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Covers in 5-uniform intersecting families with covering number three

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Abstract

Let k be an integer. It is known that the maximum number of three-covers of a k-uniform intersecting family with covering number three is $k^3 - 3k^2 + 6k - 4$ for $k = 3, 4$ or $k \geq 9$. In this paper, we prove that the same holds for $k = 5$, and show that a 5-uniform intersecting family with covering number three which has 76 three-covers is uniquely determined.

1 Introduction

Throughout this paper, we let X denote a finite set. We let 2^X denote the family of all subsets of X and, for an integer $k \geq 1$, we let $\binom{X}{k}$ denote the family of those subsets of X which have cardinality k. A family $\mathcal{F} \subseteq 2^X$ is said to be k-uniform if $\mathcal{F} \subseteq \binom{X}{k}$. Let $\mathcal{F} \subseteq 2^X$ be a k-uniform family. We say that $\mathcal{F}$ is *intersecting* if $F \cap G \neq \emptyset$ for all $F, G \in \mathcal{F}$. A set $C \subseteq X$ is called a *cover* of $\mathcal{F}$ if it intersects with every member of $\mathcal{F}$, i.e., $C \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. Let $\mathcal{C}(\mathcal{F}) := \{C : C \text{ is a cover of } \mathcal{F}\}$. The *covering number* of $\mathcal{F}$, denoted by $\tau(\mathcal{F})$, is defined by $\tau(\mathcal{F}) := \min_{C \in \mathcal{C}(\mathcal{F})} |C|$. Note that if $\mathcal{F}$ is intersecting, then we have $\tau(\mathcal{F}) \leq k$ because $\mathcal{F} \subseteq \mathcal{C}(\mathcal{F})$. For an integer $t \geq 1$, we define $\mathcal{C}_t(\mathcal{F}) := \mathcal{C}(\mathcal{F}) \cap \binom{X}{t}$. Note that if $t < \tau(\mathcal{F})$, then $\mathcal{C}_t(\mathcal{F}) = \emptyset$. Also, as was pointed out by Gyárfás, we have $|\mathcal{C}_t(\mathcal{F})| \leq k^t$ for $t = \tau(\mathcal{F})$ (this fact can be verified by induction on $\tau(\mathcal{F})$ by arguing as in the proof of Lemma 2.1 (i) in Section 2).

Let t, k be integers with $k \geq t \geq 1$, and assume that $|X|$ is sufficiently large compared with t and k. Define

$$p_t(k) := \max \left\{ |\mathcal{C}_t(\mathcal{F})| : \mathcal{F} \subseteq 2^X \text{is } k\text{-uniform and intersecting, and } \tau(\mathcal{F}) = t \right\}.$$

It can be noted that if $|X|$ is sufficiently large, then the value of $p_t(k)$ does not depend on $|X|$ because every k-uniform family $\mathcal{F}$ with $\tau(\mathcal{F}) = t$ satisfies $|\mathcal{C}_t(\mathcal{F})| \leq k^t$, which is mentioned at the end of the preceding paragraph. For the significance of the function $p_t(k)$, we refer the reader to [1] and [4]. Here, we just mention that the function $p_{t-1}(k)$ plays an important role in the determination of the more natural function $f_{k,t}(n)$ defined by

$$f_{k,t}(n) := \max\left\{|\mathcal{F}| : \mathcal{F} \subseteq 2^X \text{is } k\text{-uniform and intersecting, and } \tau(\mathcal{F}) = t\right\},$$

where $n = |X|$.

Clearly $p_1(k) = k$ for every $k \geq 1$. For $t \geq 2$, in Frankl, Ota and Tokushige [5], it is conjectured that $p_t(k) = k^t - \binom{t}{2}k^{t-1} + O(k^{t-2})$ $(k \to \infty)$, and the conjecture is settled affirmatively for $t = 4, 5$. For $t \geq 6$, it is proved in the same paper that $p_t(k) \leq k^t - \frac{1}{\sqrt{2}}\lfloor\frac{t-1}{2}\rfloor^{\frac{3}{2}}k^{t-1} + O(k^{t-2})$. For $t = 2$, the following precise result is proved in [2].

Theorem A (Frankl [2]) *Let $k \geq 2$. Then $p_2(k) = k^2 - k + 1$.*

The value of $p_3(k)$ is determined for $k \geq 9$ in [3], for $k = 3$ in [4], and for $k = 4$ in [1].

Theorem B (Frankl, Ota and Tokushige [3, 4], Chiba, Furuya, Matsubara and Takatou [1]) *Let $k = 3$ or 4, or $k \geq 9$. Then $p_3(k) = k^3 - 3k^2 + 6k - 4$.*

We now describe examples related to Theorems A and B.

Example 1 Let $k \geq 2$. Fix $2k-1$ elements a_i, b_j $(1 \leq i \leq k$ and $1 \leq j \leq k-1)$ of X. Set $A := \{a_1, a_2, \ldots, a_k\}$, $B := \{b_1, b_2, \ldots, b_{k-1}, a_1\}$ and $C := \{b_1, b_2, \ldots, b_{k-1}, a_2\}$, and define $\mathcal{F}_1^{(k)} := \{A, B, C\}$. Then $\mathcal{F}_1^{(k)}$ is k-uniform and intersecting, $\tau(\mathcal{F}_1^{(k)}) = 2$, and $|\mathcal{C}_2(\mathcal{F}_1^{(k)})| = k^2 - k + 1$.

Example 2 Let $k \geq 3$. Fix $3(k-1)$ elements a_i, b_i, c_i $(1 \leq i \leq k-1)$ of X. For each $i = 1, 2$, set $A_i := \{a_1, a_2, \ldots, a_{k-1}, c_i\}$, $B_i := \{b_1, b_2, \ldots, b_{k-1}, a_i\}$ and $C_i := \{c_1, c_2, \ldots, c_{k-1}, b_i\}$, and define $\mathcal{F}_2^{(k)} := \{A_1, A_2, B_1, B_2, C_1, C_2\}$. Then $\mathcal{F}_2^{(k)}$ is k-uniform and intersecting, $\tau(\mathcal{F}_2^{(k)}) = 3$, and $|\mathcal{C}_3(\mathcal{F}_2^{(k)})| = (k-1)^3 + 3(k-1) = k^3 - 3k^2 + 6k - 4$.

The following two theorems are proved in [1], [2], [3] and [4]. They are stronger than Theorems A and B.

Theorem C (Frankl [2]) *Let $k \geq 2$, and let $\mathcal{F} \subseteq \binom{X}{k}$ be an intersecting family with $\tau(\mathcal{F}) = 2$. Then $|\mathcal{C}_2(\mathcal{F})| \leq k^2 - k + 1$, with equality if and only if $\mathcal{F}$ is isomorphic to $\mathcal{F}_1^{(k)}$.*

Theorem D (Frankl et al. [3, 4], Chiba et al. [1]) *Let $k = 3$ or 4 or $k \geq 9$, and let $\mathcal{F} \subseteq \binom{X}{k}$ be an intersecting family with $\tau(\mathcal{F}) = 3$. Then $|\mathcal{C}_3(\mathcal{F})| \leq k^3 - 3k^2 + 6k - 4$, with equality if and only if $\mathcal{F}$ is isomorphic to $\mathcal{F}_2^{(k)}$.*

It is natural to conjecture that Theorems B and D hold for $5 \leq k \leq 8$ as well.

In this paper, we take up the case where $k = 5$, and prove the following theorem, confirming that Theorem B holds for $k = 5$.

Theorem 1 *We have* $p_3(5) = 76$.

We actually prove the following stronger result, which is an analogue of Theorem D;

Theorem 2 *Let* $\mathcal{F} \subseteq \binom{X}{5}$ *be an intersecting family with* $\tau(\mathcal{F}) = 3$. *Then* $|\mathcal{C}_3(\mathcal{F})| \leq 76$, *with equality if and only if* $\mathcal{F}$ *is isomorphic to* $\mathcal{F}_2^{(5)}$.

We add that in the proof of Theorem D for $k \geq 9$ in [3], part of the verification of an inequality was done by computer for small values of k. This suggests that it is difficult to determine $p_3(k)$ for $k = 8$ (and 7). However, Proposition 2.2, which we prove in Section 2, holds for all $k \geq 5$. Thus it is expected that Proposition 2.2 will shed some light on the determination of $p_3(k)$ for $6 \leq k \leq 8$.

Our notation is standard except for the following. Let $\mathcal{A} \subseteq 2^X$ and $Y, Z \in 2^X - \{\emptyset\}$ with $Y \cap Z = \emptyset$, and write $Y = \{y_1, y_2, \ldots, y_l\}$ and $Z = \{z_1, z_2, \ldots, z_m\}$. We define $\mathcal{A}[y_1 y_2 \ldots y_l] = \mathcal{A}[Y] := \{A \in \mathcal{A} : Y \subseteq A\}$, $\mathcal{A}(\bar{y}_1 \bar{y}_2 \ldots \bar{y}_l) = \mathcal{A}(\overline{Y}) := \{A \in \mathcal{A} : Y \cap A = \emptyset\}$ and $\mathcal{A}(\bar{z}_1 \bar{z}_2 \ldots \bar{z}_m)[y_1 y_2 \ldots y_l] = \mathcal{A}(\overline{Z})[Y] := \{A \in \mathcal{A} : Y \subseteq A \text{ and } Z \cap A = \emptyset\} = \mathcal{A}(\overline{Z}) \cap \mathcal{A}[Y]$.

2 Uniform Intersecting Families

In this section, we prove a proposition concerning k-uniform intersecting families with covering number three for $k \geq 5$.

The following observation will be used implicitly throughout this paper.

Observation 1 *Let* k *be an integer with* $k \geq 3$, *and let* $\mathcal{F} \subseteq \binom{X}{k}$ *be an intersecting family with* $\tau(\mathcal{F}) = 3$. *Then* $\mathcal{F}(\bar{x}\bar{y}) \neq \emptyset$ *for all* $x, y \in X$.

The following result is also useful for our proof.

Lemma 2.1 *Let* k *be an integer with* $k \geq 3$, *and let* $\mathcal{F} \subseteq \binom{X}{k}$ *be an intersecting family with* $\tau(\mathcal{F}) = 3$. *Then the following hold.*

(i) *We have* $|(\mathcal{C}_3(\mathcal{F}))[x]| \leq k^2 - k + 1$ *for all* $x \in X$.

(ii) *We have* $|(\mathcal{C}_3(\mathcal{F}))[xy]| \leq k$ *for all* $x, y \in X$ *with* $x \neq y$.

