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Strong amicable orthogonal designs and amicable Hadamard matrices

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Dedicated to Kathy J Horadam on the occasion of her 60th Birthday

Abstract

Amicable orthogonal designs have renewed interest because of their use in mobile communications. We show the existence of strong amicable orthogonal designs, $\mathrm{AOD}(n : 1, n-1; 1, n-1)$, for $n = p^r + 1$, $p \equiv 3 \pmod 4$ a prime and for $n = 2^r$, n a non-negative integer in a form more suitable for communications. Unfortunately the existence of amicable Hadamard matrices is not enough to demonstrate the existence of strong amicable orthogonal designs.

1 Introduction

Interest in amicable orthogonal designs was renewed by the paper of Tarokh, Jafarkhani and Calderbank [14] which showed how they could be used in mobile communications. A delightful introduction to the use of orthogonal designs for CDMA codes for communications has been given by Adams [1]. We notice that for communications the matrices need not be square and may have combinations of complex or quaternion elements. Amicability increases the number of messages which can be transmitted simultaneously but suitable designs have been difficult to find.

Since orthogonal designs and amicable orthogonal designs were first studied in the 1970's [2, 3, 4, 5, 6] a number of authors, [13, 7, 15, 16, 10], have constructed various variants of orthogonal designs to find those that may be optimal in some sense, for example having as many variables as possible or no zeros.

However amicability has always proved far more difficult.

We will always use I_n for the $n \times n$ identity matrix and J_n for the $n \times n$ matrix with all entries $+1$.

An *Hadamard matrix*, H, of order n is a square ± 1 matrix whose rows (and columns) are pairwise orthogonal, that is $HH^\top = nI_n$. Hadamard matrices of order

n are conjectured to exist for all orders $n \equiv 0 \pmod 4$. A weighing matrix, $W = W(n,k)$, of order n and weight k, has elements $0, \ \pm 1$ and satisfies $WW^\top = kI_n$. If an Hadamard matrix, M, can be written in the form $M = I + S$ where $S^\top = -S$, then M is said to be a *skew-Hadamard matrix*. Skew Hadamard matrices are also conjectured to exist for all orders $n \equiv 0 \pmod 4$.

If $M = I + S$ is a skew Hadamard matrix, of order n, and N is an Hadamard matrix also of order n and $MN^\top = NM^\top$, then M and N will be said to be *amicable Hadamard matrices*. If $MN^\top = -NM^\top$, then M and N will be said to be special or anti-amicable Hadamard matrices [12, p296]. Seberry and Yamada [11, p535] give amicable and skew Hadamard matrices which were known in 1992.

However, compared with the knowledge regarding the existence of Hadamard matrices very little is known regarding the existence of skew-Hadamard matrices and amicable Hadamard matrices.

An *orthogonal design or OD*, X of order n and type $(s_1, \cdots, s_m)$, s_i positive integers, is an $n \times n$ matrix with entries $\{0, \pm x_1, \cdots, \pm x_m\}$ (the x_i are commuting indeterminates) satisfying

$$XX^\top = (\sum_{i=1}^{m} s_i x_i^2) I_n,$$

where I_n is the identity matrix of order n. This is denoted as $OD(n; s_1, \cdots, s_m)$.

Such generically orthogonal matrices have played a significant role in the construction of Hadamard matrices (see, e.g., Geramita and Seberry [5] and they have been extensively used in the study of weighing matrices (e.g. Geramita and Geramita [2]).

The existence of structures such as skew-Hadamard matrices and amicable Hadamard matrices seemed a fruitful base to look for the existence of amicable orthogonal designs. Here we consider strong amicable orthogonal designs which are closest to Hadamard matrices (few or no zero entries, without the symmetry condition in amicable Hadamard matrices) and find new constructions.

2 Basic Results

Definition 1 [**Amicable Matrices**] Two square matrices of order n, A and B, are said to be *amicable* if $AB^\top = BA^\top$.

