Irreversible k-threshold and majority conversion processes on complete multipartite graphs and graph products

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Abstract

In graph theoretical models of the spread of disease through populations, the spread of opinion through social networks, and the spread of faults through distributed computer networks, vertices are in two states, either black or white, and these states are dynamically updated at discrete time steps according to the rules of the particular conversion process used in the model. This paper considers the irreversible k-threshold and majority conversion processes. In an irreversible k-threshold (respectively, majority) conversion process, a vertex is permanently colored black in a

certain time period if at least k (respectively, at least half) of its neighbors were black in the previous time period. A k-conversion set (respectively, dynamic monopoly) is a set of vertices which, if initially colored black, will result in all vertices eventually being colored black under a k-threshold (respectively, majority) conversion process. We answer several open problems by presenting bounds and some exact values of the minimum number of vertices in k-conversion sets and dynamic monopolies of complete multipartite graphs, as well as of Cartesian and tensor products of two graphs.

1   Introduction

Graph theoretical models have been used in the past decade to study the spreads of (a) disease through a population, (b) opinion through a social network, and (c) faults in a distributed computer network [2, 5, 6, 7, 8, 11, 13]. In these models, each vertex represents a person/object that is either colored black (e.g., infected, holding a certain opinion, corrupted) or white (e.g., uninfected, not holding a certain opinion, not corrupted), and each edge represents a relevant interaction between two people/objects. At discrete time steps, the colors of the vertices are updated under some conversion process. Many natural questions then arise. For example, what is the minimum number of initially black vertices in a graph that will eventually cause all vertices to turn black? Or, given an initially black subset of vertices in a graph, how many discrete time steps will it take to turn all of its vertices black? These questions may have relevance in applications to the design of immunization and containment strategies to prevent the spread of a disease, opinion, or fault to all of the vertices in the graph. While many other related questions can be asked, this paper focuses on the first question, sometimes referred to as the *target set selection problem* in the context of viral marketing and the spread of influence [1, 12], under two distinct conversion processes.

In an *irreversible k-threshold conversion process*, if a white vertex has at least k black neighbors at time $t - 1$, it will be permanently colored black at time t. Irreversible k-threshold conversion processes have been used to model the spread of disease and opinion through social networks [5, 6, 11]. A *k-conversion set* is a set of vertices that, when colored black at time $t = 0$, will eventually cause the graph to turn completely black under an irreversible k-threshold conversion process. In this paper, we consider the fundamental problem of determining exact values or upper bounds for the minimum number of vertices in k-conversion sets of a graph G. This number is denoted by $\min_k(G)$. A k-conversion set with exactly $\min_k(G)$ vertices will be called *minimum k-conversion set*. Determining $\min_k(G)$ is NP-complete for fixed values of $k \geq 2$ [6, 11]. Previously, exact values of $\min_k(G)$ for all possible k have been found when G is a path, cycle, complete multipartite graph, or tree [6]. In the same article, some exact values and upper bounds of $\min_k(G)$ for certain values of k have also been determined when G is a planar, cylindrical, or toroidal square grid.

In an *irreversible majority conversion process*, if at least half of the neighbors of a white vertex are black at time $t-1$, the vertex will be permanently colored black at time t. Although irreversible majority conversion processes may be used to model the spread of opinion through a social network, these processes are more often used to model the propagation of computer faults [2, 3, 7, 13, 14]. For example, in a distributed computer network, nodes communicate with each other to maintain uncorrupted copies of the same variables. These nodes may determine the state of a variable based on a majority vote of their neighbors' states of the same variable. A set of vertices that, when colored black at time $t = 0$, will eventually cause the entire graph to be colored black under an irreversible majority conversion process is called a *dynamic monopoly*, or *dynamo*. Dynamos were first introduced by Peleg [14]. We are concerned with such sets of minimum number of vertices, called *minimum dynamos*, and we let $\min_D(G)$ denote the number of vertices in a minimum dynamo of a graph G. Since the problem of determining $\min_D(G)$ is NP-hard [12], the literature has focused on certain families of graphs for which exact values (e.g., generalized Petersen graphs and line graphs of complete bipartite graphs [16]) or bounds on $\min_D(G)$ could be determined (e.g., planar, cylindrical, and toroidal triangular grids [2]; toroidal square grids and related toroidal structures [7]). The papers [1, 4] independently verified that the $\min_D(G)$ for general graphs is at most $|V(G)|/2$ but this general upper bound has been recently improved to $\alpha'(G)$ which is defined as the maximum number of independent edges in G [9]. More specifically, this latter paper shows that any graph G admits a *monopoly* with at most $\alpha'(G)$ vertices. A *monopoly* D of G is a subset of vertices in which every vertex of G has at least half of its neighbors in D. In particular, monopolies are dynamos that, when colored black at a certain time step, will cause the entire graph to be colored black in the next time step under an irreversible majority conversion process.