Proof. To prove (i), let $x \in X$. We may assume $(\mathcal{C}_3(\mathcal{F}))[x] \neq \emptyset$. Take $C \in (\mathcal{C}_3(\mathcal{F}))[x]$. Let $\mathcal{F}' = \mathcal{F}(\bar{x})$. Then $\mathcal{F}'$ is a k-uniform intersecting family, and $C - \{x\}$ is a cover of $\mathcal{F}'$. Hence $C - \{x\} \in \mathcal{C}_2(\mathcal{F}')$. This implies $\tau(\mathcal{F}') = 2$, and it follows from Theorem C that $|\mathcal{C}_2(\mathcal{F}')| \leq k^2 - k + 1$. Since $C \in (\mathcal{C}_3(\mathcal{F}))[x]$ is arbitrary, we get $|(\mathcal{C}_3(\mathcal{F}))[x]| = |\{C - \{x\} : C \in (\mathcal{C}_3(\mathcal{F}))[x]\}| \leq |\mathcal{C}_2(\mathcal{F}')| \leq k^2 - k + 1$. Similarly, if $x, y \in X$ and $x \neq y$, then since we clearly have $|\mathcal{C}_1(\mathcal{F}(\bar{x}\bar{y}))| \leq k$ by Observation 1, we get $|(\mathcal{C}_3(\mathcal{F}))[xy]| = |\{C - \{x, y\} : C \in C \in (\mathcal{C}_3(\mathcal{F}))[xy]\}| \leq |\mathcal{C}_1(\mathcal{F}(\bar{x}\bar{y}))| \leq k$. □

By the definition of $\mathcal{F}_2^{(k)}$, we see that $\mathcal{F}_2^{(k)}$ has the property that the intersection of any two members of $\mathcal{F}_2^{(k)}$ has cardinality 1 or $k - 1$. We consider a k-uniform intersecting family with covering number three having this property, and prove the following proposition, which is the main result of this section.

Proposition 2.2 *Let k be an integer with $k \geq 5$. Let $\mathcal{F} \subseteq \binom{X}{k}$ be an intersecting family with $\tau(\mathcal{F}) = 3$, and suppose that*

$$|F \cap G| = 1 \text{ or } k-1 \text{ for all } F, G \in \mathcal{F} \text{ with } F \neq G. \tag{2.1}$$

Then $|\mathcal{C}_3(\mathcal{F})| \leq k^3 - 3k^2 + 6k - 4$, with equality if and only if $\mathcal{F}$ is isomorphic to $\mathcal{F}_2^{(k)}$.

Proof. If $\mathcal{F} \cong \mathcal{F}_2^{(k)}$, then $|\mathcal{C}_3(\mathcal{F})| = k^3 - 3k^2 + 6k - 4$ (see Example 2). Thus it suffices to prove $\mathcal{F} \cong \mathcal{F}_2^{(k)}$, assuming that $|\mathcal{C}_3(\mathcal{F})| \geq k^3 - 3k^2 + 6k - 4$. Let $\mathcal{C} = \mathcal{C}_3(\mathcal{F})$.

First we show that no three members of $\mathcal{F}$ satisfy the property that the intersection of any two of them, as well as that of all of them, has cardinality one.

Claim 2.3 *Let $F, G \in \mathcal{F}$, and suppose that $|F \cap G| = 1$. Then $|H \cap (F \cup G)| \neq 1$ for every $H \in \mathcal{F}$.*

Proof. Write $F \cap G = \{u\}$. Suppose that there exists $H \in \mathcal{F}$ such that $|H \cap (F \cup G)| = 1$. Then $H \cap (F \cup G) = \{u\}$. Let $v \in F - \{u\}$ and $F' \in \mathcal{F}(\bar{u}\bar{v})$. Since $|F' \cap F| < k-1$, $|F' \cap F| = 1$ by (2.1). Write $F' \cap F = \{w_1\}$. Since $w_1 \notin G \cup H$, $u \notin F'$ and $(G - \{u\}) \cap (H - \{u\}) = \emptyset$, it follows from (2.1) that $|F' \cap G| = |F' \cap H| = 1$. Write $F' \cap G = \{w_2\}$. Then $|\mathcal{C}| = |\mathcal{C}[u]| + |\mathcal{C}(\bar{u})[w_1]| + |\mathcal{C}(\bar{u}\bar{w}_1)[w_2]| + |\mathcal{C}(\bar{u}\bar{w}_1\bar{w}_2)|$. Hence by Lemma 2.1 (i), $|\mathcal{C}| \leq (k^2 - k + 1) + |G - \{u\}| \cdot |H - \{u\}| + |F - \{u, w_1\}| \cdot |H - \{u\}| + |F - \{u, w_1\}| \cdot |G - \{u, w_2\}| \cdot |H \cap F'| = (k^2 - k + 1) + (k-1)^2 + (k-2)(k-1) + (k-2)^2 = 4k^2 - 10k + 8 < k^3 - 3k^2 + 6k - 4$, which contradicts the assumption that $|\mathcal{C}| \geq k^3 - 3k^2 + 6k - 4$. □

Next we show that $\mathcal{F}$ contains two members whose intersection has cardinality $k-1$.

Claim 2.4 *There exist $F, G \in \mathcal{F}$ such that $|F \cap G| = k - 1$.*

Proof. Suppose that $|F \cap G| = 1$ for all $F, G \in \mathcal{F}$ with $F \neq G$. Let $F, G \in \mathcal{F}$, and write $F \cap G = \{u\}$. Let $H \in \mathcal{F}(\bar{u})$, and write $H \cap F = \{v_1\}$ and $H \cap G = \{w_1\}$. Let $F' \in \mathcal{F}(\bar{v}_1\bar{w}_1)$. If $u \in F'$, then $|F' \cap (F \cup G)| = |\{u\}| = 1$, which contradicts Claim 2.3. Thus $u \notin F'$. Write $F' \cap F = \{v_2\}$ and $F' \cap G = \{w_2\}$. By Lemma 2.1 (ii), $|\mathcal{C}(\bar{u})[v_1w_2]| \leq |\mathcal{C}[v_1w_2]| \leq k$ and $|\mathcal{C}(\bar{u}\bar{v}_1)[v_2w_1]| \leq |\mathcal{C}[v_2w_1]| \leq k$. Hence by Lemma 2.1 (i), $|\mathcal{C}| = |\mathcal{C}[u]| + \big(|\mathcal{C}(\bar{u})[v_1w_2]| + |\mathcal{C}(\bar{u}\bar{w}_2)[v_1]|\big) + \big(|\mathcal{C}(\bar{u}\bar{v}_1)[v_2w_1]| + |\mathcal{C}(\bar{u}\bar{v}_1\bar{w}_1)[v_2]|\big) + |\mathcal{C}(\bar{u}\bar{v}_1\bar{v}_2)[w_1]| + |\mathcal{C}(\bar{u}\bar{v}_1\bar{v}_2\bar{w}_1)[w_2]| + |\mathcal{C}(\bar{u}\bar{v}_1\bar{v}_2\bar{w}_1\bar{w}_2)| \leq (k^2 - k + 1) + \big(k + |G - \{u, w_2\}| \cdot |F' - \{w_2\}|\big) + \big(k + |G - \{u, w_1\}| \cdot |H - \{v_1, w_1\}|\big) + |F - \{u, v_1, v_2\}| \cdot |F' - \{v_2\}| + |F - \{u, v_1, v_2\}| \cdot |H - \{v_1, w_1\}| + |F - \{u, v_1, v_2\}| \cdot |G - \{u, w_1, w_2\}| \cdot |F' \cap H| = (k^2 - k + 1) + \big(k + (k-2)(k-1)\big) + \big(k + (k-2)^2\big) + (k-3)(k-1) + (k-3)(k-2) + (k-3)^2 = 6k^2 - 21k + 25 < k^3 - 3k^2 + 6k - 4$, a contradiction. □

We now prove three claims concerning two members of $\mathcal{F}$ whose intersection has cardinality $k-1$.

Claim 2.5 *Let $F, G, H \in \mathcal{F}$, and suppose that $|F \cap G| = k-1$ and $|H \cap (F \cup G)| = 1$. Then $|F' \cap (F \cup G \cup H)| \geq 3$ for every $F' \in \mathcal{F}$.*

Proof. Write $H \cap (F \cup G) = \{u\}$. Suppose that there exists $F' \in \mathcal{F}$ such that $|F' \cap (F \cup G \cup H)| \leq 2$. If $u \in F'$, then it follows from (2.1) that $|F' \cap (F \cup H)| = |\{u\}| = 1$, which contradicts Claim 2.3. Thus $u \notin F'$. This implies that $|F' \cap (F \cup G \cup H)| = 2$ and $|F' \cap (F \cup G)| = |F' \cap H| = 1$. Write $F' \cap (F \cup G) = \{v\}$ and $F' \cap H = \{w\}$. By Lemma 2.1 (ii), $|\mathcal{C}(\bar{u})[v]| \leq \sum_{x \in H - \{u\}} |\mathcal{C}(\bar{u})[vx]| \leq |H - \{u\}| \cdot k$. Hence by Lemma 2.1, $|\mathcal{C}| = |\mathcal{C}[u]| + |\mathcal{C}(\bar{u})[v]| + \big(|\bigcup_{x \in (F \cap G) - \{u,v\}} \mathcal{C}(\bar{u}\bar{v})[wx]| + |\bigcup_{x \in (F \cap G) - \{u,v\}} \mathcal{C}(\bar{u}\bar{v}\bar{w})[x]|\big) + |\mathcal{C}(\overline{F \cap G})| \leq (k^2 - k + 1) + |H - \{u\}| \cdot k + \big(|(F \cap G) - \{u, v\}| \cdot k + |(F \cap G) - \{u, v\}| \cdot |H - \{u, w\}| \cdot |F' - \{v, w\}|\big) + |F - (F \cap G)| \cdot |G - (F \cap G)| \cdot |H \cap F'| = (k^2 - k + 1) + (k - 1)k + \big((k-3)k + (k-3)(k-2)(k-2)\big) + 1 \cdot 1 \cdot 1 = k^3 - 4k^2 + 11k - 10 < k^3 - 3k^2 + 6k - 4$, a contradiction. □

Claim 2.6 *Let $F, G \in \mathcal{F}$, and suppose that $|F \cap G| = k - 1$. Then $F \cap G \not\subseteq H$ for every $H \in \mathcal{F} - \{F, G\}$.*