We extend the concept of amicability, which is the desirable property for communication codes, to orthogonal designs:

Definition 2 [**Amicable Orthogonal Designs**] Two orthogonal designs, A, an $OD(n; s_1, s_2, \cdots, s_t)$, on the real commuting variables $x_1, x_2, \cdots, x_t$ and B, an $OD(n; r_1, r_2, \cdots, r_u)$ on the real commuting variables $y_1, y_2, \cdots, y_u$ are said to be *amicable orthogonal designs* or AOD$(n : s_1, s_2, \cdots, s_t; r_1, r_2, \cdots, r_u)$ if $AB^\top = BA^\top$.

The total number of variables in a pair of amicable orthogonal designs was solved, in another context, by Kawade and Iwahori in 1950 (see [5, Section 5.3]).

Definition 3 [Amicable Hadamard matrices] Two Hadamard matrices $M = I + S$, and N, of order n, will be said to be amicable Hadamard matrices if M and N are amicable, that is $MN^\top = NM^\top$ and $S^\top = -S$. (We note that M and N are also AOD$(n : 1, n-1; n)$: loosely we say M and N are an amicable Hadamard pair.)

Now we introduce the concept of *strong amicable Hadamard matrices*

Definition 4 [Strong Amicable Hadamard Matrices] Two Hadamard matrices, M and N, of order n which are amicable, so $MN^\top = NM^\top$, and can be written as $M = I + S$, where I is the identity matrix and $S^\top = -S$ is a skew-symmetric weighing matrix ($W(n, n-1)$) and N, which can be written in the form $N = U + V$, U a symmetric monomial matrix and V is a symmetric weighing matrix ($W(n, n-1)$) will be said to be *strongly amicable Hadamard matrices*. (In fact M and N are also AOD$(n : 1, n-1; 1, n-1)$).

Robinson's work quoted in [5, Lemma 5.135] indicated that finding more AOD was likely to prove difficult.

Example 1 Let A be the $OD(4; 1, 1, 2)$ on the real commuting variables $x_1,\ x_2,\ x_3$ and B the $OD(4; 1, 1, 2)$ on the real commuting variables $y_1,\ y_2,\ y_3$ given below.

$$A = \begin{bmatrix} x_1 & x_2 & x_3 & x_3 \\ -x_2 & x_1 & x_3 & -x_3 \\ -x_3 & -x_3 & x_1 & x_2 \\ -x_3 & x_3 & -x_2 & x_1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} y_1 & y_2 & y_3 & y_3 \\ y_2 & -y_1 & y_3 & -y_3 \\ y_3 & y_3 & -y_2 & -y_1 \\ y_3 & -y_3 & -y_1 & y_2 \end{bmatrix}$$

Then $AB^\top = BA^\top$ and so A and B are amicable orthogonal designs or AOD$(4 : 1, 1, 2; 1, 1, 2)$.

Example 2 Amicable Hadamard Matrices

Furthermore, replacing x_1, by **1**, y_1 by **-1** and x_2, x_3, y_2 and y_3 by 1, in Example 1, we obtain the following two amicable Hadamard matrices. (The **1** is just for emphasis.)

$$A = \begin{bmatrix} \mathbf{1} & 1 & 1 & 1 \\ -1 & \mathbf{1} & 1 & -1 \\ -1 & -1 & \mathbf{1} & 1 \\ -1 & 1 & -1 & \mathbf{1} \end{bmatrix}$$

$$\text{and} \qquad B = \begin{bmatrix} -\mathbf{1} & 1 & 1 & 1 \\ 1 & \mathbf{1} & 1 & -1 \\ 1 & 1 & -1 & \mathbf{1} \\ 1 & -1 & \mathbf{1} & 1 \end{bmatrix}$$

Example 3 Strong Amicable Hadamard Matrices We note that if the amicable Hadamard matrices, of the previous examples, are written as

$$A = I + S = I + \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 \\ -1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{bmatrix}$$

$$\text{and} \quad B = U + V = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}$$

then clearly their associated $W(n, n-1)$, are skew-symmetric and symmetric respectively. Thus $A = I + S$ and $B = U + V$ are strongly amicable Hadamard matrices.