In Section 2, we determine exact values of $\min_k(G)$ and $\min_D(G)$ when G is a complete multipartite graph, improving and building on related results in prior work by Dreyer and Roberts [6]. As many graphs can be expressed as products of simpler graphs, it would be desirable to find relationships between the size of a minimum k-conversion set (respectively, minimum dynamo) of a given product of graphs and the sizes of minimum k-conversion sets (respectively, minimum dynamos) of individual graphs that are factors in the product. We will focus on the Cartesian and tensor products of graphs. In Section 3, we provide upper bounds of the same graph invariants when G is a Cartesian product of two graphs. Section 4 contains analogous results for the tensor product of two graphs, and in this case, the upper bounds are shown to be tight. A brief conclusion and discussion of future work are included in Section 5.

As both of the conversion processes considered in this paper are irreversible, that is, once a white vertex is colored black it will remain permanently black, we will omit the word *irreversible* and we will often use *color(ed)* (respectively, *uncolor(ed)*) to mean color(ed) black (respectively, color(ed) white) for brevity. Let X be a k-conversion set (respectively, dynamo) of a graph G with set of vertices $V(G)$. We will sometimes simply say *initially color X* to mean that all the vertices in X are

initially colored black at time $t = 0$ and all the vertices in $V(G) - X$ are white; we will also say that X *colors* G or, equivalently, that G *is colored by* X in T time steps, if when X is initially colored, all the vertices in $V(G)$ become colored by time step T under a k-threshold (respectively, majority) conversion process.

2   Exact values of $\min_k(G)$ and $\min_D(G)$ of complete multipartite graphs G

For $m \geq 2$, let us denote by $K_{p_1,p_2,\ldots,p_m}$ the *complete multipartite graph* with m partite sets with $p_1, p_2, \ldots, p_m$ vertices, respectively. Such graphs may be useful in modeling computer networks wherein most pairs of computers are connected, however certain subsets of computers share no connections. Dreyer and Roberts [6] provided exact values of $\min_k(K_{p_1,p_2,\ldots,p_m})$ through a series of related results. Here, we provide an equivalent unified result with a straight-forward statement and a simpler proof. For the remainder of this paper, we will use the standard notation $|X|$; for the cardinality of a set X.

Lemma 2.1 *Let G be a graph with n vertices and let X be the set of all vertices with degree less than k. Then, $\min_k(G) \geq |X|$; in particular, $\min_k(G) = n$ if $n \leq k$.*

PROOF: Since all vertices in X have degree less than k, X must be contained in any k-conversion set, and therefore $\min_k(G) \geq |X|$. If $n \leq k$, then $X = V(G)$ and therefore $n \geq \min_k(G) \geq |X| = |V(G)| = n$. $\square$

Theorem 2.2 *Let $G = K_{p_1,p_2,\ldots,p_m}$ be a complete multipartite graph with $p_1 + p_2 + \ldots + p_m = n$ and $p_1 \geq p_2 \geq \ldots \geq p_m$. Let X be the set of all vertices with degree less than k. Then,*

$$\min_k(G) = \begin{cases} \max\{|X|, k\}, & n > k \\ n, & n \leq k. \end{cases}$$

PROOF: If $n \leq k$, then $\min_k(G) = n$ by Lemma 2.1. Let us consider the case $n > k$. For each $i = 1, 2, \ldots, m$, let V_i be the partite set of G with p_i vertices. Note that a vertex belongs to X if and only if its entire partite set belongs to X. Therefore X is the union of some partite sets in G. If $k \leq |X|$, then X is a k-conversion set since X is a union of whole partite sets and therefore all the vertices in $V(G) - X$ are adjacent to the vertices in X (of which there are at least k), so they would become colored in time step $t = 1$ if X is initially colored; hence $\min_k(G) \leq |X|$. From Lemma 2.1, we can conclude that $\min_k(G) = |X|$.