Proof. Suppose that there exists $H \in \mathcal{F} - \{F, G\}$ such that $F \cap G \subseteq H$. Write $F - G = \{a\}$. Let $b \in F \cap G$ and $F' \in \mathcal{F}(\bar{a}\bar{b})$. Then $|F' \cap F| < k - 1$, and hence by (2.1), $|F' \cap (F \cap G)| = |F' \cap F| = 1$, which implies $|F' \cap G| = |F' \cap H| = 1$ again by (2.1). Write $F' \cap F = \{u\}$. Note that $F' \cap (F \cup G \cup H) = \{u\}$. Let $c \in F' - \{u\}$ and $F'' \in \mathcal{F}(\bar{u}\bar{c})$. Since $|F'' \cap F'| < k - 1$, $|F'' \cap F'| = 1$ by (2.1). Write $F'' \cap F' = \{v\}$. If $|F'' \cap F| = k - 1$, then $F'' = (F - \{u\}) \cup \{v\}$, and hence $|F'' \cap G| = |(F \cap G) - \{u\}| = k - 2$, which contradicts (2.1). Thus $|F'' \cap F| = 1$ by (2.1). Similarly $|F'' \cap G| = |F'' \cap H| = 1$. If $F'' \cap (F \cap G) \neq \emptyset$, then $|F'' \cap (F \cup G)| = 1$, and hence $|F'' \cap (F \cup G \cup F')| = |F'' \cap (F \cup G)| + |\{v\}| = 2$, which contradicts Claim 2.5. Thus $F'' \cap (F \cap G) = \emptyset$. Hence $|\bigcup_{x \in (F \cap G) - \{u\}} \mathcal{C}(\bar{u}\bar{v})[x]| \leq |(F \cap G) - \{u\}| \cdot |F' - \{u, v\}| \cdot |F'' - \{v\}|$. Since the four sets $F - (F \cap G)$, $G - (F \cap G)$, $H - (F \cap G)$ and $F' - (F \cap G)$ are pairwise disjoint, we also have $\mathcal{C}(\overline{F \cap G}) = \emptyset$. Consequently, by Lemma 2.1, $|\mathcal{C}| = |\mathcal{C}[u]| + \big(|\bigcup_{x \in (F \cap G) - \{u\}} \mathcal{C}(\bar{u})[vx]| + |\bigcup_{x \in (F \cap G) - \{u\}} \mathcal{C}(\bar{u}\bar{v})[x]|\big) + |\mathcal{C}(\overline{F \cap G})| \leq (k^2 - k + 1) + \big(|(F \cap G) - \{u\}| \cdot k + |(F \cap G) - \{u\}| \cdot |F' - \{u, v\}| \cdot |F'' - \{v\}|\big) + 0 = (k^2 - k + 1) + \big((k-2)k + (k-2)(k-2)(k-1)\big) + 0 = k^3 - 3k^2 + 5k - 3 < k^3 - 3k^2 + 6k - 4$, a contradiction. □

Claim 2.7 *Let $F, G \in \mathcal{F}$, and suppose that $|F \cap G| = k - 1$. Then $|H \cap F| = |H \cap G| = 1$ for every $H \in \mathcal{F} - \{F, G\}$.*

Proof. Write $F - G = \{v\}$, $G - F = \{w\}$ and $F \cap G = \{u_1, \ldots u_{k-1}\}$. Suppose that there exists $H \in \mathcal{F} - \{F, G\}$ such that $|H \cap F| \neq 1$ or $|H \cap G| \neq 1$. By symmetry, we may assume $|H \cap F| \neq 1$. Then $|H \cap F| = k - 1$ by (2.1), and hence $|H \cap G| \geq |H \cap F \cap G| \geq k - 2$, which implies $|H \cap G| = k - 1$ by (2.1). In view of Claim 2.6, we may assume $u_{k-1} \notin H$. Then $H = \{v, w, u_1, \ldots, u_{k-2}\}$. Let $F' \in \mathcal{F}(\bar{v}\bar{w})$. By (2.1), $|F' \cap H| = |F' \cap \{u_1, u_2, \ldots, u_{k-2}\}| = 1$. We may assume $F' \cap H = \{u_1\}$. Then $F' \cap F = F' \cap G = \{u_1\}$ by (2.1), and hence $F' \cap (F \cup G \cup H) = \{u_1\}$. Consequently, by

Lemma 2.1, $|\mathcal{C}| = |\mathcal{C}[u_1]| + |\bigcup_{2\leq i\leq k-2} \mathcal{C}(\bar{u}_1)[u_i]| + \mathcal{C}(\overline{F\cap G\cap H})[u_{k-1}]| + |\mathcal{C}(\overline{F\cap G})| \leq (k^2-k+1) + (k-3)\cdot|F' - \{u_1\}|\cdot k + |H - (F\cap G\cap H)|\cdot|F' - \{u_1\}| + |F-(F\cap G)|\cdot |G-(F\cap G)|\cdot|F'-\{u_1\}| = (k^2-k+1)+(k-3)(k-1)k+2(k-1)+1\cdot 1\cdot(k-1) = k^3-3k^2+5k-2 < k^3-3k^2+6k-4$, a contradiction. $\square$

Now by Claim 2.4, there exist $F_1, F_2 \in \mathcal{F}$ such that $|F_1 \cap F_2| = k-1$. Write $F_1 - F_2 = \{b_1\}$, $F_2 - F_1 = \{b_2\}$, and $F_1 \cap F_2 = \{c_1, c_2, \ldots, c_{k-1}\}$. Let $F_3 \in \mathcal{F}(\bar{b}_1\bar{b}_2)$. Then by Claim 2.7, $|F_3 \cap (F_1 \cup F_2)| = |F_3 \cap \{c_1, c_2, \ldots, c_{k-1}\}| = 1$. We may assume $F_3 \cap (F_1 \cup F_2) = \{c_1\}$. Let $F_4 \in \mathcal{F}(\bar{b}_1\bar{c}_1)$. Then by Claim 2.7, $|F_4 \cap F_1| = |F_4 \cap F_2| = 1$, which implies $|F_4 \cap \{c_2, c_3, \ldots c_{k-1}\}| = 1$. We may assume $F_4 \cap \{c_2, c_3, \ldots c_{k-1}\} = \{c_2\}$. Then $F_4 \cap (F_1 \cup F_2) = \{c_2\}$. Since $|F_1 \cap F_2| = k-1$ and $|F_3 \cap (F_1 \cup F_2)| = 1$, $|F_4 \cap (F_1 \cup F_2 \cup F_3)| \geq 3$ by Claim 2.5. This implies that $|F_4 \cap F_3| \neq 1$, and hence $|F_4 \cap F_3| = k-1$ by (2.1). Write $F_3 \cap F_4 = \{a_1, a_2, \ldots, a_{k-1}\}$. Let $F_5 \in \mathcal{F}(\bar{c}_1\bar{c}_2)$. Then we can argue as above with F_1 and F_2 replaced by F_3 and F_4, and F_3 replaced by F_5, to get $|F_5 \cap (F_3 \cup F_4)| = |F_5 \cap \{a_1, a_2, \ldots, a_{k-1}\}| = 1$. We may assume $F_5 \cap (F_3 \cup F_4) = \{a_1\}$. By Claim 2.7, $|F_5 \cap F_1| = |F_5 \cap F_2| = 1$. If $F_5 \cap \{c_3, c_4, \ldots, c_{k-1}\} \neq \emptyset$, then $|F_5 \cap (F_1 \cup F_2)| = 1$, and hence $|F_5 \cap (F_1 \cup F_2 \cup F_3)| = |F_5 \cap (F_1 \cup F_2)| + |F_5 \cap \{a_1, a_2, \ldots, a_{k-1}\}| = 2$, which contradicts Claim 2.5. Thus $F_5 \cap \{c_3, c_4, \ldots, c_{k-1}\} = F_5 \cap \{c_1, c_2, \ldots, c_{k-1}\} = \emptyset$, and hence $F_5 \cap (F_1 \cup F_2) = \{b_1, b_2\}$. Let $F_6 \in \mathcal{F}(\bar{c}_1\bar{a}_1)$. Then it follows from Claim 2.7 that $|F_6 \cap (F_3 \cup F_4)| = |F_6 \cap \{a_2, a_3, \ldots, a_{k-1}\}| = 1$. We may assume $F_6 \cap (F_3 \cup F_4) = \{a_2\}$. Note that $c_2 \notin F_6$, which implies $F_6 \in \mathcal{F}(\bar{c}_1\bar{c}_2)$. Hence we obtain $F_6 \cap (F_1 \cup F_2) = \{b_1, b_2\}$ by arguing as above with F_5 replaced by F_6. Since $\{b_1, b_2\} \subseteq F_5 \cap F_6$, $|F_5 \cap F_6| = k-1$ by (2.1). Note that $F_6 = (F_5 - \{a_1\}) \cup \{a_2\}$.

Now set $\mathcal{F}' = \{F_1, F_2, F_3, F_4, F_5, F_6\}$. Then $\mathcal{F}' \cong \mathcal{F}_2^{(k)}$, using $(F_5 \cap F_6) - \{b_1, b_2\} = \{b_3, \ldots, b_{k-1}\}$, and the notation in Example 2, with $C_1 = F_1$, $C_2 = F_2$, $A_1 = F_3$, $A_2 = F_4$, $B_1 = F_5$ and $B_2 = F_6$.

Suppose that $\mathcal{F} \neq \mathcal{F}'$, and $F \in \mathcal{F} - \mathcal{F}'$. By Claim 2.7,

$$|F \cap F_i| = 1 \text{ for all } i \text{ with } 1 \leq i \leq 6. \tag{2.2}$$

If $F \cap \{b_1, b_2\} \neq \emptyset$, then by (2.2), we get $F \cap (F_1 \cap F_2) = \emptyset$ and $\{b_1, b_2\} \subseteq F$, and hence $|F \cap F_5| \geq |\{b_1, b_2\}| = 2$, which contradicts (2.2). Thus $F \cap \{b_1, b_2\} = \emptyset$, and hence $|F \cap (F_1 \cup F_2)| = 1$ by (2.2). Similarly $|F \cap (F_3 \cup F_4)| = 1$. Therefore $|F \cap (F_1 \cup F_2 \cup F_3)| \leq |F \cap (F_1 \cup F_2)| + |F \cap F_3| \leq 2$, which contradicts Claim 2.5. Consequently $\mathcal{F} = \mathcal{F}' \cong \mathcal{F}_2^{(k)}$.