Example 4 [Strong Amicable Orthogonal Designs SAOD(n)] Let A be the $OD(4; 1, 3)$ on the real commuting variables $x_1,\ x_2$ and B the $OD(4; 1, 3)$ on the real commuting variables $y_1,\ y_2$ from Example 1 given above. Then

$$A = \begin{bmatrix} x_1 & x_2 & x_2 & x_2 \\ -x_2 & x_1 & x_2 & -x_2 \\ -x_2 & -x_2 & x_1 & x_2 \\ -x_2 & x_2 & -x_2 & x_1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} y_1 & y_2 & y_2 & y_2 \\ y_2 & -y_1 & y_2 & -y_2 \\ y_2 & y_2 & -y_2 & -y_1 \\ y_2 & -y_2 & -y_1 & y_2 \end{bmatrix}.$$

Now $AB^\top = BA^\top$ and so A and B are strongly amicable orthogonal designs, SAOD(4), or AOD(4 : 1, 3; 1, 3).

Definition 5 [Strong $AOD(n : 1, n-1; 1, n-1)$ **Form]** Suppose M and N, are AOD(n : $1, n-1; 1, n-1$), where $M = x_1 I + x_2 S$, and $N = y_1 P + y_2 Q$, with $S^\top = -S$, $P^\top = P$ and $Q^\top = Q$ so $N^\top = N$, ($x_1,\ y_1$ occur once in each each row and each column of the relevant matrix, and $x_2,\ y_2$, which occur $n-1$ times per row and column of the relevant each matrix, are real commuting variables). Then M and N will be said to be in *amicable orthogonal designs in strong form, SAOD(n).*

Notation A strong AOD($n : 1, n-1; 1, n-1$) will be denoted as SAOD(n).

3 Results

In this section we give some previously observed results and some results which have not been previously explicitly emphasized. The following results are quoted from [9], and given here for future reference.

Lemma 1 *Suppose there exist* $AOD(n : s_1, s_2, \cdots, s_t; r_1, r_2, \cdots, r_u)$*, A and B, written, on the commuting variables* x_i*,* $i = 1, \cdots, t$ *and* y_j*,* $j = 1, \cdots, u$*). We write* $A = x_1A_1 + x_2A_2 + \cdots x_tA_t$*,* $x_i \neq x_j$ *and* $B = y_1B_1 + y_2B_2 + \cdots y_uB_u$*,* $y_k \neq y_\ell$*, then*

$$A_iB_j^\top = B_jA_i^\top \quad \text{for any } i,\ j.$$

That is, the individual terms are pairwise amicable.

Proof. Since A and B are orthogonal designs we have

$$A_iA_j^\top = -A_jA_i^\top \text{ and } B_kB_\ell^\top = -B_\ell B_k^\top,$$

that is, they are pairwise anti-amicable. Further, by definition,

$$A_kA_k^\top = s_kI_n,\ B_\ell k B_\ell k^\top = r_\ell k I_n.$$

Since A and B are AOD we have, by definition

$$(x_1A_1 + \cdots + x_tA_t)(y_1B_1 + \cdots + y_uB_u)^\top = (y_1B_1 + \cdots + y_uB_u)(x_1A_1 + \cdots + x_tA_t)^\top.$$

We expand this equation and consider the terms which involve j and k. Now from the left hand side of the equation we have

$$\begin{aligned} LHS &= x_jy_jA_jB_j^\top + x_ky_jA_kB_j^\top + x_jy_kA_jB_k^\top + x_ky_kA_kB_k^\top \\ &= x_jy_jB_jA_j^\top + x_ky_jB_jA_k^\top + x_jy_kB_kA_j^\top + x_ky_kB_kA_k^\top \\ &= RHS. \end{aligned}$$