Finally, we examine the remaining case where $n > k > |X|$. To show that $\min_k(G) = k$ under these conditions, it suffices to exhibit a k-conversion set S with k vertices since the inequality $\min_k(G) \geq k$ is obvious. Color all vertices in X, and in addition, color a subset of $V(G) - X$ with $k - |X|$ vertices, where this subset is chosen to

contain a union of entire partite sets, along with a possibly empty proper subset of some partite set V_j. This set S of k initially black vertices is non-empty because $n - |X| > k - |X| > 0$. Moreover, S is a union of partite sets, except possibly for the proper subset of V_j because X is the union of partite sets. Every vertex in $V(G)$, except possibly for the vertices in V_j, will be colored by time $t = 1$ since every white vertex not in V_j is connected to the $|X|$ vertices in X and the $k - |X|$ vertices in $S - X$, giving a total of k vertices. The only possible vertices that could still be white at time $t = 1$ would be in V_j, but these would be colored at time $t = 2$. This is because the vertices in V_j are connected to every vertex outside V_j, which are all black by time $t = 1$, and since V_j is not part of X, its vertices have degree at least k. Thus, S is a k-conversion set of size k as desired. $\qquad\square$

We now extend this previous result to a majority conversion process and find $\min_D(K_{p_1, p_2, \ldots, p_m})$.

Theorem 2.3 *Let $G = K_{p_1, p_2, \ldots, p_m}$ be a complete multipartite graph with $p_1 + p_2 + \ldots + p_m = n$ and $p_1 \geq p_2 \geq \ldots \geq p_m$. Then,*

$$\min_D(G) = \left\lceil \frac{n - p_1}{2} \right\rceil.$$

PROOF: For each $i = 1, 2, \ldots, m$, note that the vertices in the partite set V_i with p_i vertices have degree $n - p_i \geq n - p_1$. So any dynamo of G must have at least $\left\lceil \frac{n-p_1}{2} \right\rceil$ vertices, and therefore $\min_D(G) \geq \left\lceil \frac{n-p_1}{2} \right\rceil$. To show the equality holds, it suffices to exhibit a dynamo with $\left\lceil \frac{n-p_1}{2} \right\rceil$ vertices. To do this, initially color any subset D of $V(G) - V_1$ with $\left\lceil \frac{n-p_1}{2} \right\rceil$ vertices. Since the vertices in V_1 have $n - p_1$ neighbors, all vertices in V_1 will become colored at time $t = 1$. Therefore, at least $p_1 + \left\lceil \frac{n-p_1}{2} \right\rceil = \left\lceil \frac{n+p_1}{2} \right\rceil$ vertices will be colored by time $t = 1$. At this point, since any partite set V_i for $i = 2, 3, \ldots, m$ has at most p_i black vertices, any vertex in any V_i is adjacent to at least $\left\lceil \frac{n+p_1}{2} \right\rceil - p_i$ black vertices; since $\left\lceil \frac{n-p_i}{2} \right\rceil$ vertices are required to color any vertex in V_i (because a vertex in V_i has $n - p_i$ neighbors), all uncolored vertices will become colored at time $t = 2$ as $\left\lceil \frac{n+p_1}{2} \right\rceil - p_i = \left\lceil \frac{n+p_1-2p_i}{2} \right\rceil \geq \left\lceil \frac{n-p_i}{2} \right\rceil$. Hence, D is a dynamo with $\left\lceil \frac{n-p_1}{2} \right\rceil$ vertices as desired. $\qquad\square$

3 Upper bounds of $\min_k(G \square H)$ and $\min_D(G \square H)$ of the Cartesian product $G \square H$

In this section, we will provide general upper bounds of $\min_k(G \square H)$ and $\min_D(G \square H)$ where $G \square H$ denotes the Cartesian product of two graphs G and H. Let us first recall the definition of $G \square H$ and some related concepts.