This completes the proof of Proposition 2.2. $\square$

3 Proof of Theorem 2

Throughout the rest of this paper, let $\mathcal{F} \subseteq \binom{X}{5}$ be an intersecting family with $\tau(\mathcal{F}) = 3$, and let $\mathcal{C} = \mathcal{C}_3(\mathcal{F})$. We first restate Lemma 2.1 for the case where $k = 5$.

Claim 3.1 *Let $x, y \in X$ with $x \neq y$. Then the following hold.*

(i) *We have* $|\mathcal{C}[x]| \le 21$.

(ii) *We have* $|\mathcal{C}[xy]| \le 5$.

In view of Proposition 2.2, we may assume that there exist $F, G \in \mathcal{F}$ such that $2 \le |F \cap G| \le 3$. In order to prove Theorem 2, it suffices to show that $|\mathcal{C}| < 5^3 - 3\cdot 5^2 + 6\cdot 5 - 4 = 76$. By way of contradiction, suppose that $|\mathcal{C}| \ge 76$. We start with claims concerning three members F, G, H of $\mathcal{F}$ such that $2 \le |F \cap G \cap H| \le |F \cap G| \le 3$.

Claim 3.2 *Let $F, G, H \in \mathcal{F}$, and suppose that $2 \le |F \cap G| \le 3$ and $|F \cap G| = |F \cap G \cap H|$. Then $H \subseteq F \cup G$.*

Proof. Set $t = |(F \cup G) \cap H|$, $t_1 = |F \cap G \cap H|$, $t_2 = |(F - G) \cap H|$ and $t_3 = |(G - F) \cap H|$. Suppose that $H \not\subseteq F \cup G$. Then $t = t_1 + t_2 + t_3 \le 4$. Note that $2 \le t_1 \le 3$ by the assumption, and hence $2 \le t \le 4$. Consequently, by Claim 3.1, $|\mathcal{C}| = \left|\bigcup_{x \in F\cap G\cap H} \mathcal{C}[x]\right| + \left|\bigcup_{x\in (F-G)\cap H} \mathcal{C}(\overline{F \cap G \cap H})[x]\right| + \left|\bigcup_{x\in(G-F)\cap H} \mathcal{C}(\overline{F \cap H})[x]\right| + |\mathcal{C}(\overline{(F \cup G)\cap H})| \le 21t_1 + t_2\cdot(|G| - t_1)\cdot 5 + t_3 \cdot (|F| - (t_1 + t_2))\cdot 5 + (|F| - (t_1+t_2))(|G| - (t_1+t_3))(|H| - t) = 21t_1 + (t - t_1)(5 - t_1)\cdot 5 - t_3t_2 5 + ((5-t_1)^2 - (5-t_1)(t-t_1) + t_2t_3)(5-t) = (5-t_1)t^2 - 5(5-t_1)t + 5t_1^2 - 29t_1 + 125 - t_2t_3t \le (5-t_1)\left(t - \frac{5}{2}\right)^2 + 5t_1^2 - \frac{91}{4}t_1 + \frac{375}{4} \le (5-t_1)\left(4 - \frac{5}{2}\right)^2 + 5t_1^2 - \frac{91}{4}t_1 + \frac{375}{4} = 5\left(t_1 - \frac{5}{2}\right)^2 + \frac{295}{4} = 5\cdot\frac{1}{4} + \frac{295}{4} = 75$, a contradiction. □

The following claim is stronger than Claim 3.2.

Claim 3.3 *Let $F_0, G_0, H_0 \in \mathcal{F}$, and suppose that $2 \le |F_0 \cap G_0| \le 3$ and $2 \le |F_0 \cap G_0 \cap H_0| \le 3$. Then $H_0 \subseteq F_0 \cup G_0$.*

Proof. Suppose that $H_0 \not\subseteq F_0 \cup G_0$. By Claim 3.2, we have $|F_0 \cap G_0| = 3$ and $|F_0 \cap G_0 \cap H_0| = 2$. Write $F_0 \cap G_0 = \{u_1, u_2, u_3\}$. We may assume $F_0 \cap G_0 \cap H_0 = \{u_1, u_2\}$. Set $t_1 = |(F_0 - G_0) \cap H_0|$ and $t_2 = |(G_0 - F_0) \cap H_0|$. We may assume $t_1 \ge t_2$. Since $H_0 \not\subseteq F_0 \cup G_0$ and $|F_0 \cap G_0 \cap H_0| = 2$, $t_1 + t_2 \le 2$. Hence $(t_1, t_2) \in \{(0,0), (1,0), (1,1), (2,0)\}$. By Claim 3.1,

$$\begin{aligned}
|\mathcal{C}| &= |\mathcal{C}[u_1] \cup \mathcal{C}[u_2]| + |\mathcal{C}(\bar{u}_1\bar{u}_2)[u_3]| + \left|\bigcup_{x\in(F_0-G_0)\cap H_0} \mathcal{C}(\bar{u}_1\bar{u}_2\bar{u}_3)[x]\right| \\
&\quad + \left|\bigcup_{x\in(G_0-F_0)\cap H_0} \mathcal{C}(\overline{F_0 \cap (G_0 \cup H_0)})[x]\right| + |\mathcal{C}(\overline{((F_0 \cup G_0)\cap H_0) \cup \{u_3\}})| \\
&\le 2\cdot 21 + |H_0 - (F_0 \cap G_0)|\cdot 5 + t_1 \cdot |G_0 - F_0| \cdot 5 + t_2 \cdot (|F_0 - G_0| - t_1)\cdot 5 \\
&\qquad + (|F_0 - G_0| - t_1)(|G_0 - F_0| - t_2)(|H_0 - (F_0 \cap G_0)| - t_1 - t_2) \\
&= 57 + 10t_1 + 10t_2 - 5t_1t_2 + (2 - t_1)(2 - t_2)(3 - t_1 - t_2).
\end{aligned}$$

If $(t_1, t_2) \in \{(0,0), (1,0), (1,1)\}$, then this implies $|\mathcal{C}| \le 73$, a contradiction. Thus $(t_1, t_2) = (2, 0)$. This implies $(F_0 \cup G_0) \cap H_0 = F_0 \cap H_0 = F_0 - \{u_3\}$. Write $H_0 - F_0 = \{y\}$. Let $F' \in \mathcal{F}(\bar{u}_3\bar{y})$. If $2 \le |F_0 \cap F'| \le 3$, then since $|F_0 \cap F' \cap H_0| = |(F_0 - \{u_3\}) \cap F'| = |F_0 \cap F'|$, applying Claim 3.2 with $F = F_0$, $G = F'$ and

$H = H_0$, we get $H_0 \subseteq F_0 \cup F'$, which contradicts the fact that $y \notin F_0 \cup F'$. Hence $|F_0 \cap F'| = 1$ or $|F_0 \cap F'| = 4$. Write $F_0 - G_0 = \{v_1, v_2\}$. If $|F_0 \cap F'| = 4$, then $F_0 \cap H_0 = F_0 \cap F' = H_0 \cap F' = \{u_1, u_2, v_1, v_2\}$, and hence by Claim 3.1, $|\mathcal{C}| = |\mathcal{C}[u_1] \cup \mathcal{C}[u_2]| + |\mathcal{C}(\bar{u}_1\bar{u}_2)[v_1] \cup \mathcal{C}(\bar{u}_1\bar{u}_2)[v_2]| + |\mathcal{C}(\bar{u}_1\bar{u}_2\bar{v}_1\bar{v}_2)| \leq 2 \cdot 21 + 2 \cdot |G_0 - \{u_1, u_2\}| \cdot 5 + |F_0 - \{u_1, u_2, v_1, v_2\}| \cdot |H_0 - \{u_1, u_2, v_1, v_2\}| \cdot |F' - \{u_1, u_2, v_1, v_2\}| = 73$, a contradiction. Thus $|F_0 \cap F'| = 1$. Suppose that $F' \cap \{u_1, u_2\} \neq \emptyset$. We may assume $F' \cap \{u_1, u_2\} = \{u_1\}$. Then $H_0 \cap F' = \{u_1\}$. Hence by Claim 3.1, $|\mathcal{C}| = |\mathcal{C}[u_1]| + |\mathcal{C}(\bar{u}_1)[u_2]| + |\mathcal{C}(\bar{u}_1\bar{u}_2)[u_3]| + |\mathcal{C}(\bar{u}_1\bar{u}_2\bar{u}_3)| \leq 21 + |F' - \{u_1\}| \cdot 5 + |H_0 - \{u_1, u_2\}| \cdot |F' - \{u_1\}| + |F_0 - \{u_1, u_2, u_3\}| \cdot |G_0 - \{u_1, u_2, u_3\}| \cdot 5 = 73$, a contradiction. Thus $F' \cap \{u_1, u_2\} = \emptyset$. This implies that $|F' \cap \{v_1, v_2\}| = |F' \cap F_0| = 1$. We may assume $F' \cap \{v_1, v_2\} = \{v_1\}$. Then $H_0 \cap F' = \{v_1\}$. Consequently, by Claim 3.1, $|\mathcal{C}| = |\mathcal{C}[u_1] \cup \mathcal{C}[u_2]| + |\mathcal{C}(\bar{u}_1\bar{u}_2)[v_1]| + |\mathcal{C}(\bar{u}_1\bar{u}_2\bar{v}_1)[u_3]| + |\mathcal{C}(\bar{u}_1\bar{u}_2\bar{u}_3\bar{v}_1)| \leq 2 \cdot 21 + |G_0 - \{u_1, u_2\}| \cdot 5 + |H_0 - \{u_1, u_2, v_1\}| \cdot |F' - \{v_1\}| + |F_0 - \{u_1, u_2, u_3, v_1\}| \cdot |G_0 - \{u_1, u_2, u_3\}| \cdot 5 = 75$, a contradiction. □

We need to consider, slightly more generally, three members F, G, H of $\mathcal{F}$ such that $2 \leq |F \cap G| \leq 3$ and $F \cap G \cap H \neq \emptyset$. The case where $|F \cap G| = 3$ will be dealt with in Claim 3.10 and Claim 3.11. Here we take up the case where $|F \cap G| = 2$. Our aim is to show that such members satisfy $F - G \subseteq H$ or $G - F \subseteq H$ (see Claim 3.7). For this purpose, we prove the following three claims.