Considering equality of variables we have

$$A_iB_j^\top = B_jA_i^\top \text{ for all } i,\ j.$$

Since A and B are orthogonal designs we also have

$$A_jA_k^\top = -A_kA_j^\top \text{ and } B_jB_k^\top = -B_kB_j^\top,$$

and also

$$A_jA_j^\top = s_jI_n,\ B_kB_k^\top = r_kI_n, \text{ for all } s_j,\ r_k.$$

This is the desired result. □

Corollary 1 *There exist strong amicable orthogonal designs, $AOD(n : 1, n-1; 1, n-1)$, A and B on the commuting real variables x_i and y_i, i = 1, 2, respectively, if and only if there exist two amicable orthogonal designs of the form $M = x_1I + x_2S$, where $S^\top = -S$ and $N = y_1P + y_2Q$, where $N^\top = N$, that is both P and Q are symmetric and S is skew symmetric.*

4 Multiplication of Strong Amicable Orthogonal Designs

Seberry and Yamada [11, p. 535] give amicable Hadamard matrices which were known in 1992. We note the family of strong amicable Hadamard matrices (and indeed amicable orthogonal designs) which were fore-shadowed in Geramita, Pullman and Seberry Wallis [4], but first explicitly defined in [9] for AOD($2^t; 1, 2^t-1; 1, 2^t-1$). We now observe that the structure can be maintained in carefully chosen AOD. Indeed we have a new, more general, theorem which has this Geramita-Pullman-Seberry family as a corollary.

Theorem 1 **[Multiplication Theorem for Strong Amicable Orthogonal Designs]** *If there are strongly amicable orthogonal designs of orders n_1 and n_2 there are strongly amicable orthogonal designs of order n_1n_2. (The theorem also holds if "orthogonal designs" is replaced by "Hadamard matrices".)*

Proof. We write x_1, x_2, y_1, y_2, u_1, u_2, v_1 and v_2, for pairs of commuting real, variables. Using the definition we write the strongly amicable orthogonal designs of order n_i, $i = 1$ or 2, in the form $M_{n_i} = x_1 I_{n_i} + x_2 S_{n_i}$, where $S_{n_i}^\top = -S_{n_i}$ and $N_{n_i} = y_1 P_{n_i} + y_2 Q_{n_i}$, where $N_{n_i}^\top = N_{n_i}$. Then

$$M = u_1 I_{n_1} \times I_{n_2} + u_2(I_{n_1} \times S_{n_2} + S_{n_1} \times N_{n_2})$$

$$\text{and } N = v_1(P_{n_1} \times P_{n_2}) + v_2(P_{n_1} \times Q_{n_2} + Q_{n_1} \times N_{n_2})$$

are the required SAODs of order n_1n_2. □

There are other amicable orthogonal design parameters which can be obtained from the previous proof but we will not pursue them here. A more direct proof of the following corollary appears in [9] using cores as in the next section.

Corollary 2 *Let t be a positive integer. Then there exist SAOD$(2^t : 1, 2^t - 1; 2^t - 1)$ and strongly amicable Hadamard matrices for every 2^t.*

Proof. We note that, writing the real variables x_1, x_2, y_1, and y_2 as ± 1 gives amicable Hadamard matrices of order 2 from the AOD$(2 : 1, 1; 1, 1)$ given below

$$A = \begin{bmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} y_1 & y_2 \\ y_2 & -y_1 \end{bmatrix}.$$

Then iterated use of Theorem 1 gives the result. □

5 Construction of SAODs Using Cores

The following theorem, due to Paley [8], is quoted from Geramita and Seberry [5, Theorem 5.52]

Theorem 2 *Let $q \equiv 3 \pmod 4$ be a prime power. Then there exist strong AOD$(p + 1; 1, p; 1, p)$.*

