AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **53** (2012), Pages 141–150

Reconstruction of bipartite graphs and triangle-free graphs with connectivity two*

S. Monikandan J. Balakumar

Department of Mathematics
Manonmaniam Sundaranar University
Tirunelveli, Tamilnadu
INDIA
monikandans@gmail.com baladcm@gmail.com

Abstract

A graph is said to be *reconstructible* if it is determined up to isomorphism from the collection of all its one-vertex deleted unlabeled subgraphs. The *Reconstruction Conjecture* (RC) asserts that all graphs on at least three vertices are reconstructible. In this paper, we prove that all triangle-free graphs G with connectivity two such that $\mathrm{diam}(G) = 2$ or $\mathrm{diam}(G) = \mathrm{diam}(\overline{G}) = 3$ are reconstructible. We also prove that all 2-connected bipartite graphs G such that $\mathrm{diam}(G) = 2$ or $\mathrm{diam}(G) = \mathrm{diam}(\overline{G}) = 3$ are reconstructible and show that RC is true if and only if all 2-connected graphs H containing an odd cycle such that $\mathrm{diam}(H) = 2$ or $\mathrm{diam}(H) = \mathrm{diam}(\overline{H}) = 3$ are reconstructible.

1 Introduction

All graphs in this paper are finite, simple and undirected. We use the terminology in Harary [5]. The *degree* of a vertex v of a graph G is denoted by $\deg_G v$ (or $\deg v$); a vertex u with $\deg u = 1$ is an *endvertex.* The *maximum degree*, the number of components, and the *connectivity* of a graph G are denoted by $\Delta(G)$, $\omega(G)$, and $\kappa(G)$ respectively. Graphs containing no K_3 (a complete graph on 3 vertices) as a subgraph are called *triangle-free* graphs. We call a graph with just one vertex *trivial* and all other graphs *nontrivial.* A *vertex-deleted subgraph* (or *card*) $G - v$ of a graph G is the unlabeled subgraph obtained from G by deleting v and all edges incident with v. The collection of all cards of G is called the *deck* of G. A graph H is called a *reconstruction* of G if H has the same deck as G. A graph is said to be *reconstructible* if it is isomorphic to all its reconstructions. A family $\mathscr{F}$ of graphs is *recognizable* if,

* Research supported by the DST, Govt. of India. Grant No. SR/S4/MS: 628/09.

for each $G \in \mathscr{F}$, every reconstruction of G is also in $\mathscr{F}$, and *weakly reconstructible* if, for each graph $G \in \mathscr{F}$, all reconstructions of G that are in $\mathscr{F}$ are isomorphic to G. A family $\mathscr{F}$ of graphs is *reconstructible* if, for any graph $G \in \mathscr{F}$, G is reconstructible (i.e. if $\mathscr{F}$ is both recognizable and weakly reconstructible). A parameter p defined on graphs is reconstructible if, for any graph G, it takes the same value on every reconstruction of G. The *Reconstruction Conjecture* (RC) asserts that all graphs on at least three vertices are reconstructible. We refer the reader to [2, 3, 8, 9] for surveys of work done on RC and related problems.

Yang Yongzhi [12] settled Problem 3 listed in the survey [2] when he proved that every connected graph is reconstructible if and only if every 2-connected graph is reconstructible. Gupta et al. [4] have proved that the RC is true if and only if all connected graphs G such that $\operatorname{diam}(G) = 2$ or $\operatorname{diam}(G) = \operatorname{diam}(\overline{G}) = 3$ are reconstructible. Using these results, recently the following reduction was proved.

Theorem A ([10]). *The RC is true if and only if all 2-connected graphs G such that* $\operatorname{diam}(G) = 2$ *or* $\operatorname{diam}(G) = \operatorname{diam}(\overline{G}) = 3$ *are reconstructible.*

In their survey [2], Bondy and Hemminger suggested:

Problem B ([2]). Show that bipartite graphs are reconstructible.

But in view of Theorem A, to attempt Problem B, it is enough to consider all 2-connected bipartite graphs G such that $\operatorname{diam}(G) = 2$ or $\operatorname{diam}(G) = \operatorname{diam}(\overline{G}) = 3$. In this paper, we prove that this class of bipartite graphs is reconstructible. We also prove that all triangle-free graphs G with $\kappa(G) = 2$ such that $\operatorname{diam}(G) = 2$ or $\operatorname{diam}(G) = \operatorname{diam}(\overline{G}) = 3$ are reconstructible.