Claim 3.4 *Let $F, G, H \in \mathcal{F}$ with $H \neq F, G$, and suppose that $|F \cap G| = 2$. Then $|F \cap G \cap H| \leq 1$.*

Proof. Suppose that $|H \cap F \cap G| = 2$. Then by Claim 3.3, $H \subseteq F \cup G$. Since $H \neq F, G$, we have $H \not\supseteq F - G$ and $H \not\supseteq G - F$. Since $|H - (F \cap G)| = 3$, we have $|H \cap (F - G)| = 1$ or $|H \cap (G - F)| = 1$. We may assume that $|H \cap (F - G)| = 1$. Then $|H \cap F| = |H \cap (F - G)| + |H \cap F \cap G| = 3$ and $|H \cap F \cap G| = 2$. Hence, applying Claim 3.3 with $F_0 = H$, $G_0 = F$ and $H_0 = G$, we obtain $G \subseteq H \cup F$. This contradicts the assertion that $G - F \not\subseteq H$. □

Claim 3.5 *Let $F, G, H \in \mathcal{F}$, and suppose that $|F \cap G| = 2$ and $|F \cap G \cap H| = 1$. Then $|(F - G) \cap H| + |(G - F) \cap H| \geq 3$, that is to say, $|(F \cup G) \cap H| \geq 4$.*

Proof. Set $t_1 = |H \cap (F - G)|$ and $t_2 = |H \cap (G - F)|$. We may assume $t_1 \geq t_2$. Suppose that $t_1 + t_2 \leq 2$. Then $(t_1, t_2) \in \{(0, 0), (1, 0), (1, 1), (2, 0)\}$. By Claim 3.1,

$$\begin{aligned}
|\mathcal{C}| &= \Big| \bigcup_{x \in F \cap G} \mathcal{C}[x] \Big| + \Big| \bigcup_{x \in H \cap (F-G)} \mathcal{C}(\overline{F \cap G})[x] \Big| + \Big| \bigcup_{x \in H \cap (G-F)} \mathcal{C}(\overline{F \cap (G \cup H)})[x] \Big| \\
&\qquad + \big| \mathcal{C}(\overline{(F \cap G) \cup (H \cap (F \cup G))}) \big| \\
&\leq 2 \cdot 21 + t_1 \cdot |G - F| \cdot 5 + t_2 \cdot (|F - G| - t_1) \cdot 5 \\
&\qquad + (|F - G| - t_1)(|G - F| - t_2)(|H - (F \cap G)| - (t_1 + t_2)) \\
&= 78 + 3(t_1 + t_2)^2 - 6(t_1 + t_2) - t_1 t_2 (t_1 + t_2) - t_1 t_2.
\end{aligned}$$

If $(t_1, t_2) \in \{(1,0),(1,1)\}$, then this implies $|\mathcal{C}| \le 75$, a contradiction. Thus $(t_1, t_2) \in \{(0,0),(2,0)\}$. Write $F \cap G = \{u_1, u_2\}$. Then $|H \cap \{u_1, u_2\}| = |H \cap F \cap G| = 1$. We may assume that $H \cap \{u_1, u_2\} = \{u_1\}$. Let $F' \in \mathcal{F}(\bar{u}_1\bar{u}_2)$.

First we consider the case where $(t_1, t_2) = (0,0)$. In this case, $H \cap (F \cup G) = \{u_1\}$. Set $t_3 = |F' \cap H|$. Note that $1 \le t_3 \le 3$ because $F' \cap F \ne \emptyset$, $F' \cap G \ne \emptyset$ and $F' \in \mathcal{F}(\bar{u}_1\bar{u}_2)$. Hence by Claim 3.1, $|\mathcal{C}| = |\mathcal{C}[u_1]| + \big(\big|\bigcup_{x \in F' \cap H} \mathcal{C}(\bar{u}_1)[u_2 x]\big| + |\mathcal{C}(\overline{\{u_1\} \cup (F' \cap H)})[u_2]|\big) + |\mathcal{C}(\bar{u}_1\bar{u}_2)| \le 21 + \big(t_3 \cdot 5 + (|H - \{u_1\}| - t_3)(|F'| - t_3)\big) + |F - \{u_1, u_2\}| \cdot |G - \{u_1, u_2\}| \cdot |H - \{u_1\}| = 77 + t_3^2 - 4t_3 = (t_3 - 2)^2 + 73 \le 74$, a contradiction.

Next we consider the case where $(t_1, t_2) = (2,0)$. Note that $(G-F) \cap F' = G \cap F' \ne \emptyset$ because $F' \in \mathcal{F}(\bar{u}_1\bar{u}_2)$. Since $|H \cap (G-F)| = t_2 = 0$, this implies $(G-F-H) \cap F' \ne \emptyset$. Write $F - G = \{v_1, v_2, v_3\}$. Then $|H \cap \{v_1, v_2, v_3\}| = t_1 = 2$. We may assume that $H \cap \{v_1, v_2, v_3\} = \{v_1, v_2\}$. Suppose that $\{v_1, v_2\} \not\subseteq F'$. We may assume $v_1 \notin F'$. Set $t_4 = |G \cap F'|$ $(= |(G-F) \cap F'|)$ and $t_5 = |(H - \{u_1, v_1, v_2\}) \cap F'|$. Then $1 \le t_4 \le 3$ and $0 \le t_5 \le 2$. Hence by Claim 3.1, $|\mathcal{C}| = |\mathcal{C}[u_1] \cup \mathcal{C}[v_2]| + \big(\big|\bigcup_{x \in G \cap F'} \mathcal{C}(\bar{u}_1\bar{v}_2)[v_1 x]\big| + |\mathcal{C}(\overline{\{u_1, v_2\} \cup (G \cap F')})[v_1]|\big) + \big(\big|\bigcup_{x \in (H - \{u_1, v_1, v_2\}) \cap F'} \mathcal{C}(\bar{u}_1\bar{v}_1\bar{v}_2)[u_2 x]\big|$
$+ |\mathcal{C}(\overline{\{u_1, v_1, v_2\} \cup (H \cap F')})[u_2]|\big) + |\mathcal{C}(\bar{u}_1\bar{u}_2\bar{v}_1\bar{v}_2)| \le 2 \cdot 21 + (t_4 \cdot 5 + (|G - \{u_1\}| - t_4)(|F'| - t_4)) + (t_5 \cdot 5 + (|H - \{u_1, v_1, v_2\}| - t_5)(|F'| - t_5)) + |F - \{u_1, u_2, v_1, v_2\}| \cdot |G - \{u_1, u_2\}| \cdot |H - \{u_1, v_1, v_2\}| = (t_4 - 2)^2 + (t_5 - 1)^2 + 73 \le 75$, a contradiction. Thus $\{v_1, v_2\} \subseteq F'$. Consequently $|F \cap H| = |\{u_1, v_1, v_2\}| = 3$ and $|F \cap H \cap F'| = |\{v_1, v_2\}| = 2$, and it therefore follows from Claim 3.3 that $F' \subseteq F \cup H$, which contradicts the earlier assertion that $F' \cap (G - F - H) \ne \emptyset$. □

Claim 3.6 *Let $F, G, H \in \mathcal{F}$, and suppose that $|F \cap G| = 2$, $F \cap G \cap H \ne \emptyset$, $F - G \not\subseteq H$ and $G - F \not\subseteq H$. Then $|F \cap G \cap H| = 1$ and $H \subseteq F \cup G$.*

Proof. By Claim 3.4 $|F \cap G \cap H| = 1$. Write $F = \{u_1, u_2, v_1, v_2, v_3\}$ and $G = \{u_1, u_2, w_1, w_2, w_3\}$. Suppose that $H \not\subseteq F \cup G$. Then by Claim 3.5, $|(F \cup G) \cap H| = (|(F-G) \cap H| + |(G-F) \cap H|) + |F \cap G \cap H| = 4$. Since $F - G \not\subseteq H$ and $G - F \not\subseteq H$, we have $|(F-G) \cap H| \le 2$ and $|(G-F) \cap H| \le 2$. Thus by symmetry, we may assume that $(F \cup G) \cap H = \{u_1, v_1, v_2, w_1\}$. Let $F' \in \mathcal{F}(\bar{v}_3\bar{w}_1)$.

Subclaim 3.6.1 *We have $\{v_1, v_2\} \not\subseteq F'$.*

Proof. Suppose that $\{v_1, v_2\} \subseteq F'$. Then $|F' \cap F \cap H| \ge |\{v_1, v_2\}| = 2$. By Claim 3.4, $|F' \cap \{u_1, u_2\}| = |F' \cap (F \cap G)| \le 1$, and hence $|F' \cap F| = |\{v_1, v_2\}| + |F' \cap \{u_1, u_2\}| \le 3$. Consequently, applying Claim 3.3 with $F_0 = F'$, $G_0 = F$ and $H_0 = H$, we get $H \subseteq F' \cup F$, which contradicts the fact that $w_1 \notin F' \cup F$. □

Subclaim 3.6.2 *We have $F \cap G \cap F' \ne \emptyset$.*

Proof. Suppose that $F \cap G \cap F' = \emptyset$. Then we have $F' \cap \{v_1, v_2\} = F' \cap F \ne \emptyset$ and $F' \cap \{w_2, w_3\} = F' \cap G \ne \emptyset$. We may assume that $v_1, w_2 \in F'$. By Subclaim 3.6.1, $v_2 \notin F'$. Hence $F' \cap F = \{v_1\}$. Write $H - (F \cup G) = \{a\}$. If $a \in F'$, then

$H \cap F' = \{v_1, a\}$ and $H \cap F' \cap F = \{v_1\}$, and hence $|(H \cup F') \cap F| \geq 4$ by Claim 3.5, which contradicts the fact that $(H \cup F') \cap F = \{u_1, v_1, v_2\}$. Thus $a \notin F'$, which implies $F' \cap H = \{v_1\}$. Hence by Claim 3.1, $|\mathcal{C}| = |\mathcal{C}[u_1]| + |\mathcal{C}(\bar{u}_1)[v_1]| + |\mathcal{C}(\bar{u}_1\bar{v}_1)[u_2]| + |\mathcal{C}(\bar{u}_1\bar{u}_2\bar{v}_1)[w_1]| + |\mathcal{C}(\bar{u}_1\bar{u}_2\bar{v}_1\bar{w}_1)[v_2]| + |\mathcal{C}(\bar{u}_1\bar{u}_2\bar{v}_1\bar{v}_2\bar{w}_1)| \leq 21 + |G - \{u_1\}| \cdot 5 + |H - \{u_1, v_1\}| \cdot |F' - \{v_1\}| + |F - \{u_1, u_2, v_1\}| \cdot |F' - \{v_1\}| + |G - \{u_1, u_2, w_1\}| \cdot 5 + |F - \{u_1, u_2, v_1, v_2\}| \cdot |G - \{u_1, u_2, w_1\}| \cdot |H - \{u_1, v_1, v_2, w_1\}| = 73$, a contradiction. □

