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Stellar subdivisions and Stanley-Reisner rings of Gorenstein complexes*

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Abstract

Unprojection theory analyzes and constructs complicated commutative rings in terms of simpler ones. Our main result is that, on the algebraic level of Stanley-Reisner rings, stellar subdivisions of Gorenstein* simplicial complexes correspond to unprojections of type Kustin-Miller. As an application of our methods we study the minimal resolution of Stanley–Reisner rings associated to stacked polytopes, recovering results of Terai, Hibi, Herzog and Li Marzi.

1 Introduction

Stanley-Reisner rings of simplicial complexes form an important class of commutative rings whose theory has provided spectacular applications to combinatorics; see [28, 38] and [14, Chapter 5]. The Stanley-Reisner ring of a simplicial complex Δ, defined as the quotient of a polynomial ring by a certain ideal, depends only on the

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combinatorics of Δ. Given a combinatorial operation on Δ which produces another simplicial complex, it is natural to ask how the Stanley-Reisner ring of the new complex is related to that of Δ. Stellar subdivision, which is one of the simplest ways to subdivide a simplicial complex, is such an operation. It has been used successfully, for instance, to give a method for transforming the boundary of a polytope into that of any other polytope of the same dimension by operations which preserve interesting invariants [19], to construct polytopes whose f-vectors, or flag f-vectors, span a certain 'Euler' or 'Dehn-Sommerville' space [21, Chapter 9] [4] and to construct simplicial polytopes with prescribed face lattices [26, 36].

On a different tone, unprojection theory aims to analyze and construct commutative rings in terms of simpler ones. The first kind of unprojection which appeared in the literature is that of type Kustin-Miller, studied originally by Kustin and Miller [23] and later by Reid and the second author [32, 34]. Starting from a codimension one ideal I of a Gorenstein ring R such that the quotient R/I is Gorenstein, Kustin-Miller unprojection uses the information contained in $\operatorname{Hom}_R(I, R)$ to construct a new Gorenstein ring S which is 'birational' to R and corresponds to the 'contraction' of $V(I) \subset \operatorname{Spec} R$. It has been used in the classification of Tor algebras in Gorenstein codimension 4 [25]; in the birational geometry of Fano 3-folds [16, 17]; in the study of Mori flips [11, 12]; in the study of algebraic surfaces of general type [29], [31]; in the construction of weighted K3 surfaces and Fano 3-folds [1], [10]; and in the construction of Calabi–Yau 3-folds of high codimension [5, 30]. A general discussion of unprojection theory and its applications is contained in [35], while a precise general definition of unprojection is proposed in [33]. The Kustin-Miller unprojection and the associated complex construction has been implemented in the package KustinMiller [7] for the computer algebra system Macaulay2 [20].

The main objective of this paper is to show that the Stanley-Reisner rings of stellar subdivisions of a Gorenstein* simplicial complex Δ can be constructed from the Stanley-Reisner ring of Δ by unprojections of type Kustin-Miller. As an application, we inductively calculate the minimal graded free resolution of the Stanley-Reisner rings of the boundary simplicial complexes of stacked polytopes, recovering results by Terai and Hibi [39] and Herzog and Li Marzi [22].

To state our main result, we need to introduce some notation and terminology (see Section 2 for more details). We denote by $k[\Delta]$ the Stanley-Reisner ring of a simplicial complex Δ with coefficients in a fixed field k. Recall that Δ is said to be Gorenstein* over k if $k[\Delta]$ is Gorenstein and given a vertex i of Δ there exists $\sigma \in \Delta$ such that $\sigma \cup \{i\}$ is not a face of Δ.

Given a face σ of Δ, we denote by Δ_σ the stellar subdivision of Δ on σ, by x_σ the square-free monomial in $k[\Delta]$ with support σ and by J_σ the annihilator of the principal ideal of $k[\Delta]$ generated by x_σ. Recall also from [34, Definition 1.2] that if $I = (f_1, \dots, f_r) \subset R$ is a homogeneous codimension 1 ideal of a graded Gorenstein ring R such that the quotient R/I is Gorenstein, then there exists $\phi \in \operatorname{Hom}_R(I, R)$ such that ϕ together with the inclusion $I \hookrightarrow R$ generate $\operatorname{Hom}_R(I, R)$ as an R-module. The Kustin-Miller unprojection ring of the pair $I \subset R$ is defined as the quotient of $R[y]$ by the ideal generated by the elements $yf_i - \phi(f_i)$, where y is a new variable.