Definition 3.1 *The Cartesian product of two disjoint graphs G and H is the graph $G \square H$ such that $V(G \square H) = V(G) \times V(H)$ and two vertices (u, u') and (v, v') are adjacent if and only if either $u = v$ and u' is adjacent to v' in H, or $u' = v'$ and*

u is adjacent to v in G. For a fixed vertex v in H, let G_v be the subgraph of $G\Box H$ induced by the vertices (u, v) for every vertex u in G; note that G_v is isomorphic to G. Similarly, for a fixed vertex u in G, let H_u be the subgraph of $G\Box H$ induced by the vertices (u, v) for every vertex v in H; note that H_u is isomorphic to H.

We first provide an upper bound on $\min_k(G\Box H)$.

Theorem 3.2 *Let G and H be two graphs. Then,*

$$\min_k(G\Box H) \leq \min_k(G)\min_k(H).$$

PROOF: Let S_G and S_H be minimum k-conversion sets of G and H, respectively. Let S be the set of $\min_k(G)\min_k(H)$ vertices (u, v) in $G\Box H$ where u is in S_G and v is in S_H. To verify the proposed upper bound, it suffices to show that S is a k-conversion set of $G\Box H$.

Initially color S. For a fixed vertex u in S_G, H_u contains vertices (u, v) for all v in S_H. Since these vertices are initially colored and S_H is a k-conversion set of H, then all vertices in H_u will eventually be colored as H_u is isomorphic to H and the coloration of H by S_H induces the coloration of H_u by the set of vertices (u, v) for v in S_H.

To conclude the proof, it suffices to show that the union of all H_u for u in S_G is also a k-conversion set of $G\Box H$. Initially color the vertices in this union. For each v in H, G_v contains vertices (u, v) for all u in S_G. Since all the vertices in H_u for u in S_G are colored and since S_G is a k-conversion set of G, then all the vertices in G_v will eventually get colored because G_v is isomorphic to G and the coloration of G by S_G induces the coloration of G_v by the set of vertices (u, v) for u in S_G. Therefore, we showed that all vertices in $G\Box H$ will get colored and consequently S is a k-conversion set of $G\Box H$ as desired. $\Box$

We will now provide the result analogous to Theorem 3.2 under a majority conversion process. In what follows, given a graph G and a non-empty subset X of $V(G)$, we denote by $\deg_G(u)$ the degree of a vertex u in G, and define $\deg_X(u)$ as the number of neighbors of u in X.

Theorem 3.3 *Let G and H be two graphs. Then,*

$$\min_D(G\Box H) \leq \min_D(G)|V(H)| + \min_D(H)|V(G)| - \min_D(G)\min_D(H).$$

PROOF: Let D_G and D_H be two minimum dynamos of G and H, respectively. Let D be the set of vertices (u, v) in $G\Box H$ where either u is in D_G or v is in D_H, or both. (Refer to Figure 1 for a concrete example). Note that D has $|V(H)|$ vertices (u, v) for each u in D_G, $|V(G)|$ vertices (u, v) for each v in D_H, and we must subtract $\min_D(G)\min_D(H)$ vertices to account for double-counting the vertices (u, v) where u and v are in D_G and D_H, respectively. Thus,

$$
\begin{aligned}
|D| &= \min_D(G)|V(H)| + \min_D(H)|V(G| - \min_D(G)\min_D(H) \\
&\geq \min_D(G)\min_D(H) + \min_D(H)\min_D(G) - \min_D(G)\min_D(H) \\
&= \min_D(G)\min_D(H) > 0
\end{aligned}
$$

and therefore, D is not empty. To verify the proposed bound, it suffices to show that D is a dynamo of $G\square H$.
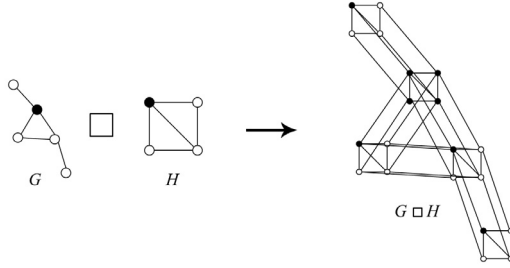

Figure 1: Graphs G, H, $G\square H$ with dynamos D_G, D_H, D, respectively. (dynamos are colored black), where $|D| = \min_D(G)|V(H)| + \min_D(H)|V(G)| - \min_D(G)\min_D(H)$ as constructed in the proof of Theorem 3.3.