By Claim 3.4 and Subclaim 3.6.2, $|\{u_1, u_2\} \cap F'| = |F \cap G \cap F'| = 1$, and hence it follows from Claim 3.5 and Subclaim 3.6.1 that $|\{v_1, v_2\} \cap F'| = 1$ and $\{w_2, w_3\} \subseteq F'$. We may assume $\{v_1, v_2\} \cap F' = \{v_1\}$. If $\{u_1, u_2\} \cap F' = \{u_1\}$, then $|F \cap F'| = |\{u_1, v_1\}| = 2$ and $|F \cap F' \cap H| = |\{u_1, v_1\}| = 2$, and hence $H \subseteq F \cup F'$ by Claim 3.3, which contradicts the fact that $w_1 \notin F \cup F'$. Thus $\{u_1, u_2\} \cap F' \neq \{u_1\}$, which implies that $\{u_1, u_2\} \cap F' = \{u_2\}$, and hence $F \cap F' = \{u_2, v_1\}$. Consequently, by Claim 3.1, $|\mathcal{C}| = |\mathcal{C}[u_1]| + |\mathcal{C}(\bar{u}_1)[u_2]| + |\mathcal{C}(\bar{u}_1\bar{u}_2)[v_1]| + |\mathcal{C}(\bar{u}_1\bar{u}_2\bar{v}_1)[w_1]| + |\mathcal{C}(\bar{u}_1\bar{u}_2\bar{v}_1\bar{w}_1)[v_2]| + |\mathcal{C}(\bar{u}_1\bar{u}_2\bar{v}_1\bar{v}_2\bar{w}_1)| \leq 21 + |H - \{u_1\}| \cdot 5 + |G - \{u_1, u_2\}| \cdot 5 + |F - \{u_1, u_2, v_1\}| \cdot |F' - \{u_2, v_1\}| + |G - \{u_1, u_2, w_1\}| \cdot 5 + |F - \{u_1, u_2, v_1, v_2\}| \cdot |G - \{u_1, u_2, w_1\}| \cdot |H - \{u_1, v_1, v_2, w_1\}| = 74$, a contradiction.

This completes the proof of Claim 3.6. □

Claim 3.7 *Let $F, G, H \in \mathcal{F}$, and suppose that $|F \cap G| = 2$ and $F \cap G \cap H \neq \emptyset$. Then $F - G \subseteq H$ or $G - F \subseteq H$.*

Proof. Suppose that $F - G \not\subseteq H$ and $G - F \not\subseteq H$. By Claim 3.6, $|F \cap G \cap H| = 1$ and $H \subseteq F \cup G$. Write $F = \{u_1, u_2, v_1, v_2, v_3\}$ and $G = \{u_1, u_2, w_1, w_2, w_3\}$. We may assume that $H = \{u_1, v_1, v_2, w_1, w_2\}$. Let $F' \in \mathcal{F}(\bar{v}_3\bar{w}_1)$. Arguing as in the proof of Subclaim 3.6.1, we see that $\{v_1, v_2\} \not\subseteq F'$. Since we also have $|\{u_1, u_2\} \cap F'| = |F \cap G \cap F'| \leq 1$ by Claim 3.4, $|(F \cup G) \cap F'| \leq |\{v_1, v_2\} \cap F'| + |\{u_1, u_2\} \cap F'| + |\{w_2, w_3\}| \leq 4$, which means that $F' \not\subseteq F \cup G$. Since $v_3 \in (F - G) - F'$ and $w_1 \in (G - F) - F'$, this together with Claim 3.6 implies that $\{u_1, u_2\} \cap F' = F \cap G \cap F' = \emptyset$. Hence $\{v_1, v_2\} \cap F' = F \cap F' \neq \emptyset$. We may assume that $F' \cap \{v_1, v_2\} = \{v_1\}$. Then $F \cap F' = \{v_1\}$. Suppose that $w_2 \in F'$. Then $|H \cap F'| = |\{v_1, w_2\}| = 2$ and $|H \cap F' \cap G| = |\{w_2\}| = 1$. Since $|F' - H| = 3$ and $|G - H| = 2$, $(F' - H) - G \neq \emptyset$. We also have $v_2 \notin (H - F') - G$. Consequently $G \subseteq H \cup F'$ by Claim 3.6, which contradicts the fact that $u_2 \notin H \cup F'$. Thus $w_2 \notin F'$, which implies $H \cap F' = \{v_1\}$. Therefore by Claim 3.1, $|\mathcal{C}| = |\mathcal{C}[u_1]| + |\mathcal{C}(\bar{u}_1)[v_1]| + |\mathcal{C}(\bar{u}_1\bar{v}_1)[u_2]| + |\mathcal{C}(\bar{u}_1\bar{u}_2\bar{v}_1)[w_1]| + |\mathcal{C}(\bar{u}_1\bar{u}_2\bar{v}_1\bar{w}_1)[w_2]| + |\mathcal{C}(\bar{u}_1\bar{u}_2\bar{v}_1\bar{w}_1\bar{w}_2)| \leq 21 + |G - \{u_1\}| \cdot 5 + |H - \{u_1, v_1\}| \cdot |F' - \{v_1\}| + |F - \{u_1, u_2, v_1\}| \cdot |F' - \{v_1\}| + |F - \{u_1, u_2, v_1\}| \cdot |F' - \{v_1\}| + |G - \{u_1, u_2, w_1, w_2\}| \cdot |H - \{u_1, v_1, w_1, w_2\}| \cdot 5 = 74$, which is a contradiction. □

Now fix $F_1, F_2 \in \mathcal{F}$ with $2 \leq |F_1 \cap F_2| \leq 3$, and set $i_0 = |F_1 \cap F_2|$. Write $F_1 = \{a_1, a_2, a_3, a_4, a_5\}$ and $F_2 = \{b_1, b_2, b_3, b_4, b_5\}$ so that $a_i = b_i$ for each $1 \leq i \leq i_0$ and $\{a_i : i_0 + 1 \leq i \leq 5\} \cap \{b_i : i_0 + 1 \leq i \leq 5\} = \emptyset$. We consider the cases where $i_0 = 2$ and $i_0 = 3$ separately. In each case, we get a contradiction.

Case 1: $i_0 = 2$.

Note that $F_1 = \{a_1, a_2, a_3, a_4, a_5\}$ and $F_2 = \{a_1, a_2, b_3, b_4, b_5\}$. The following claim follows from Claim 3.7.

Claim 3.8 *Let* $a \in \{a_3, a_4, a_5\}$ *and* $b \in \{b_3, b_4, b_5\}$, *and let* $F \in \mathcal{F}(\bar{a}\bar{b})$. *Then* $F \cap F_1 \cap F_2 = F \cap \{a_1, a_2\} = \emptyset$ *and* $|F \cap (F_1 \cup F_2)| = 2$.

Proof. By symmetry, we may assume that $a = a_3$ and $b = b_3$. If $F \cap \{a_1, a_2\} \neq \emptyset$, then it follows from Claim 3.7 that $F_1 - F_2 \subseteq F$ or $F_2 - F_1 \subseteq F$, which contradicts the fact that $a_3, b_3 \notin F$. Thus $F \cap \{a_1, a_2\} = \emptyset$, and hence $F \cap \{a_4, a_5\} = F \cap F_1 \neq \emptyset$ and $F \cap \{b_4, b_5\} = F \cap F_2 \neq \emptyset$. We may assume that $a_4, b_4 \in F$. Suppose that $F \cap \{a_5, b_5\} \neq \emptyset$. We may assume $a_5 \in F$. Let $F' \in \mathcal{F}(\bar{a}_3\bar{b}_4)$. Then arguing as above, we get $F' \cap \{a_1, a_2\} = \emptyset$. This implies $F' \cap \{a_4, a_5\} \neq \emptyset$. Note that $|F_1 \cap F| = |\{a_4, a_5\}| = 2$ and $F_1 \cap F \cap F' \neq \emptyset$. Hence by Claim 3.7, $F_1 - F \subseteq F'$ or $F - F_1 \subseteq F'$, which contradicts the fact that $a_3, b_4 \notin F'$. Thus $F \cap \{a_5, b_5\} = \emptyset$. This implies that $|F \cap (F_1 \cup F_2)| = |\{a_4, b_4\}| = 2$. □

Let $F_3 \in \mathcal{F}(\bar{a}_3\bar{b}_3)$. By Claim 3.8, we have $|F_3 \cap F_1| = |F_3 \cap \{a_4, a_5\}| = 1$ and $|F_3 \cap F_2| = |F_3 \cap \{b_4, b_5\}| = 1$. We may assume that $a_4, b_4 \in F_3$. Let $F_4 \in \mathcal{F}(\bar{a}_4\bar{b}_4)$. By Claim 3.8, $|F_4 \cap F_1| = |F_4 \cap \{a_3, a_5\}| = 1$ and $|F_4 \cap F_2| = |F_4 \cap \{b_3, b_5\}| = 1$. We may assume that $a_3, b_3 \in F_4$. Let $F_5 \in \mathcal{F}(\bar{a}_3\bar{b}_4)$.

Claim 3.9 *We have* $F_5 \cap (F_1 \cup F_2) = \{a_5, b_5\}$.

