph into Hamilton cycles and n-cycles. For example, Bryant and Maenhaut [9] show that for all odd positive integers m, the complete graph K_m can be decomposed into h Hamilton cycles and t triangles (i.e., C_3's) if and only if $hm + 3t = m(m-1)/2$. More recently, Jordon [17] settled the corresponding problem for Hamilton cycles and 5-cycles. Corollaries 22 and 23 can be used to obtain some decompositions of complete graphs into Hamilton cycles and 2-regular graphs.

As noted earlier, $\langle\{m/2\}\rangle_m$ is a 1-factor in K_m when m is even. Otherwise, for $1 \le i < m/2$, it is easy to see that $\langle\{i\}\rangle_m$ consists of δ vertex-disjoint $C_{m/\delta}$'s, where $\delta = \gcd(i, m)$. Thus, $\langle\{i\}\rangle_m$ is a Hamilton cycle if and only if i and m are relatively prime. A special case of a celebrated result by Bermond, Favron, and Mahéo [5] tells us when $\langle\{i, j\}\rangle_m$ can be decomposed into two Hamilton cycles.

Lemma 25. (Bermond et al. [5]) *For positive integers i, j, and m with $i < j < m/2$, the graph $\langle\{i, j\}\rangle_m$ can be decomposed into two Hamilton cycles if and only if* $\gcd(i, j, m) = 1$.

We will make use of the following corollary to the above lemma.

Corollary 26. *Let t and m be positive integers with $t < m/2$ and let $L = [1, t]$. Then $\langle L\rangle_m$ can be decomposed into t Hamilton cycles.*

Proof. If t is even, let $Q_i = \{2i-1, 2i\}$ for $1 \le i \le t/2$. Then $Q = \{Q_i\colon\ 1 \le i \le t/2\}$ is a partition of L. Since the elements of each Q_i are relatively prime, the circulant graph $\langle Q_i\rangle_m$ can be decomposed into two Hamilton cycles for $1 \le i \le t/2$. If t is odd, let $Q_1 = \{1\}$ and for $2 \le i \le (t+1)/2$, let $Q_i = \{2(i-1), 2i-1\}$. Again, $Q = \{Q_i\colon\ 1 \le i \le (t+1)/2\}$ is a partition of L. Now $\langle Q_1\rangle_m$ is a Hamilton cycle and for $2 \le i \le (t+1)/2$, each $\langle Q_i\rangle_m$ can be decomposed into two Hamilton cycles. Thus the result holds. ∎

We will also make use of the following result of Dean [10, 11].

Lemma 27. *For integers r, s, t, and n with $r < s < t < n/2$,* $\gcd(r, s, t, n) = 1$, *and either n is odd or* $\gcd(x, n) = 1$ *for some $x \in \{r, s, t\}$, the graph $\langle\{r, s, t\}\rangle_n$ can be decomposed into three Hamilton cycles.*

First we state a basic lemma about decompositions using ρ_k-labelings.

Lemma 28. *Let G be a graph of size n that admits a ρ_k-labeling for some positive integer k. Then there exists a $2(k-1)$-regular spanning subgraph H of $K_{2(n+k)-1}$ that can be decomposed into $k-1$ Hamilton cycles such that $K_{2(n+k)-1} - H$ has a cyclic G-decomposition.*

Proof. Let H be the spanning subgraph of $K_{2(n+k)-1}$ induced by edge-lengths $[1, k-1]$. By Corollary 26, we can decompose H into $k-1$ Hamilton cycles. By Theorem 14, G decomposes $K_{2(n+k)-1} - H$ cyclically. ∎

If the graph has a σ_k-labeling, then more can be done.

Lemma 29. *Let G be a graph of size n that admits a σ_k-labeling for some positive integer k and let t be a nonnegative integer. Then there exists a cyclic G-decomposition of $K_{2(n+k+t)-1} - H$, where H is a $2(t+k-1)$-regular spanning subgraph that can be decomposed into $t+k-1$ Hamilton cycles. Moreover, there exists a cyclic G-decomposition of $K_{2(n+k+t)} - H'$, where H' is a $2(t+k-1)+1$-regular spanning subgraph that can be decomposed into a 1-factor and $t+k-1$ Hamilton cycles.*