Theorem 1.1 *Suppose that Δ is a Gorenstein* simplicial complex and that $\sigma \in \Delta$ is a face of dimension $d-1$ for some $d \geq 2$. Let z be a new variable of degree $d-1$ and set $M = \mathrm{Hom}_{k[\Delta][z]}((J_\sigma, z), k[\Delta][z])$.*

(a) M is generated as a $k[\Delta][z]$-module by the elements i and ϕ_σ, where $i\colon (J_\sigma, z) \to k[\Delta][z]$ is the natural inclusion morphism, and ϕ_σ is uniquely specified by $\phi_\sigma(z) = x_\sigma$ and $\phi_\sigma(u) = 0$ for $u \in J_\sigma$.

(b) Denote by S the Kustin-Miller unprojection ring of the pair $(J_\sigma, z) \subset k[\Delta][z]$. Then z is an S-regular element and $k[\Delta_\sigma]$ is isomorphic to $S/(z)$ as a k-algebra.

An example demonstrating Theorem 1.1 is the following. Assume Δ is the boundary simplicial complex of the 2-simplex and σ is a facet of Δ. In coordinates, $k[\Delta] = k[x_1, x_2, x_3]/(x_1x_2x_3)$, $\sigma = \{1, 2\}$ and $J_\sigma = 0 : (x_1x_2) = (x_3)$. Then

$$S = \frac{k[x_1, \ldots, x_4, z]}{(x_4z - x_1x_2, x_4x_3)},$$

where x_4 denotes the new unprojection variable. Notice that when $z = 0$, $S|_{z=0}$ is isomorphic to $k[\Delta_\sigma]$, while when $a \in k^*$, $S|_{z=a}$ is isomorphic (as ungraded k-algebra) to $k[\Delta]$. A toric face ring interpretation of S is discussed in Example 1.

The paper is organised as follows: Theorem 1.1 is proved in Section 3. Section 2 includes some definitions and background related to the concepts which appear in Theorem 1.1. Section 4 contains an interpretation of Theorem 1.1 using the theory of toric face rings. In Section 5, we apply Theorem 1.1 to inductively calculate the minimal graded free resolutions of the Stanley-Reisner rings of the boundary simplicial complexes of stacked polytopes, which were originally given in [22]. The graded Betti numbers of these rings were first calculated in [39]. We conclude with some remarks and directions for future research.

The applications of unprojection theory to Stanley-Reisner rings are not limited to the case of stellar subdivisions, and in the paper [6] we use unprojection techniques for an inductive treatment of Stanley-Reisner rings associated to cyclic polytopes.

2 Preliminaries

Let m be a positive integer and set $E = \{1, 2, \ldots, m\}$. An (abstract) *simplicial complex* on the vertex set E is a collection Δ of subsets of E such that (i) all singletons $\{i\}$ with $i \in E$ belong to Δ and (ii) $\sigma \subset \tau \in \Delta$ implies $\sigma \in \Delta$. The elements of Δ are called *faces* and those maximal with respect to inclusion are called *facets*. The dimension of a face σ is defined as one less than the cardinality of σ. The dimension of Δ is the maximum dimension of a face. The complex Δ is called *pure* if all facets of Δ have the same dimension. Any abstract simplicial complex Δ has a geometric realization, which is unique up to linear homeomorphism. When we refer to a topological property of Δ, we mean the corresponding property of the geometric realization of Δ.

For any subset ρ of E, we denote by x_ρ the square-free monomial in the polynomial ring $k[x_1, \ldots, x_m]$ with support ρ. The ideal I_Δ of $k[x_1, \ldots, x_m]$ which is generated by the square-free monomials x_ρ with $\rho \notin \Delta$ is called the *Stanley-Reisner ideal* of Δ. The *face ring*, or *Stanley-Reisner ring*, $k[\Delta]$ of Δ over k, is defined as the quotient ring of $k[x_1, \ldots, x_m]$ by the ideal I_Δ. For a face σ of Δ denote by $\mathrm{lk}_\Delta(\sigma) = \{\tau : \tau \cup \sigma \in \Delta, \tau \cap \sigma = \emptyset\}$ the *link*, and by $\mathrm{star}_\Delta(\sigma) = \{\tau : \tau \cup \sigma \in \Delta\}$ the *star* of σ in Δ. Given a face σ of Δ of dimension at least 1, the *stellar subdivision* of Δ on σ is the simplicial complex Δ_σ on the vertex set $E \cup \{m+1\}$ obtained from Δ by removing all faces containing σ and adding all sets of the form $\tau \cup \{m+1\}$, where $\tau \in \Delta$ does not contain σ and $\tau \cup \sigma \in \Delta$. The complex Δ_σ is homeomorphic to Δ. We denote by J_σ the ideal $(0 : (x_\sigma))$ of $k[\Delta]$, in other words

$$J_\sigma = \{y \in k[\Delta] \colon y x_\sigma = 0\}.$$

The complex Δ is said to be Gorenstein* (over k) if $k[\Delta]$ is a Gorenstein ring and given a vertex i of Δ there exists $\sigma \in \Delta$ such that $\sigma \cup \{i\}$ is not a face of Δ.