2 Bipartite graphs

There is a fundamental lemma in RC due to Kelly [6].

Kelly's Lemma. *For any two graphs F and G with $|V(F)| < |V(G)|$, the number of subgraphs of G isomorphic to F is reconstructible.*

The following three lemmas related to RC are well known.

Lemma 1 ([2]). *The connectivity $\kappa(G)$ of a graph G is reconstructible.*

Lemma 2 (Tutte [11]). *The number of nonseparable spanning subgraphs of G with a given number of edges is reconstructible.*

Lemma 3 ([4]). *Graphs G with* $\operatorname{diam}(G) = 2$ *and graphs H with* $\operatorname{diam}(H) = \operatorname{diam}(\overline{H}) = 3$ *are recognizable.*

Since bipartite graphs of diameter two are precisely complete bipartite graphs, using this we prove the following theorem.

Theorem 1. *Bipartite graphs G of diameter two are reconstructible.*

Proof. *Recognition:* If C is a cycle of order strictly less than n (the order of G), then, using Kelly's Lemma, we can determine whether C is a subgraph of G or not.

Moreover, if C is a cycle of order n, then C is a nonseparable graph with n edges and hence, using Lemma 2, we can determine whether C is a subgraph of G or not. In particular, we can determine whether G contains an odd cycle or not. Therefore bipartite graphs are recognizable. The recognition of bipartite graphs of diameter two now follows by Lemma 3.

Weak reconstruction: Let G be a bipartite graph of diameter two. Then G is complete bipartite, say $K_{r,s}$, where $r \geq s$. Since the degree sequence of G is reconstructible, the values of r and s are known. Since trees are reconstructible [6], we can assume that $s > 1$. Then G is 2-connected and so every card $G - v$ is connected. But since every connected graph has a unique bipartition [1], the set of vertices of every card $G - v$ can be uniquely bipartitioned into (X_v, Y_v). In particular, there must exist a card $G - w$ with a unique bipartition (X_w, Y_w) such that $|X_w| > |Y_w| = s - 1$. Now G can be obtained uniquely by augmenting $G - w$ as adding a new vertex to $G - w$ and joining it with all the vertices in the (identifiable) set X_w. ∎

Theorem 2. *All* 2*-connected bipartite graphs* G *such that* $\mathrm{diam}(G) = \mathrm{diam}(\overline{G}) = 3$ *are reconstructible.*

Proof. Since bipartite graphs are recognizable, the recognition follows by Lemmas 1 and 3.

Weak reconstruction: Since every connected graph has a unique bipartition [1], every card of G being 2-connected, is connected and so it has a unique bipartition. Hence a 2-connected graph G is a bipartite graph with bipartition (X, Y) such that $|X| = r$ and $|Y| = s$ if and only if the deck of G consisting of $r + s$ cards, all are bipartite, such that partite sets of r cards in the deck have sizes $r - 1$ and s and that of the remaining s cards have sizes r and $s - 1$. Therefore we can determine the values of r and s from the deck of G. Without loss of generality, we may assume that $r \geq s$.

Clearly, in $\overline{G}$, the induced subgraphs $\langle X \rangle$ and $\langle Y \rangle$ are isomorphic to K_r and K_s, respectively. Since $\mathrm{diam}(\overline{G}) = 3$, $d_{\overline{G}}(x, y) = 3$ for some $x \in X$ and $y \in Y$. Consequently, in $\overline{G}$, the vertex x is adjacent to no vertex of Y and the vertex y is adjacent to no vertex of X. Hence, in G, the vertex x is adjacent to every vertex of Y and y is adjacent to every vertex of X. Consequently, there must exist a card $G - v$ with a unique bipartition (X_v, Y_v) such that $|X_v| > |Y_v| = s - 1$. Now G can be obtained uniquely (up to isomorphism) from the card $G - v$ by adding a new vertex w to $G - v$ and joining w with all the vertices in the (identifiable) set X_v. ∎

Corollary 1. *All graphs on at least three vertices are reconstructible if and only if all* 2*-connected graphs* G *containing an odd cycle such that* $\mathrm{diam}(G) = 2$ *or* $\mathrm{diam}(G) = \mathrm{diam}(\overline{G}) = 3$ *are reconstructible.*

Proof. This follows by Theorem A and Theorems 1 and 2. ∎

When U and W are disjoint subsets of the vertex set $V(G)$ of a graph G, $U \sim W$ means that every vertex in U is adjacent to every vertex in W, and when $u \notin W$, $u \sim W$ means that u is adjacent to every vertex in W. Similarly, by $u \sim w$ we mean the vertices u and w are adjacent; otherwise we denote $u \not\sim w$. Let G be a triangle-free graph with $\kappa(G) = 2$. For a vertex cut $S = \{x_1, x_2\}$ of G, we denote the components of $G - S$ by $C_1, C_2, \ldots, C_{\omega(G-S)}$, and in each C_i, we define the following four possible subsets:

$C_i(x_1, x_2)$: set of vertices that are adjacent to both x_1 and x_2.