Proof. Let H be the spanning subgraph of $K_{2(n+k+t)-1}$ induced by edge-lengths $[1, k-1] \cup [n+k, n+k+t-1]$. By Corollary 26, we can decompose $\langle [1, k-1] \rangle_{2(n+k+t)-1}$ into $k-1$ Hamilton cycles. If t is even, then $Q = \{\{n+k+2i-2, n+k+2i-1\} \colon 1 \le i \le t/2\}$ is a partition of $[n+k, n+k+t-1]$. If t is odd, then $Q = \{\{n+k+2i-2, n+k+2i-1\} \colon 1 \le i \le (t-1)/2\} \cup \{n+k+t-1\}$ is a partition of $[n+k, n+k+t-1]$. In either case, the elements of Q that are pairs of consecutive integers induce graphs that can be decomposed into Hamilton cycles by Lemma 25. Moreover, when t is odd, the graph $\langle \{n+k+t-1\} \rangle_{2(n+k+t)-1}$ is a Hamilton cycle since the $\gcd(n+k+t-1, 2(n+k+t)-1) = 1$. By Theorem 16, G decomposes $K_{2(n+k+t)-1} - H$ cyclically.

A similar argument can be used to obtain the decomposition of $K_{2(n+k+t)} - H'$, where H' is the graph induced by edge-lengths $[1, k-1] \cup [n+k, n+k+t]$. As before, $\langle [1, k-1] \rangle_{2(n+k+t)}$ can be decomposed into $k-1$ Hamilton cycles. If t is even, then $[n+k, n+k+t-1]$ can be partitioned into pairs of consecutive integers. By Lemma 25, the subgraphs induced by these pairs can be decomposed into Hamilton cycles. The subgraph $\langle \{n+k+t\} \rangle_{2(n+k+t)}$ is the 1-factor. If $t > 1$ is odd, then $[n+k, n+k+t-4]$ can be partitioned into pairs of consecutive integers and thus $\langle [n+k, n+k+t-4] \rangle_{2(n+k+t)}$ can be decomposed into Hamilton cycles (if $t = 3$, then the circulant is empty and the decomposition is trivial). Moreover, $\langle [n+k+t-3, n+k+t-1] \rangle_{2(n+k+t)}$ can be decomposed into 3 Hamilton cycles by Lemma 27. If $t = 1$ and $n+k$ is odd, then $\langle \{n+k\} \rangle_{2(n+k+t)}$ is a Hamilton cycle and thus $\langle \{n+k, n+k+1\} \rangle_{2(n+k+t)}$ can be decomposed into a Hamilton cycle and a 1-factor. Here, the subgraph $\langle \{n+k+t\} \rangle_{2(n+k+t)}$ is the 1-factor. Finally, if $t = 1$ and $n+k$ is even, then $\langle \{n+k, n+k+t\} \rangle_{2(n+k+t)}$ is isomorphic to $C_{n+k+t} \times K_2$ and can thus be decomposed into a Hamilton cycle and a 1-factor. ∎

If the graph G in the previous two lemmas is bipartite and admits a uniformly-ordered k-labeling, then we have the following.

Lemma 30. *Let G be a graph of size n that admits a ρ_k^{++}-labeling for some positive integer k. Then there exists a cyclic G-decomposition of $K_{2(nx+k)-1} - H$, where H is a $2(k-1)$-regular spanning subgraph that can be decomposed into $k-1$ Hamilton cycles.*

Lemma 31. *Let G be a graph of size n that admits a σ_k^{++}-labeling for some positive integer k and let t be a nonnegative integer. Then there exists a cyclic G-decomposition of $K_{2(nx+k+t)-1} - H$, where H is a $2(t+k-1)$-regular spanning subgraph that can be decomposed into $t+k-1$ Hamilton cycles. Moreover, there exists* *a cyclic G-decomposition of $K_{2(nx+k+t)} - H'$, where H' is a $(2(t+k-1)+1)$-regular spanning subgraph that can be decomposed into a 1-factor and $t+k-1$ Hamilton cycles.*

Since every 2-regular bipartite graph admits a ρ_k^{++}-labeling for every positive integer k, we have the following.

Corollary 32. *Let G be a 2-regular bipartite graph of size n and let k be a positive integer. There exists a cyclic G-decomposition of $K_{2(nx+k)-1} - H$, where H is a $2(k-1)$-regular spanning subgraph that can be decomposed into $k-1$ Hamilton cycles.*

In conclusion, we remark that Corollary 32 contributes towards a solution of a conjecture of Alspach [4] that there exists a decomposition of K_n (n odd) into cycles of lengths $m_1, m_2, \ldots, m_t$ whenever $3 \le m_i \le n$ for $1 \le i \le t$ and $m_1 + m_2 + \cdots + m_t = n(n-1)/2$. Alspach's Conjecture was settled recently by Bryant, Horsley, and Pettersson [8].

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