It is known [38, Section II.5] that Δ is Gorenstein* if and only if for any $\sigma \in \Delta$ (including the empty face) we have

$$\widetilde{H}_i(\mathrm{lk}_\Delta(\sigma), k) \;\cong\; \begin{cases} k, & \text{if } i = \dim(\mathrm{lk}_\Delta(\sigma)) \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

where $\widetilde{H}_*(\mathrm{lk}_\Delta(\sigma), k)$ denotes simplicial homology of $\mathrm{lk}_\Delta(\sigma)$ with coefficients in the field k. By [14, Corollary 5.1.5], any Gorenstein* complex Δ is pure. It follows from (1) that the Gorenstein* property is inherited by links. In particular, any codimension 1 face of Δ is contained in exactly 2 facets of Δ. The class of Gorenstein* complexes includes all triangulations of spheres.

Assume R is a polynomial ring over a field k with the degrees of all variables positive, and M is a finitely generated graded R-module. For $i \in \mathbb{Z}$, we denote by $M(i)$ the graded R-module with $M(i)_j = M_{i+j}$ for all i, j. Let

$$0 \to F_g \to F_{g-1} \to \ldots \to F_1 \to F_0 \to M \to 0$$

be the minimal graded free resolution of M as R-module. Write

$$F_i = \oplus_j R(-j)^{b_{ij}},$$

then b_{ij} is called the *ij-th graded Betti number* of M, and we also denote it by $b_{ij}(M)$. For more details about free resolutions and Betti numbers see, for example, [18, Sections 19, 20].

Assume R is a ring. An element $r \in R$ will be called *R-regular* if the multiplication by r map $R \to R, u \mapsto ru$ is injective. A sequence $r_1, \ldots, r_n$ of elements of R will be called a *regular R-sequence* if r_1 is R-regular, and, for $2 \leq i \leq n$, we have that r_i is $R/(r_1, \ldots, r_{i-1})$-regular.

3 Proof of Theorem 1.1

In this section, Δ denotes an $(n-1)$-dimensional simplicial complex on the vertex set $\{1, 2, \dots, m\}$.

Remark 1 We will use the fact that $k[\Delta]$ has no nonzero nilpotent elements and that if I_1, I_2 are monomial ideals of $k[\Delta]$, then so is the ideal quotient

$$(I_1 : I_2) \;=\; \{y \in k[\Delta] \colon yI_2 \subset I_1\}.$$

Remark 2 Assume that Δ is Gorenstein*. If e is a vertex of Δ and $\sigma \in \Delta$ is a face that does not contain e, then there exists a facet of Δ that contains σ but not e. Indeed, let τ_1 be a facet of Δ containing σ. If τ_1 contains e, then there exists a facet τ_2 distinct from τ_1 containing $\tau_1 \setminus \{e\}$. This facet contains σ and does not contain e.

Proposition 3.1 *Let Δ be a Gorenstein* simplicial complex on the vertex set $\{1, 2, \dots, m\}$ and let σ be a face of Δ of dimension at least 1. The ideal J_σ is a codimension 0 ideal of $k[\Delta]$ and the quotient $k[\Delta]/J_\sigma$ is Gorenstein. Moreover,*

$$(0 : J_\sigma) = (x_\sigma).$$

Proof. The first claim is well-known, cf. [18, Theorem 21.23], and the second follows from the observation that $k[\Delta]/J_\sigma = k[x_1, \dots, x_m]/I$, where

$$I = I_{\mathrm{star}_\Delta(\sigma)} + (x_i : i \text{ is not a vertex of } \mathrm{star}_\Delta(\sigma)),$$

and the fact that $\mathrm{lk}_\Delta(\sigma)$ is also Gorenstein*.

We now prove that $(0 : J_\sigma) = (x_\sigma)$. Using [27, Proposition 5.2.3] it is enough to prove that (I_Δ, x_σ) is the ideal of a pure simplicial complex, and this is immediate from Remark 2. □

Remark 3 The conclusion of Proposition 3.1 is not true under the weaker hypothesis that $k[\Delta]$ is Gorenstein. For a counterexample consider

$$\Delta = \{\{1, 2\}, \{1, 3\}, \{1\}, \{2\}, \{3\}, \emptyset\}$$

and $\sigma = \{1, 2\}$. We have $k[\Delta] = k[x_1, x_2, x_3]/(x_2x_3)$, $J_\sigma = (0 : x_1x_2) = (x_3)$, but $(0 : J_\sigma) = (x_2)$.

Let $\sigma \in \Delta$ be a face of dimension $d-1$ for some $d \geq 2$. We recall that the stellar subdivision Δ_σ of Δ on σ is a simplicial complex on the vertex set $\{1, 2, \dots, m+1\}$. We will use the (easy) fact that

$$k[\Delta_\sigma] \cong \frac{k[x_1, \dots, x_{m+1}]}{(I_\Delta, x_\sigma, x_{m+1}u_1, \dots, x_{m+1}u_r)}, \tag{2}$$

where $\{u_1, \ldots, u_r\}$ is a generating set of monomials for the ideal J_σ of $k[\Delta]$.