First, we need to partition the vertices of $G\square H$ based on the coloration of G by D_G as follows. Suppose it takes T time steps to fully color G by D_G and for each $t = 0, 1, \ldots, T$, let $H(t)$ be the subgraph of $G\square H$ induced by the vertices (u, v) where u is a vertex that becomes colored at time step t in G. Note that each $H(t)$ is the disjoint union of subgraphs H_u (recall Definition 3.1) for u in $H(t)$, and $H(0), H(1), \ldots, H(t)$ partitions the vertices in $G\square H$. (Refer to Figure 2 for the coloration of the individual graphs G, H, and $G\square H$ in Figure 1, and to Figure 3 for the corresponding subgraphs $H(t)$ for $t = 0, 1, \ldots, T$.)

Initially color D. We will show that all vertices in $G\square H$ will eventually be colored by proving that each $H(t)$ for $t = 0, 1, \ldots, T$ will eventually become colored. We proceed by induction on t. As the base case, note that every vertex in $H(0)$ has its first coordinate in D_G and consequently belongs to D which is assumed to be initially colored. Now assume that all of the vertices in $H(0), H(1), \ldots, H(t-1)$ have been colored and let us show that the vertices in $H(t)$ will eventually become colored. Consider an arbitrary uncolored vertex (u, v) in $H(t)$. Note that (u, v) has $\deg_{G\square H - H_u}(u, v) = \deg_G(u)$ neighbors not in H_u and $\deg_{H_u}(u, v) = \deg_H(v)$ neighbors in H_u and therefore $\deg_{G\square H}(u, v) = \deg_G(u) + \deg_H(v)$. At least half of the neighbors of (u, v) which are not in H_u must be in the union of $H(i)$ for $i = 0, 1, \ldots, t-1$ which contains only colored vertices. Moreover, the set (u, v')
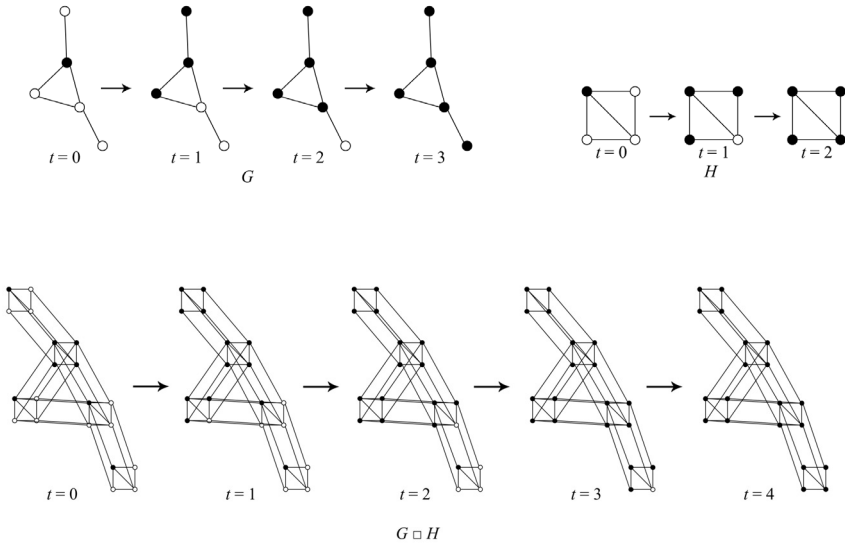
Figure 2: Individual colorations of graphs G, H, $G \square H$ of Figure 1 by dynamos D_G, D_H, D, respectively.

with v' in D_H is a dynamo of the subgraph H_u so half of the neighbors of (u, v) in this subgraph will eventually get colored similarly and at least as fast as H is colored by D_H. Combining these colorations of the neighbors of (u, v) in H_u and out of H_u we can conclude that half of all the neighbors of (u, v) will eventually be colored, and consequently (u, v) will become colored. Thus every vertex in $H(t)$ will be eventually colored. It now follows by induction that D is a dynamo of $G \square H$ providing the desired upper bound on $\min_D(G \square H)$. $\square$

There exist graphs G and H so that the upper bound on $\min_D(G \square H)$ provided in Theorem 3.3 is tight. For example, if G consists of n isolated vertices and H consists of m isolated vertices, then $G \square H$ consists of nm isolated vertices and the equality for the bound in Theorem 3.3 will follow. If we require G and H to not contain isolated vertices, Theorem 3.6 below offers a smaller upper bound on $\min_D(G \square H)$. We must first prove Lemma 3.4 and Corollary 3.5.