To illustrate we use Paley's original construction [8, 12] to form a skew symmetric matrix, Q, with zero diagonal and non-diagonal entries ± 1 satisfying $QJ = 0$, $QQ^\top = qI - J$, $Q^\top = -Q$. This matrix Q is called the *Paley core*. Then with

$$e = \{\underbrace{1, \cdots, 1}_{q}\},$$

and x_1, x_2 commuting variables, the A_1 now given is one of a pair of strongly amicable Hadamard matrices in standard form; the A_2 is one of a pair of strongly amicable orthogonal designs also in standard form:

$$A_1 = \begin{pmatrix} 1 & e \\ -e^\top & I+Q \end{pmatrix}, \quad A_2 = \begin{pmatrix} x_1 & x_2 e \\ -x_2 e^\top & x_1 I + x_2 Q \end{pmatrix}.$$

The second matrix of each amicable pair is made by choosing the back-circulant (or type 2, see [12, p 284] for the definition of type 2) version of Q. We use a symmetric monomial matrix P, of order $q+1$, usually based on the back-diagonal matrix R of order q, with y_1 and y_2 commuting variables. So we use

$$P = \begin{pmatrix} -1 & 0 \\ 0^\top & R \end{pmatrix},$$

R the back diagonal matrix of order $q-1$ and form B_1 and $B_2 = B_1 P$

$$B_1 = \begin{pmatrix} -1 & e \\ e^\top & (I+Q)R \end{pmatrix}, \quad B_2 = \begin{pmatrix} -y_1 & y_2 e \\ y_2 e^\top & (y_1 I + y_2 Q)R \end{pmatrix}.$$

Thus A_1 and B_1 are strongly amicable Hadamard matrices and A_2 and B_2 are strongly amicable orthogonal designs in standard form.

5.1 Cores

In the above illustration we showed how to use Paley cores to make strongly amicable Hadamard matrices. In fact we showed more, for Theorem 1, when used carefully can preserve the substructure of the cores. Thus we have

Corollary 3 *There exist strongly amicable orthogonal designs, $AOD(n : 1, n-1; 1, n-1)$, A and B on the commuting real variables x_i and y_i, $i = 1, 2$, respectively, if and only if there exist two amicable cores, of order $n-1$, of the orthogonal designs of the form $M = x_1 I + x_2 S$, where $S^\top = -S$ and $N = y_1 U + y_2 V$, where $N^\top = N$, that is both U and V are symmetric and S is skew symmetric.*

Further more $II^\top = UU^\top = I_{n-1}$, $SS^\top = VV^\top = nI_{n-1} - J_{n-1}$, $SJ = VJ = 0$.

These are the required cores for the main result of Seberry [9]. We state this as

Proposition 1 Powers of Cores *If there exist $SAOD(n : 1, n-1; 1, n-1)$ and a suitable amicabilizer (written as R above) then there exists $SAOD((n-1)^r + 1)$ for every odd integer $r > 0$.*

We note without engineering explanation that

Proposition 2 *Using the circulant difference set $SBIBD(2^t - 1, 2^{t-1} - 1, 2^{t-2} - 1)$ to form the core of a $SAOD(2^t : 1, 2^t - 1; 1, 2^t - 1)$ allows a more efficient construction for practical purposes.*

6 Concluding Remarks

Seberry and Yamada [11, p541-542] give a list of constructions for skew-Hadamard matrices known in 1992. There are very few further constructions known.

From Seberry and Yamada [11, p535], Geramita, Pullman and Seberry Wallis [4] and this paper, strong amicable Hadamard matrices and strong amicable orthogonal designs SAOD exist for the following orders:

Summary 1 *AOD*$(n:1,n-1;1,n-1)$, *SAOD, (or SAOD) exist for orders*

Key	Order	Method
x_1	2^t	t a non − negative integer; Corollary 2.
x_2	p^r+1	$p^r \equiv 3 \pmod 4$ is a prime power; Theorem 2.
x_3	$(n-1)^r+1$	n is the order of SAOD with suitable cores
		$r>0$ is any odd integer; Proposition 1.
x_4	nh	$n, h,$ are the orders of SAODs; Theorem 1.

The constraints on finding amicable orthogonal designs, even using the most promising candidates, that is those from skew-Hadamard matrices, makes the further construction of SAOD most challenging.

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