Proof of Theorem 1.1. Clearly there exists a unique element ϕ_σ of M satisfying $\phi_\sigma(z) = x_\sigma$ and $\phi_\sigma(u) = 0$ for $u \in J_\sigma$. Given $f \in M$, we write $f(z) = w_1 z + w_2$ with $w_1 \in k[\Delta][z]$ and $w_2 \in k[\Delta]$ and set $g = f - w_1 i \in M$, so that $g(z) = w_2$. For $u \in J_\sigma$ we have

$$zg(u) = g(zu) = ug(z) = uw_2 \in k[\Delta].$$

Hence $g(u) = 0$ for all $u \in J_\sigma$, which implies $w_2 \in (0 : J_\sigma)$. By Proposition 3.1 we have $(0 : J_\sigma) = (x_\sigma)$. As a consequence, there exist $w \in k[\Delta]$ such that $w_2 = wx_\sigma$ and hence $g = w\phi_\sigma$. This proves part (a) of the theorem.

By Proposition 3.1, the ring $k[\Delta]/J_\sigma$ is Gorenstein of the same dimension as $k[\Delta]$. Therefore (J_σ, z) is a codimension 1 homogeneous ideal of the graded Gorenstein ring $k[\Delta][z]$, so the general theory of [34] applies. Using part (a) we get

$$S \cong \frac{k[x_1, \ldots, x_{m+1}, z]}{(I_\Delta, x_{m+1}z - x_\sigma, x_{m+1}u_1, \ldots, x_{m+1}u_r)},$$

where the new variable x_{m+1} has degree equal to 1. It follows from (2) that $S/(z) \cong k[\Delta_\sigma]$. By [34, Theorem 1.5], S is Gorenstein of dimension equal to the dimension of $k[\Delta][z]$. As a consequence $\dim S/(z) = \dim S - 1$ and therefore z is an S-regular element. This completes the proof of the theorem. □

Remark 4 It follows from Theorem 1.1 that S is a 1-parameter deformation ring of $k[\Delta_\sigma]$, compare [18, Exerc. 18.18]. The fact that such a deformation ring of $k[\Delta_\sigma]$ exists is a special case of more general results due to Altmann and Christophersen [2, 3].

4 Toric face ring interpretation

Is it clear that Theorem 1.1 is equivalent to the following theorem.

Theorem 4.1 *Suppose that Δ is a Gorenstein* simplicial complex and that $\sigma \in \Delta$ is a face of dimension $d-1$ for some $d \geq 2$. Let $z_1, \ldots, z_{d-1}$ be $d-1$ new variables of degree 1 and set $M_1 = \operatorname{Hom}_{k[\Delta][z_1,\ldots,z_{d-1}]}((J_\sigma, z_1z_2\cdots z_{d-1}), k[\Delta][z_1, \ldots, z_{d-1}])$.*

- *(a) M_1 is generated as a $k[\Delta][z_1, \ldots, z_{d-1}]$-module by the elements i and ϕ_σ, where $i\colon (J_\sigma, z_1z_2\cdots z_{d-1}) \to k[\Delta][z_1, \ldots, z_{d-1}]$ is the natural inclusion morphism, and ϕ_σ is uniquely specified by $\phi_\sigma(z_1z_2\cdots z_{d-1}) = x_\sigma$ and $\phi_\sigma(u) = 0$ for $u \in J_\sigma$.*
- *(b) Denote by S_1 the Kustin-Miller unprojection ring of the pair $(J_\sigma, z_1z_2\cdots z_{d-1}) \subset k[\Delta][z_1, \ldots, z_{d-1}]$. Then $z_1, z_2, \ldots, z_{d-1}$ is an S_1-regular sequence, and $k[\Delta_\sigma]$ is isomorphic to $S_1/(z_1, z_2, \ldots, z_{d-1})$ as a k-algebra.*

We remark that, unlike in Theorem 1.1, in Theorem 4.1 all variables have degree 1 which is the usual grading in the theory of Stanley-Reisner rings. Compare also [6, Section 4], where a similar product $z_1 z_2$ appears in a natural way when relating unprojection and cyclic polytopes.