Lemma 3.4 *Let G be a graph without isolated vertices, and let D be a minimal dynamo of G (meaning that removing any vertex from D results in a set that is no longer a dynamo). Then the set of vertices $V(G) - D$ is a dynamo of G. In addition, the dynamo $V(G) - D$ colors all the vertices in G by time step $t = 1$.*

PROOF: Initially color D. If a vertex u in D has $\deg_D(u) \geq \deg_G(u)/2 > 0$ (recall that G does not have isolated vertices), then $D - u$ would be a dynamo of G because
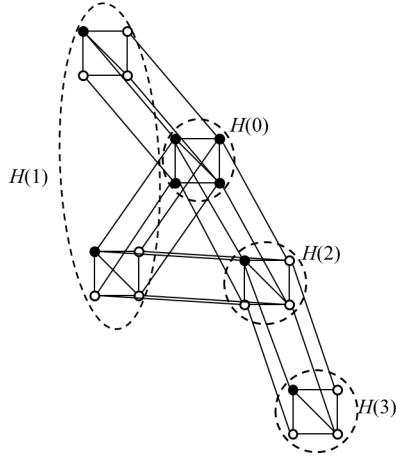
Figure 3: $V(G \Box H)$ of Figure 1 partitioned into $H(t)$, $t = 0, 1, \ldots, T$ as in the proof of Theorem 3.6.

if all the vertices in $D - u$ were initially black and u was initially white then its $\deg_D(u)$ black neighbors in $D - u$ would cause u to become black in the next time step. This would contradict the minimality of the dynamo D. Hence, $\deg_D(u) < \deg_G(u)/2$ for all vertices u in D. Note that this implies $D \neq V(G)$ since otherwise $\deg_G(u) = \deg_D(u) < \deg_G(u)/2$ for any u in D, a contradiction.

Now, initially color $V(G) - D$. Previously, we have shown that for each vertex u in D, $\deg_D(u) < \deg_G(u)/2$ which is equivalent to $\deg_{V(G)-D}(u) \geq \deg_G(u)/2$, and therefore u will be colored in the next time step. Thus, $V(G) - D$ is a dynamo of G and all the vertices in G become colored by time step $t = 1$. $\qquad \square$

Corollary 3.5 *Let G be a graph without isolated vertices. Then, $\min_D(G) \leq |V(G)|/2$.*

PROOF: Let D be a minimum (hence minimal) dynamo of G. Suppose for contradiction that $\min_D(G) = |D| > |V(G)|/2$. By Lemma 3.4, $V(G) - D \neq \emptyset$ is a dynamo of G and

$$|V(G) - D| = |V(G)| - |D| < |V(G)| - |V(G)|/2 = |V(G)|/2 < |D| = \min_D(G)$$

contradicting the minimality of the dynamo D. Therefore, $\min_D(G) \leq |V(G)|/2$. $\square$

We note that the previous corollary was independently obtained by Chang and Lyuu [4] and also by Ackerman et al.[1], however our derivation is simpler. We should note that the following stronger result has also been obtained in [4]: any graph G with

no isolated vertices contains a dynamic monopoly with $\lceil |V(G)|/2 \rceil$ vertices under an irreversible *strict* majority conversion process in which a white vertex with more than half of its neighbors colored black at time $t - 1$, will be permanently colored black at time t. This general upper bound was improved to $\alpha'(G) + 1$ in [10] (recall from Section 1 that $\alpha'(G)$ is the maximum number of independent edges in G).

We are now prepared to present Theorem 3.6 which provides a smaller upper bound for $\min_D(G \square H)$ than Theorem 3.3 when G and H do not contain isolated vertices.