Proof. It follows from Claim 3.8 that $|F_5 \cap F_1| = |F_5 \cap \{a_4, a_5\}| = 1$ and $|F_5 \cap F_2| = |F_5 \cap \{b_3, b_5\}| = 1$. Suppose that $F_5 \cap F_1 = \{a_4\}$. Then by Claim 3.1, $|\mathcal{C}| = |\mathcal{C}[a_1] \cup \mathcal{C}[a_2]| + \big(|\mathcal{C}(\bar{a}_1\bar{a}_2)[a_4b_3]| + |\mathcal{C}(\bar{a}_1\bar{a}_2\bar{b}_3)[a_4]|\big) + \big(|\mathcal{C}(\bar{a}_1\bar{a}_2\bar{a}_4)[a_3b_4]| + |\mathcal{C}(\bar{a}_1\bar{a}_2\bar{a}_4\bar{b}_4)[a_3]|\big) + |\mathcal{C}(\bar{a}_1\bar{a}_2\bar{a}_3\bar{a}_4)\ [b_3]| + |\mathcal{C}(\bar{a}_1\bar{a}_2\bar{a}_3\bar{a}_4\bar{b}_3)| \leq 2 \cdot 21 + (5 + |F_2 - \{a_1, a_2, b_3\}| \cdot |F_4 - \{b_3\}|) + (|F_5 - \{a_4\}| + |F_2 - \{a_1, a_2, b_4\}| \cdot |F_3 - \{a_4, b_4\}|) + |F_1 - \{a_1, a_2, a_3, a_4\}| \cdot |F_3 - \{a_4\}| + |F_1 - \{a_1, a_2, a_3, a_4\}| \cdot |F_2 - \{a_1, a_2, b_3\}| \cdot |F_4 - \{a_3, b_3\}| = 75$, a contradiction. Thus $F_5 \cap F_1 \neq \{a_4\}$, and hence $F_5 \cap F_1 = \{a_5\}$. Similarly $F_5 \cap F_2 = \{b_5\}$. Hence $F_5 \cap (F_1 \cup F_2) = \{a_5, b_5\}$. □

We can now complete the discussion for Case 1. Recall that $F_3 \cap (F_1 \cup F_2) = \{a_4, b_4\}$ and $F_4 \cap (F_1 \cup F_2) = \{a_3, b_3\}$ and, by Claim 3.9, we have $F_5 \cap (F_1 \cup F_2) = \{a_5, b_5\}$. In particular, $|F_3 \cap F_5| \leq 3$. Therefore by Claim 3.1, $|\mathcal{C}| = |\mathcal{C}[a_1] \cup \mathcal{C}[a_2]| + \big(|\mathcal{C}(\bar{a}_1\bar{a}_2)[a_3b_4]| + |\mathcal{C}(\bar{a}_1\bar{a}_2\bar{b}_4)[a_3b_3]| + |\mathcal{C}(\bar{a}_1\bar{a}_2\bar{b}_3\bar{b}_4)[a_3]|\big) + \big(|\mathcal{C}(\bar{a}_1\bar{a}_2\bar{a}_3)[a_4b_3]| + |\mathcal{C}(\bar{a}_1\bar{a}_2\bar{a}_3\bar{b}_3)[a_4]|\big) + |\mathcal{C}(\bar{a}_1\bar{a}_2\bar{a}_3\bar{a}_4)[b_3]| + |\mathcal{C}(\bar{a}_1\bar{a}_2\bar{a}_3\bar{a}_4\bar{b}_3)| \leq 2 \cdot 21 + (5 + |F_3 \cap F_5| + |F_2 - \{a_1, a_2, b_3, b_4\}| \cdot |F_3 - \{b_4\}|) + (5 + |F_2 - \{a_1, a_2, b_3\}| \cdot |F_4 - \{a_3, b_3\}|) + |F_1 - \{a_1, a_2, a_3, a_4\}| \cdot |F_3 - \{a_4\}| + |F_1 - \{a_1, a_2, a_3, a_4\}| \cdot |F_2 - \{a_1, a_2, b_3\}| \cdot |F_4 - \{a_3, b_3\}| \leq 75$, which is a contradiction.

This completes the discussion for Case 1.

Case 2: $i_0 = 3$.

We have shown that Case 1 leads to a contradiction. Thus

$$|F \cap G| \neq 2 \text{ for any } F, G \in \mathcal{F}. \tag{3.1}$$

With the aid of (3.1), we first prove a result corresponding to Claim 3.7 (see Claim 3.11).

Claim 3.10 *Let $F, G, H \in \mathcal{F}$ with $H \neq F, G$, and suppose that $|F \cap G| = 3$ and $F \cap G \cap H \neq \emptyset$. Then $|F \cap G \cap H| = 1$.*

Proof. Write $F = \{u_1, u_2, u_3, v_1, v_2\}$ and $G = \{u_1, u_2, u_3, w_1, w_2\}$. Suppose that $2 \leq |F \cap G \cap H| \leq 3$. By Claim 3.3, $H \subseteq F \cup G$.

First we consider the case where $|F \cap G \cap H| = 3$. Since $H \neq F, G$, $|H \cap \{v_1, v_2\}| = |H \cap \{w_1, w_2\}| = 1$. We may assume that $H = \{u_1, u_2, u_3, v_1, w_1\}$. Let $F' \in \mathcal{F}(\bar{v}_1 \bar{w}_1)$. Then by (3.1), $|F' \cap H| = |F' \cap \{u_1, u_2, u_3\}| = 1$ or 3. Suppose that $|F' \cap H| = 1$. We may assume $F' \cap H = \{u_1\}$. By (3.1), this implies $F' \cap F = \{u_1\}$ and $F' \cap G = \{u_1\}$. Hence by Claim 3.1, $|\mathcal{C}| = |\bigcup_{1 \leq i \leq 3} \mathcal{C}[u_i]| + |\mathcal{C}(\bar{u}_1 \bar{u}_2 \bar{u}_3)[v_1]| + |\mathcal{C}(\bar{u}_1 \bar{u}_2 \bar{u}_3 \bar{v}_1)| \leq 3 \cdot 21 + |G - \{u_1, u_2, u_3\}| \cdot |F' - \{u_1\}| + |F - \{u_1, u_2, u_3, v_1\}| \cdot |H - \{u_1, u_2, u_3, v_1\}| \cdot |F' - \{u_1\}| = 75$, a contradiction. Thus $F' \cap H = \{u_1, u_2, u_3\}$, and hence $F' \cap F \cap G = \{u_1, u_2, u_3\}$. Consequently $F' \subseteq F \cup G$ by Claim 3.3, which implies $F' = \{u_1, u_2, u_3, v_2, w_2\}$. Hence by Claim 3.1, $|\mathcal{C}| = |\bigcup_{1 \leq i \leq 3} C[u_i]| + \big(|\mathcal{C}(\bar{u}_1 \bar{u}_2 \bar{u}_3)[v_1 w_2]| + |\mathcal{C}(\bar{u}_1 \bar{u}_2 \bar{u}_3 \bar{w}_2)[v_1]|\big) + |\mathcal{C}(\bar{u}_1 \bar{u}_2 \bar{u}_3 \bar{v}_1)| \leq 3 \cdot 21 + (5 + |G - \{u_1, u_2, u_3, w_2\}| \cdot |F' - \{u_1, u_2, u_3, w_2\}|) + |F - \{u_1, u_2, u_3, v_1\}| \cdot |H - \{u_1, u_2, u_3, v_1\}| \cdot 5 = 74$, a contradiction.

Next we consider the case where $|F \cap G \cap H| = 2$. We may assume that $u_1, u_2 \in H$ and $|F \cap H| \geq |G \cap H|$. Recall that $H \subseteq F \cup G$. Hence we have $v_1, v_2 \in H$ and $|H \cap \{w_1, w_2\}| = 1$. We may assume $H = \{u_1, u_2, v_1, v_2, w_1\}$. Let $F' \in \mathcal{F}(\bar{u}_3 \bar{w}_1)$.

Subclaim 3.10.1 *We have $F' \cap \{u_1, u_2\} = \emptyset$.*

Proof. Suppose that $F' \cap \{u_1, u_2\} \neq \emptyset$. We may assume $u_1 \in F'$. Suppose that $u_2 \in F'$. Then $F \cap G \cap F' = \{u_1, u_2\}$, and hence $F' \subseteq F \cup G$ by Claim 3.3, which implies $F' = \{u_1, u_2, v_1, v_2, w_2\}$. Hence by Claim 3.1, $|\mathcal{C}| = |\mathcal{C}[u_1] \cup \mathcal{C}[u_2]| + |\mathcal{C}(\bar{u}_1 \bar{u}_2)[v_1] \cup \mathcal{C}(\bar{u}_1 \bar{u}_2)[v_2]| + |\mathcal{C}(\bar{u}_1 \bar{u}_2 \bar{v}_1 \bar{v}_2)| \leq 2 \cdot 21 + \big(|G - \{u_1, u_2\}| \cdot 5 + |G - \{u_1, u_2\}| \cdot 5\big) + |F - \{u_1, u_2, v_1, v_2\}| \cdot |H - \{u_1, u_2, v_1, v_2\}| \cdot |F' - \{u_1, u_2, v_1, v_2\}| = 73$, a contradiction. Thus $F' \cap \{u_1, u_2\} = \{u_1\}$. By (3.1), this implies $F' \cap G = \{u_1\}$. If $F' \cap \{v_1, v_2\} = \emptyset$, then $F' \cap F = F' \cap H = \{u_1\}$, and hence by Claim 3.1, $|\mathcal{C}| = |\mathcal{C}[u_1]| + |\mathcal{C}(\bar{u}_1)[u_2]| + |\mathcal{C}(\bar{u}_1 \bar{u}_2)[u_3]| + |\mathcal{C}(\bar{u}_1 \bar{u}_2 \bar{u}_3)| \leq 21 + |F' - \{u_1\}| \cdot 5 + |H - \{u_1, u_2\}| \cdot |F' - \{u_1\}| + |F - \{u_1, u_2, u_3\}| \cdot |G - \{u_1, u_2, u_3\}| \cdot |F' - \{u_1\}| = 69$, a contradiction. Thus $F' \cap \{v_1, v_2\} \neq \emptyset$. This implies that $F \cap F' = \{u_1, v_1, v_2\}$ by (3.1). Hence $F \cap F' \cap H = \{u_1, v_1, v_2\}$. Consequently $H \subseteq F \cup F'$ by Claim 3.3, which contradicts the fact that $w_1 \notin F \cup F'$. □

Recall that $F = \{u_1, u_2, u_3, v_1, v_2\}$, $G = \{u_1, u_2, u_3, w_1, w_2\}$, $H = \{u_1, u_2, v_1, v_2, w_1\}$ and $F' \in \mathcal{F}(\bar{u}_3 \bar{w}_1)$. By Subclaim 3.10.1, $F' \cap \{u_1, u_2, u_3, w_1\} = \emptyset$. Hence $F' \cap H = F' \cap \{v_1, v_2\} \neq \emptyset$. By (3.1), $|F' \cap H| = 1$. We may assume $F' \cap H = \{v_1\}$. Then by Claim 3.1, $|\mathcal{C}| = |\mathcal{C}[u_1] \cup \mathcal{C}[u_2]| + |\mathcal{C}(\bar{u}_1 \bar{u}_2)[v_1]| + |\mathcal{C}(\bar{u}_1 \bar{u}_2 \bar{v}_1)[u_3]| + |\mathcal{C}(\bar{u}_1 \bar{u}_2 \bar{u}_3 \bar{v}_1)| \leq 2 \cdot 21 + |G - \{u_1, u_2\}| \cdot 5 + |H - \{u_1, u_2, v_1\}| \cdot |F' - \{v_1\}| + |F - \{u_1, u_2, u_3, v_1\}| \cdot |G - \{u_1, u_2, u_3\}| \cdot 5 = 75$, a contradiction.