Consider the Kustin-Miller unprojection ring

$$S_1 = \frac{k[x_1, \ldots, x_{m+1}, z_1, \ldots, z_{d-1}]}{(I_\Delta, x_{m+1} z_1 \cdots z_{d-1} - x_\sigma, x_{m+1} u_1, \ldots, x_{m+1} u_r)}$$

appearing in Theorem 4.1, where as in Section 3 $\{u_1, \ldots, u_r\}$ denotes a generating set of monomials for the ideal $J_\sigma = (0 : x_\sigma)$ of $k[\Delta]$. We will now give a combinatorial interpretation of S_1 using the notion of toric face rings as defined by Stanley in [37, p. 202], compare also [13, Section 4] and [15]. Let M be a free $\mathbb{Z}$-module of rank $m + d - 1$, and consider the $\mathbb{R}$-vector space $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$. We will define a (finite, pointed) rational polyhedral fan $\mathcal{F}$ in $M_\mathbb{R}$, such that S_1 is isomorphic to the toric face ring $k[\mathcal{F}]$. For simplicity of notation we assume in the following that $\sigma = \{1, 2, \ldots, d\}$.

Denote by $e_{x,1}, \ldots, e_{x,m}, e_{z,1}, \ldots, e_{z,d-1}$ a fixed $\mathbb{Z}$-basis of M, and set

$$e_a = (e_{x,1} + \cdots + e_{x,d}) - (e_{z,1} + \cdots + e_{z,d-1}) \in M.$$

Assume $\tau = \{a_1, \ldots, a_p\}$ is a face of Δ. If σ is not a face of τ we set c_τ to be the cone in $M_\mathbb{R}$ spanned by the basis vectors

$$e_{x,a_1}, \ldots, e_{x,a_p}, e_{z,1}, \ldots, e_{z,d-1},$$

while if σ is a face of τ we set c_τ to be the cone in $M_\mathbb{R}$ spanned by the (non-affinely independent) vectors

$$e_{x,a_1}, \ldots, e_{x,a_p}, e_{z,1}, \ldots, e_{z,d-1}, e_a.$$

It is easy to see that the collection of cones $\{c_\tau \mid \tau \text{ face of } \Delta\}$ together with their faces form a fan $\mathcal{F}$ in $M_\mathbb{R}$ and that the toric face ring $k[\mathcal{F}]$ is isomorphic as a k-algebra to S_1.

Example 1 Consider the example given after the statement of Theorem 1.1. That is, let Δ be the boundary of a triangle with vertices corresponding to the variables x_1, x_2, x_3, and denote by Δ_σ the stellar subdivision of Δ with respect to the face $x_1 x_2$. We embed the fan $\mathcal{F}$ into $\mathbb{R}^3$ by assigning to the variables x_1, x_2, x_3, x_4 the rays generated by $(1, 0, 0)$, $(0, 1, 0)$, $(-1, -1, -1)$, $(0, 0, 1) \in \mathbb{Z}^3$, i.e., those of the standard fan of $\mathbb{P}^3$ as a toric variety. Then the ray associated to z is generated by $(1, 1, -1)$. The right hand side of Figure 1 visualizes the Kustin-Miller unprojection ring $S \cong k[\mathcal{F}]$ via representing each cone of the embedded fan $\mathcal{F}$ by a polytope spanning it. There are 3 polytopes of maximal dimension, spanned by $\{x_1, x_3, z\}, \{x_2, x_3, z\}$ and $\{x_1, x_4, x_2, z\}$. Notice that subdividing the cone corresponding to x_1, x_4, x_2, z into x_1, x_4, z and x_4, x_2, z amounts to passing from S to the polynomial ring in the variable z over $k[\Delta_\sigma]$.

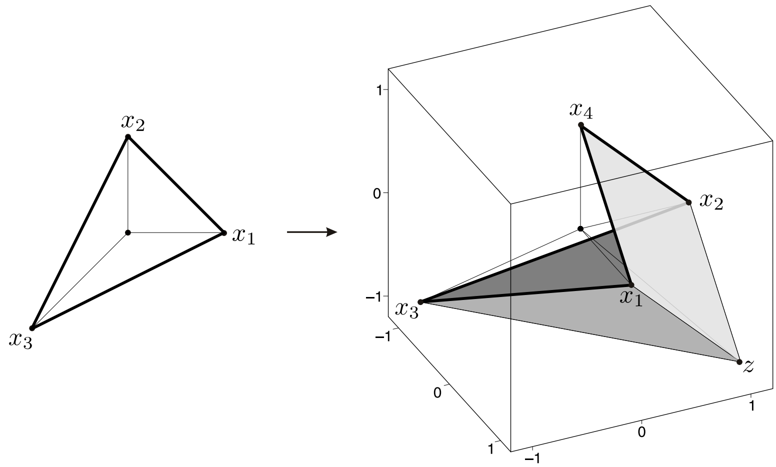

Figure 1: Unprojection via toric face rings

5 Application to stacked polytopes and final remarks

Kustin and Miller [23] described a construction that produces a graded free resolution of the Kustin-Miller unprojection ring in terms of resolutions of the initial data. We briefly discuss this construction in our setting in Subsection 5.1. We then use it in Subsection 5.2 to recover results of Terai, Hibi, Herzog and Li Marzi [39, 22] about the betti numbers and the minimal graded free resolutions of Stanley-Reisner rings associated to stacked polytopes. The final Subsection 5.3 contains remarks and open questions.