Theorem 3.6 *Let G and H be two graphs without isolated vertices. Then,*

$$\min_D(G \square H) \leq \min_D(G)|V(H)| + \min_D(H)|V(G)| - 2\min_D(G)\min_D(H).$$

PROOF: Let D_G and D_H be two minimum dynamos of G and H, respectively, and let D be the dynamo of $G \square H$ constructed in the proof of Theorem 3.3 with

$$|D| = \min_D(G)|V(H)| + \min_D(H)|V(G)| - \min_D(G)\min_D(H).$$

Let D' be the set of $\min_D(G)\min_D(H)$ vertices (u, v) in D where u and v are in D_G and D_H, respectively. Using Corollary 3.5 for G and H, we have that

$$
\begin{aligned}
|D - D'| &= \min_D(G)|V(H)| + \min_D(H)|V(G)| - 2\min_D(G)\min_D(H) \\
&\geq \min_D(G)(2\min_D(H)) + \min_D(H)(2\min_D(G)) - 2\min_D(G)\min_D(H) \\
&= 2\min_D(G)\min_D(H) > 0,
\end{aligned}
$$

and therefore $D - D'$ is not empty. To verify the proposed bound, we will show that $D - D'$ is a dynamo of $G \square H$. Initially color $D - D'$. It then suffices to show that the vertices in D' will eventually get colored since D is a dynamo of $G \square H$.

Let (u, v) be an arbitrary vertex in D'. By the definition of D', we have that u and v are in D_G and D_H, respectively. Note that (u, v) belongs to H_u and G_v (recall Definition 3.1) and the vertices in the neighborhood of (u, v) must be in exactly one of these two subgraphs, so $\deg_{G \square H}(u, v) = \deg_G(u) + \deg_H(v)$. From Lemma 3.4 we have that $V(G) - D_G$ (respectively, $V(H) - D_H$) is a dynamo of G (respectively, H) which colors all the vertices in G (respectively, H) by time step $t = 1$. This means that the vertex u (respectively, v) has at least half of its neighbors in $V(G) - D_G$ (respectively, $V(H) - D_H$). Therefore the vertex (u, v) has at least half of its neighbors in $D - D'$ which was initially colored and we can conclude that (u, v) will become colored in the next time step. This shows that $D - D'$ is also a dynamo of $G \square H$ as desired. □

4   Tight upper bounds of $\min_k(G \times H)$ and $\min_D(G \times H)$ of the tensor product $G \times H$

Recall the definition of the tensor product of two graphs.

Definition 4.1 *The tensor product of two disjoint graphs G and H is the graph $G \times H$ such that $V(G \times H) = V(G) \times V(H)$ and two vertices (u, u') and (v, v') are adjacent if and only if u is adjacent to v in G and u' is adjacent to v' in H.*

Note that each of the i_G (respectively, i_H) isolated vertices of G (respectively, H) generates $|V(H)|$ (respectively, $|V(G)|$) isolated vertices in $G \times H$, and these are the only isolated vertices in $G \times H$. Thus, $G \times H$ contains $i_G|V(H)| + i_H|V(G)| - i_G i_H$ isolated vertices which must be all contained in any k-conversion set and dynamo of $G \times H$. We can then focus on determining upper bounds of $\min_k(G \times H)$ and $\min_D(G \times H)$ when G and H are graphs without isolated vertices and later add the number of potential isolated vertices to these bounds. We will first provide an upper bound on $\min_k(G \times H)$.

Theorem 4.2 *Let G and H be two graphs without isolated vertices. Then,*

$$\min_k(G \times H) \leq \min\left\{\min_k(G)|V(H)|, \min_k(H)|V(G)|\right\}.$$

PROOF: We may assume without loss of generality that $\min_k(G)|V(H)| \leq \min_k(H)$ $|V(G)|$ because the tensor product is commutative. Let S_G be a minimum k-conversion set of G, and let S be the set of $\min_k(G)|V(H)|$ vertices (u, v) in $G \times H$ where u is in S_G. To verify the proposed upper bound, it suffices to show that S is a k-conversion set of $G \times H$.

We will first partition the vertices in $G \times H$ based on the coloration of G by S_G. Suppose it takes T time steps to fully color G by S_G and for each $t = 0, 1, \ldots, T$, let $H(t)$ be the set of vertices (u, v) where u is a vertex that becomes colored at time step t in G. The sets $H(0), H(1), \ldots, H(t)$ partition the vertices in $G \times H$.