This completes the proof of Claim 3.10. □□

Claim 3.11 *Let $F, G, H \in \mathcal{F}$, and suppose that $|F \cap G| = 3$ and $F \cap G \cap H \neq \emptyset$. Then $F - G \subseteq H$ or $G - F \subseteq H$.*

Proof. Suppose that $F - G \not\subseteq H$ and $G - F \not\subseteq H$. By Claim 3.10, $|F \cap G \cap H| = 1$. Write $F \cap G \cap H = \{u\}$. Since $F - G \not\subseteq H$ and $G - F \not\subseteq H$, we have $|H \cap (F - G)| \leq 1$ and $|H \cap (G - F)| \leq 1$. Hence by (3.1), $H \cap F = \{u\}$ and $H \cap G = \{u\}$. Write $H = \{u, x_1, x_2, x_3, x_4\}$. Let $F' \in \mathcal{F}(\bar{u}\bar{x}_1)$. Then $F' \cap H = F' \cap \{x_2, x_3, x_4\} \neq \emptyset$. We may assume $x_2 \in F'$. Then by (3.1), $H \cap F' = \{x_2\}$ or $\{x_2, x_3, x_4\}$. Suppose that $F \cap G \cap F' \neq \emptyset$. Then by Claim 3.10, $|F \cap G \cap F'| = 1$. Write $F \cap G \cap F' = \{u'\}$ and $F \cap G = \{u, u', u''\}$. If $F' \cap H = \{x_2\}$, then by Claim 3.1, $|\mathcal{C}| = |\mathcal{C}[u]| + |\mathcal{C}(\bar{u})[u']| + (|\mathcal{C}(\bar{u}\bar{u}')[u''x_2]| + |\mathcal{C}(\bar{u}\bar{u}'\bar{x}_2)[u'']|) + |\mathcal{C}(\bar{u}\bar{u}'\bar{u}'')| = 21 + |H - \{u\}| \cdot 5 + (5 + |H - \{u, x_2\}| \cdot |F' - \{u', x_2\}|) + |F - \{u, u', u''\}| \cdot |G - \{u, u', u''\}| \cdot |H - \{u\}| = 71$, a contradiction; if $F' \cap H = \{x_2, x_3, x_4\}$, then by Claim 3.1, $|\mathcal{C}| = |\mathcal{C}[u]| + |\mathcal{C}(\bar{u})[u']| + \Big(\big|\bigcup_{2 \leq i \leq 4} \mathcal{C}(\bar{u}\bar{u}')[x_i u'']\big| + |\mathcal{C}(\bar{u}\bar{u}'\bar{x}_2\bar{x}_3\bar{x}_4)[u'']|\Big) + |\mathcal{C}(\bar{u}\bar{u}'\bar{u}'')| = 21 + |H - \{u\}| \cdot 5 + (3 \cdot 5 + |H - \{u, x_2, x_3, x_4\}| \cdot |F' - \{u', x_2, x_3, x_4\}|) + |F - \{u, u', u''\}| \cdot |G - \{u, u', u''\}| \cdot |H - \{u\}| = 73$, a contradiction. Thus $F \cap G \cap F' = \emptyset$. Hence by Claim 3.1, $|\mathcal{C}| = |\mathcal{C}[u]| + \Big(\big|\bigcup_{y \in (F \cap G) - \{u\}} \big(\bigcup_{2 \leq i \leq 4} \mathcal{C}(\bar{u})[y x_i]\big)\big| + \big|\bigcup_{y \in (F \cap G) - \{u\}} \mathcal{C}(\bar{u}\bar{x}_2\bar{x}_3\bar{x}_4)[y]\big|\Big)$ $+ |\mathcal{C}(\overline{F \cap G})| \leq 21 + \big(2 \cdot 3 \cdot 5 + 2 \cdot |H - \{u, x_2, x_3, x_4\}| \cdot |F' - \{x_2\}|\big) + |F - (F \cap G)| \cdot |G - (F \cap G)| \cdot |H - \{u\}| = 75$, a contradiction. □

Recall that $F_1 = \{a_1, a_2, a_3, a_4, a_5\}$ and $F_2 = \{a_1, a_2, a_3, b_4, b_5\}$. We can now start an argument corresponding to the argument in Case 1. The following claim follows from Claim 3.11.

Claim 3.12 *Let $a \in \{a_4, a_5\}$ and $b \in \{b_4, b_5\}$, and let $F \in \mathcal{F}(\bar{a}\bar{b})$. Then $F \cap F_1 \cap F_2 = F \cap \{a_1, a_2, a_3\} = \emptyset$.*

Proof. Since $a, b \notin F$, it follows from Claim 3.11 that $\{a_1, a_2, a_3\} \cap F = F_1 \cap F_2 \cap F = \emptyset$. □

Let $F_3 \in \mathcal{F}(\bar{a}_4\bar{b}_4)$. Then by Claim 3.12, $F_3 \cap \{a_1, a_2, a_3\} = \emptyset$, and hence $a_5, b_5 \in F_3$. Write $F_3 = \{a_5, b_5, c_1, c_2, c_3\}$. Note that $c_i \in X - (F_1 \cup F_2)$ for each $1 \leq i \leq 3$. Let $F_4 \in \mathcal{F}(\bar{a}_5\bar{b}_5)$ and $F_5 \in \mathcal{F}(\bar{a}_4\bar{b}_5)$. Arguing as above, we get $F_4 \cap (F_1 \cup F_2) = \{a_4, b_4\}$ and $F_5 \cap (F_1 \cup F_2) = \{a_5, b_4\}$.

Claim 3.13 *We have $F_3 \cap F_4 = \{c_1, c_2, c_3\}$.*

Proof. By (3.1), we have either $|F_4 \cap F_3| = |F_4 \cap \{c_1, c_2, c_3\}| = 1$ or 3. Suppose that $|F_4 \cap \{c_1, c_2, c_3\}| = 1$. We may assume that $F_4 \cap \{c_1, c_2, c_3\} = \{c_1\}$. Write $F_4 = \{a_4, b_4, c_1, d_1, d_2\}$. Note that $d_i \in X - (F_1 \cup F_2 \cup F_3)$ for each $1 \leq i \leq 2$. Suppose that $c_1 \in F_5$. Since $|F_5 \cap F_3| \geq |\{a_5, c_1\}| = 2$ and $|F_5 \cap F_4| \geq |\{b_4, c_1\}| = 2$, it follows from (3.1) that $F_5 \cap \{c_2, c_3\} \neq \emptyset$ and $F_5 \cap \{d_1, d_2\} \neq \emptyset$. We may assume

$F_5 = \{a_5, b_4, c_1, c_2, d_1\}$. Then $F_3 \cap F_5 = \{a_5, c_1, c_2\}$ and $F_3 \cap F_5 \cap F_1 = \{a_5\}$. Hence, applying Claim 3.11 with $F = F_3$, $G = F_5$ and $H = F_1$, we get $F_3 - F_5 \subseteq F_1$ or $F_5 - F_3 \subseteq F_1$, which contradicts the fact that $b_4, b_5 \notin F_1$. Thus $c_1 \notin F_5$.

Now suppose that $F_5 \cap \{c_2, c_3, d_1, d_2\} \neq \emptyset$. By the symmetry of $\{c_2, c_3\}$ and $\{d_1, d_2\}$, we may assume that $F_5 \cap \{c_2, c_3\} \neq \emptyset$. Then by (3.1), $F_3 \cap F_5 = \{a_5, c_2, c_3\}$, which implies $F_3 \cap F_5 \cap F_1 = \{a_5\}$. Consequently $F_3 - F_5 \subseteq F_1$ or $F_5 - F_3 \subseteq F_1$ by Claim 3.11, which contradicts the fact that $b_4, b_5 \notin F_1$. Thus $F_5 \cap \{c_2, c_3, d_1, d_2\} = \emptyset$.

Combining the assertions in the two preceding paragraphs, we obtain $F_5 \cap \{c_1, c_2, c_3, d_1, d_2\} = \emptyset$. This implies that $F_1 - \{a_4, a_5\}$, $F_3 - \{a_5, c_1\}$, $F_4 - \{a_4, b_4, c_1\}$ and $F_5 - \{a_5, b_4\}$ are pairwise disjoint, and hence $\mathcal{C}(\bar{a}_4\bar{a}_5\bar{b}_4\bar{c}_1) = \emptyset$. Consequently by Claim 3.1, $|\mathcal{C}| = |\mathcal{C}[a_5]| + |\mathcal{C}(\bar{a}_5)[b_4]| + |\mathcal{C}(\bar{a}_5\bar{b}_4)[c_1]| + |\mathcal{C}(\bar{a}_5\bar{b}_4\bar{c}_1)[a_4]| + |\mathcal{C}(\bar{a}_4\bar{a}_5\bar{b}_4\bar{c}_1)| \leq 21 + |F_1 - \{a_5\}| \cdot |F_3 - \{a_5\}| + |F_5 - \{a_5, b_4\}| \cdot |F_1 \cap F_2| + |F_5 - \{a_5, b_4\}| \cdot |F_2 \cap F_3| + 0 = 49$, a contradiction.

This completes the proof of Claim 3.13. □

We are now in a position to complete the proof of Theorem 2. By Claim 3.13, $F_3 \cap F_4 = \{c_1, c_2, c_3\}$. Since $a_4, b_5 \notin F_5$, it follows from Claim 3.11 that $F_3 \cap F_4 \cap F_5 = \{c_1, c_2, c_3\} \cap F_5 = \emptyset$, and hence $F_5 \cap (F_1 \cup F_2 \cup F_3 \cup F_4) = \{a_5, b_4\}$. Therefore by Claim 3.1, $|\mathcal{C}| = |\mathcal{C}[a_5]| + |\mathcal{C}(\bar{a}_5)[b_4]| + |\bigcup_{1 \leq i \leq 3} \mathcal{C}(\bar{a}_5\bar{b}_4)[a_i]| + |\mathcal{C}(\bar{a}_1\bar{a}_2\bar{a}_3\bar{a}_5\bar{b}_4)| \leq 21 + |F_1 - \{a_5\}| \cdot |F_3 - \{a_5\}| + 3 \cdot |F_5 - \{a_5, b_4\}| \cdot |F_3 \cap F_4| + |F_1 - \{a_1, a_2, a_3, a_5\}| \cdot |F_2 - \{a_1, a_2, a_3, b_4\}| \cdot |F_5 - \{a_5, b_4\}| = 67$, which is a contradiction.

This completes the proof of Theorem 2.

Remark. By the same (but complicated) argument, we have verified that Theorems B and D hold for the remaining cases where $6 \leq k \leq 8$.

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