5.1 Kustin-Miller complex construction

We use the assumptions and notations of Theorem 1.1. Denote the vertex set of Δ by $\{1, 2, \dots, m\}$. We define the polynomial rings $A = k[x_1, \dots, x_m]$ and $B = A[x_{m+1}]$, with all variables of degree 1.

Let F, H be the minimal graded free resolutions of $k[\Delta]$ and A/J_σ respectively as A-modules, with

$$0 \to F_{c-1} \to F_{c-2} \to \cdots \to F_1 \to F_0 \to k[\Delta] \to 0$$

and

$$0 \to H_{c-1} \to H_{c-2} \to \cdots \to H_1 \to H_0 \to A/J_\sigma \to 0.$$

Combining Theorem 1.1 with the Kustin-Miller complex construction [23] there exists a (not necessarily minimal) resolution G of $k[D_\sigma]$ as B-module such that, when

$c \geq 3$,

$$\begin{aligned} &G_0 = F'_0, \qquad G_1 = F'_1 \oplus H'_1(-1) \oplus H'_0(-d), \\ &G_i = F'_i \oplus H'_i(-1) \oplus H'_{i-1}(-d) \oplus F'_{i-1}(-1), \qquad \text{for } 2 \leq i \leq c-2, \\ &G_{c-1} = H'_{c-1}(-1) \oplus H'_{c-2}(-d) \oplus F'_{c-2}(-1), \qquad G_c = F'_{c-1}(-1), \end{aligned}$$

where $F'_i = F_i \otimes_A B$ and $H'_i = H_i \otimes_A B$. For more details and an implementation of the Kustin-Miller construction see [8].

Remark 5 It follows from the Kustin-Miller complex construction that

$$F(k[\Delta_\sigma], t) = F(k[\Delta], t) + (t + t^2 + \cdots + t^{d-1})\, F(k[\Delta]/J_\sigma, t),$$

where $F(R, t)$ stands for the Hilbert series of R and $d-1$ is the dimension of the face σ. This equality can be rewritten as

$$h(\Delta_\sigma, t) = h(\Delta, t) + (t + t^2 + \cdots + t^{d-1})\, h(\mathrm{lk}_\Delta(\sigma), t), \tag{3}$$

where $h(\Gamma, t)$ stands for the h-polynomial [38, Section II.2] of the simplicial complex Γ. It is not hard to see that (3) holds for any pure simplicial complex Δ. Indeed, one can check directly that (3) is equivalent to the formula

$$f_j(\Delta_\sigma) = f_j(\Delta) - f_{j-d}(\mathrm{lk}_\Delta(\sigma)) + \sum_{i \geq 0} \binom{d}{j-1} f_{i-1}(\mathrm{lk}_\Delta(\sigma)),$$

where $f_j(\Gamma)$ denotes the number of j-dimensional faces of a complex Γ. That formula follows from the definition of Δ_σ.

5.2 The minimal resolution for stacked polytopes

Assume $d \geq 2$ is a fixed integer. Recall from [39, p. 448], that starting from a d-simplex one can add new vertices by building shallow pyramids over facets to obtain a simplicial convex d-polytope with m vertices, called a *stacked polytope* $P_d(m)$. We denote by $\Delta P_d(m)$ the boundary simplicial complex of the simplicial polytope $P_d(m)$. By definition, $\Delta P_d(m)$ has as elements the empty set and the sets of vertices of the proper faces of $P_d(m)$, cf. [14, Corollary 5.2.7]. There is a slight abuse of notation here, since the combinatorial type of $\Delta P_d(m)$ does not depend only on d and m but also on the specific choices of the sequence of facets we used when building the shallow pyramids. The graded Betti numbers b_{ij} of the Stanley-Reisner ring $k[\Delta P_d(m)]$ have been calculated by Terai and Hibi in [39, Theorem 1.1], and it turns out that they only depend on d and m. Later Herzog and Li Marzi [22] constructed the minimal graded free resolution of $k[\Delta P_d(m)]$.

It is clear that, for $d < m$, the simplicial complex $\Delta P_d(m+1)$ can be considered as the stellar subdivision of the boundary simplicial complex $\Delta P_d(m)$ of a stacked polytope $P_d(m)$ with respect to a facet σ of $\Delta P_d(m)$. Since σ is a facet, the ideal (J_σ, z) is generated by the regular sequence x_ρ, z, where ρ takes values in the set of vertices of $\Delta P_d(m)$ which are not vertices of σ. Hence, the minimal graded free resolution of (J_σ, z) is a Koszul complex. Combining Theorem 1.1 with the Kustin-Miller complex construction we can get, starting with the Koszul complex and the minimal graded free resolution of $k[\Delta P_d(m)]$, a graded free resolution of $k[\Delta P_d(m+1)]$. It turns out that we indeed get the minimal graded free resolution of $k[\Delta P_d(m+1)]$. For the details of an inductive proof we refer the interested reader to [9, Section 5.2]. The main idea of the proof is that, for $d \neq 3$, there are no degree 0 morphisms in the Kustin-Miller complex construction, so it is necessarily minimal.