Initially color S. We will show that all vertices in $G \times H$ will eventually be colored by proving that each $H(t)$ for $t = 0, 1, \ldots, T$ will eventually be colored. We proceed by induction on t. As the base case, note that every vertex in $H(0)$ has its first coordinate in S_G and consequently belongs to S which is assumed to be initially colored. Now suppose that all of the vertices in $H(0), H(1), \ldots, H(t-1)$ have been colored and let us show that the vertices in $H(t)$ will become colored. Consider an arbitrary uncolored vertex (u, v) in $H(t)$. Note that $\deg_{G \times H}(u, v) = \deg_G(u)\deg_H(v)$. Since u is colored in time step t in G by S_G, at least k of its $\deg_G(u)$ neighbors in G were black by time step $t-1$. Each one of these black neighbors in G will generate $\deg_H(v) > 0$ (recall that H does not have isolated vertices) neighbors of (u, v) in $G \times H$ in the union of $H(i)$, $i = 0, 1, \ldots, t-1$ which contains only colored vertices. Therefore, at least k of the $\deg_G(u)\deg_H(v)$ neighbors of (u, v) in $G \times H$ are colored, and hence, (u, v) will become colored. It now follows by induction that S is a k-conversion set of $G \times H$ providing the desired upper bound on $\min_k(G \times H)$. $\square$

If in the proof of Theorem 4.2 we replace each occurrence of $\min_k$, S_G, S, "k-conversion set", and "at least k" with, respectively, $\min_D$, D_G, D, "dynamo," and "at least half," we obtain a proof for the analogous result under a majority conversion
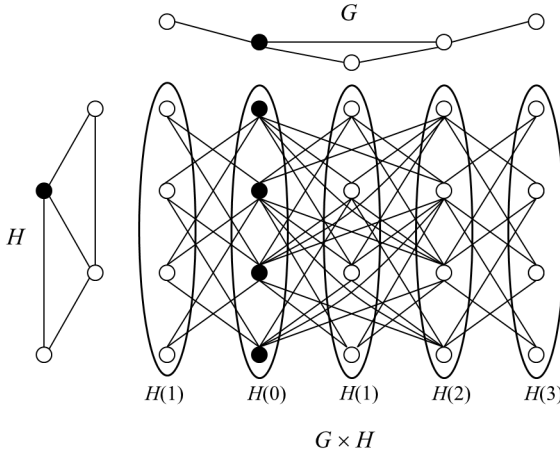
Figure 4: Graphs G, H, $G \times H$ with minimum dynamos D_G, D_H, D, respectively (dynamos are colored black), where $|D| = \min_D(G)|V|$ for Theorem 4.3; $V(G \times H)$ partitioned into $H(t)$, $t = 0, 1, \ldots, T$.

process stated in Theorem 4.3 below. We omit the details in this proof but provide a concrete example in Figure 4 to illustrate the construction of such dynamo D of $G \times H$ where $V(G \times H)$ is partitioned into $H(t)$, $t = 0, 1, \ldots, T$.

Theorem 4.3 *Let G and H be two graphs without isolated vertices. Then,*

$$\min_D(G \times H) \leq \min\{\min_D(G)|V(H)|, \min_D(H)|V(G)|\}.$$

We will close this section by providing an infinite family of graphs where the upper bounds in Theorems 4.2 and 4.3 are tight. We will use the *bipartite double cover* of a graph G which is defined as the tensor product $G \times K_2$ where K_2 is the complete graph on two vertices. We state, without proof, the following lemma due to Sampathkumar [15].

Lemma 4.4 *The bipartite double cover of any connected bipartite graph G is isomorphic to the graph $2G$, that is, two disjoint copies of G.*

Proposition 4.5 below, easily follows from Lemma 4.4.

Proposition 4.5 *Let G be a connected bipartite graph. Then, $\min_k(G \times K_2) = 2\min_k(G)$ and $\min_D(G \times K_2) = 2\min_D(G)$.*

From Proposition 4.5, one can verify the equalities in Theorems 4.2 and 4.3 if G is any connected bipartite graph with at least two vertices and $H = K_2$. We leave the details to the reader.

5 Conclusions and Future Work

In this paper, we have provided exact values of $\min_k(G)$ and $\min_D(G)$ when G is a complete multipartite graph. We have also found upper bounds on these graph invariants when G is the Cartesian or tensor product of two graphs, but these bounds are not always tight. A natural next step is to improve on these general upper bounds for the Cartesian and tensor products within specific families of graphs and ultimately to obtain exact values.

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