5.3 Final remarks and open questions

Remark 6 In [29], Neves and the second author introduced the $\binom{n}{2}$ Pfaffians format, starting from a certain hypersurface ideal. We give a monomial interpretation of the construction. Start with the boundary simplicial complex Δ of the $(n-1)$-simplex. Denote by Δ_1 the simplicial complex obtained by the stellar subdivisions of all facets of Δ. It is easy to check that the Stanley-Reisner ideal of Δ_1 is equal to $\widetilde{I}_n$, where $\widetilde{I}_n$ denotes the ideal obtained by substituting $z_i = 0$, for $1 \leq i \leq n$, and $r_{d_1,\ldots,d_n} = 1$, for $(d_1, \ldots, d_n) \in \{0,1\}^n$, to the ideal I_n defined in [29, Definition 2.2].

Similarly, in [30, Section 4.3], Neves and the second author constructed a codimension 11 Gorenstein ideal starting from a certain codimension 2 complete intersection ideal. The monomial interpretation of the construction is as follows. Denote by Δ the simplicial complex which is the join [14, p. 221] of 2 copies of the boundary simplicial complex of the 2-simplex. Δ has Stanley-Reisner ideal equal to $(x_{11}x_{12}x_{13}, x_{21}x_{22}x_{23})$ and exactly 9 facets. Denote by Δ_1 the simplicial complex obtained by the stellar subdivisions of Δ on these 9 facets. Using the notations of [30, Section 2], denote by $I_{\mathcal{L}}$ the kernel of the surjection $R[y_u \mid u \in L] \to R_{\mathcal{L}}$. It is easy to check that the Stanley-Reisner ideal of Δ_1 is equal to $\widetilde{I_{\mathcal{L}}}$, where $\widetilde{I_{\mathcal{L}}}$ denotes the ideal obtained by substituting $x_{3i} = 0$, for $1 \leq i \leq 3$, to $I_{\mathcal{L}}$.

Remark 7 It is plausible that our ideas also generalize to non-Gorenstein simplicial complexes. To do this a more detailed study of non-Gorenstein unprojections would be necessary.

Remark 8 Combining our results with those of [24] we get a link between stellar subdivisions of Gorenstein* simplicial complexes and linkage theory [27]. Is it possible to use this connection to define new combinatorial invariants of simplicial complexes?

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References

[1] S. Altinok, Graded rings corresponding to polarised K3 surfaces and $\mathbb{Q}$-Fano 3-folds, Ph.D. thesis, Univ. of Warwick, Sept. 1998, 93+ vii pp.

[2] K. Altmann and J.A. Christophersen, Deforming Stanley-Reisner rings, preprint, 2000, 40 pp, `http://arxiv.org/abs/math/0006139`math.AG/0006139 v2.

[3] K. Altmann and J.A. Christophersen, Cotangent cohomology of Stanley-Reisner rings, *Manuscripta Math.* **115** (2004), 361–378.

[4] M. Bayer and L.J. Billera, Generalized Dehn-Sommerville relations for polytopes, spheres and Eulerian partially ordered sets, *Invent. Math.* **79** (1985), 143–157.

[5] M.-A. Bertin, Examples of Calabi-Yau 3-folds of $\mathbb{P}^7$ with $\rho = 1$, *Canad. J. Math.* **61** (2009), 1050–1072.

[6] J. Böhm and S. Papadakis, On the structure of Stanley-Reisner rings associated to cyclic polytopes, *Osaka J. Math.* **49** (2012), 81–100.

[7] J. Böhm and S. Papadakis, KustinMiller — The Kustin-Miller complex construction and resolutions of Gorenstein rings, Macaulay2 package, 2010, available at `http://www.math.uni-sb.de/ag/schreyer/jb/Macaulay2/KustinMiller/html/`.

[8] J. Böhm and S. Papadakis, Implementing the Kustin-Miller complex construction, *The Journal of Software for Algebra and Geometry*, **4** (2012), 6–11.

[9] J. Böhm and S. Papadakis, Stellar subdivisions and Stanley-Reisner rings of Gorenstein complexes, preprint, 2012, 14 pp, `http://arxiv.org/abs/0912.2151` [math.AC].

[10] G. Brown, M. Kerber and M. Reid, Fano 3-folds in codimension 4, Tom and Jerry, Part I, *Compositio Math* **148** (2012), 1171–1194.

[11] G. Brown and M. Reid, Diptych varieties. I, preprint, arXiv:1208.2446v1 [math.AG].

[12] G. Brown and M. Reid, Diptych varieties. II: Polar varieties, preprint, arXiv:1208.5858v1 [math.AG].

[13] M. Brun and T. Römer, Subdivisions of toric complexes, *J. Algebraic Combin.* **21** (2005), 423–448.

[14] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, revised edition, Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, Cambridge, 1998.

[15] W. Bruns, R. Koch and T. Römer, Gröbner Bases and Betti numbers of monoidal complexes, *Michigan Math. J.* **57** (2008), 71–91.

[16] A. Corti and M. Mella, Birational geometry of terminal quartic 3-folds I, *Amer. J. Math.* **126** (2004), 739–761.

[17] A. Corti, A. Pukhlikov and M. Reid, Fano 3-fold hypersurfaces, in Explicit birational geometry of 3-folds (A. Corti and M. Reid, eds.), London Math. Soc. Lecture Note Series **281**, Cambridge University Press, Cambridge, 2000, pp. 175–258.

[18] D. Eisenbud, Commutative Algebra with a View Toward Algebraic Geometry, Graduate Texts in Mathematics **150**, Springer-Verlag, 1995.

[19] G. Ewald and G.C. Shephard, Stellar subdivisions of boundary complexes of convex polytopes, *Math. Ann.* **210** (1974), 7–16.

[20] D. Grayson and M. Stillman, Macaulay2, a software system for research in algebraic geometry, available at `http://www.math.uiuc.edu/Macaulay2/`.

[21] B. Grünbaum, Convex Polytopes, John Wiley & Sons, New York, 1967; second edition, Graduate Texts in Mathematics **221**, Springer-Verlag, New York, 2003.

[22] J. Herzog, and E. M. Li Marzi, Bounds for the Betti numbers of shellable simplicial complexes and polytopes in Commutative algebra and algebraic geometry (Ferrara), Lecture Notes in Pure and Appl. Math., 206, Dekker, New York, 1999.

[23] A. Kustin and M. Miller, Constructing big Gorenstein ideals from small ones, *J. Algebra* **85** (1983), 303–322.

[24] A. Kustin and M. Miller, Deformation and linkage of Gorenstein algebras, *Trans. Amer. Math. Soc.* **284** (1984), 501–534.

[25] A. Kustin and M. Miller, Classification of the Tor-algebras of codimension four Gorenstein local rings, *Math. Z.* **190** (1985), 341–355.

[26] C.W. Lee, The associahedron and triangulations of the n-gon, *European J. Combin.* **10** (1990), 551–560.

[27] J.C. Migliore, Introduction to Liaison Theory and Deficiency Modules, Progress in Mathematics **165**, Birkhäuser, Boston, 1998.

[28] E. Miller and B. Sturmfels, Combinatorial Commutative Algebra, Graduate Texts in Mathematics **227**, Springer, 2005.

[29] J. Neves and S. Papadakis, A construction of numerical Campedelli surfaces with $\mathbb{Z}/6$ torsion, *Trans. Amer. Math. Soc.* **361** (2009), 4999–5021.

[30] J. Neves and S. Papadakis, Parallel Kustin-Miller unprojection with an application to Calabi-Yau geometry, *Proc. London Math. Soc.* (to appear).

[31] J. Neves and R. Pignatelli, Unprojection and deformations of tertiary Burniat surfaces, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (to appear).

[32] S. Papadakis, Kustin-Miller unprojection with complexes, *J. Algebraic Geom.* **13** (2004), 249–268.

[33] S. Papadakis, Towards a general theory of unprojection, *J. Math. Kyoto Univ.* **47** (2007), 579–598.

[34] S. Papadakis and M. Reid, Kustin-Miller unprojection without complexes, *J. Algebraic Geom.* **13** (2004), 563–577.

[35] M. Reid, Graded rings and birational geometry, in Proc. of Algebraic Geometry Symposium (K. Ohno, ed.), Kinosaki, Oct. 2000, pp. 1–72, available from `http://www.warwick.ac.uk/ masda/3folds/Ki/Ki.pdf`.

[36] R. Simion, A type-B associahedron, *Adv. in Appl. Math.* **30** (2003), 2–25.

[37] R.P. Stanley, Generalized H-vectors, intersection cohomology of toric varieties, and related results, Commutative algebra and combinatorics (Kyoto, 1985), 187–213, Adv. Stud. Pure Math., 11, North-Holland, Amsterdam, 1987.

[38] R.P. Stanley, Combinatorics and Commutative Algebra, Progress in Mathematics **41**, Birkhäuser, Boston, second edition, 1996.

[39] N. Terai and T. Hibi, Computation of Betti numbers of monomial ideals associated with stacked polytopes, *Manuscripta Math.* **92** (1997), 447–453.

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