$C_i(\overline{x_1}, x_2)$: set of vertices that are adjacent to x_2 but not to x_1.

$C_i(x_1, \overline{x_2})$: set of vertices that are adjacent to x_1 but not to x_2.

$C_i(\overline{x_1}, \overline{x_2})$: set of vertices that are neither adjacent to x_1 nor to x_2.

We now prove that all triangle-free graphs G with $\kappa(G) = 2$ such that $\text{diam}(G) = 2$ or $\text{diam}(G) = \text{diam}(\overline{G}) = 3$ are reconstructible. We begin with a lemma.

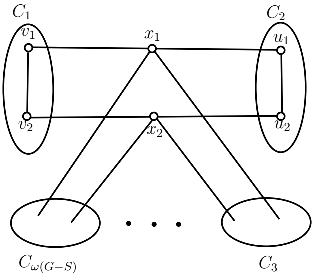

Figure 1.

Lemma 4. *Let G be a triangle-free graph of diameter two with $\kappa(G) = 2$. If S is a vertex cut of G of cardinality two, then at most one component of $G - S$ has at least two vertices.*

Proof. Assume, to the contrary, that $G - S$ has two nontrivial components, say C_1 and C_2; let $v_1v_2 \in E(C_1)$ and $u_1u_2 \in E(C_2)$ (Figure 1). Let the components of $G - S$ be $C_1, C_2, \ldots, C_{\omega(G-S)}$. Then since $\text{diam}(G) = 2$, it follows that $C_i(\overline{x_1}, \overline{x_2}) = \phi$ for all i and each vertex in C_i $(i = 1, 2, \ldots, \omega(G - S))$ is adjacent to at least one vertex in S. Since G is triangle-free, we may assume, without loss of generality, that $v_1 \sim x_1$, $v_2 \sim x_2$, $v_1 \nsim x_2$, $v_2 \nsim x_1$ and similarly we assume that $u_1 \sim x_1$, $u_2 \sim x_2$, $u_1 \nsim x_2$, $u_2 \nsim x_1$. Then v_1 and u_2 are nonadjacent and they are not adjacent to a common vertex in G, giving a contradiction to $\text{diam}(G) = 2$. ∎

Theorem 3. *Triangle-free graphs G of diameter two with $\kappa(G) = 2$ are reconstructible.*

Proof. It is known [7] that all graphs with fewer than twelve vertices are reconstructible. So we assume that $n \geq 12$.

Recognition: From Kelly's Lemma, it follows that triangle-free graphs are recognizable and so the recognition of the family of all triangle-free graphs G of diameter two with $\kappa(G) = 2$ follows by Lemmas 1 and 3.

Weak reconstruction: Suppose, for a vertex cut $S = \{x_1, x_2\}$ of G, all the components of $G - S$ are trivial. Then G has a card, say $G - v$ with a unique cutvertex and $n-2$ endvertices. Therefore G can be obtained uniquely (up to isomorphism) by adding a new vertex w to $G - v$ and joining it with all the $n-2$ endvertices and hence G is reconstructible. So, in view of Lemma 4, we assume that, for every vertex cut $S = \{x_1, x_2\}$ of G, only one component of $G - S$ is nontrivial. Let the component be C_1. Then since G is triangle-free, it follows that $C_1(x_1, \overline{x_2})$ and $C_1(\overline{x_1}, x_2)$ are independent and $C_1(x_1, x_2) = \phi$ (because if $C_1(x_1, x_2)$ were nonempty, then the three vertices a, b, c, x_1 (or x_2), where $a \in C_1(x_1, x_2)$ and $b \in C_1(\overline{x_1}, x_2) \cup C_1(x_1, \overline{x_2})$, would form a triangle in G, a contradiction). Also, since $\operatorname{diam}(G) = 2$, every vertex in $C_1(x_1, \overline{x_2})$ is adjacent to every vertex in $C_1(\overline{x_1}, x_2)$ (Figure 2).

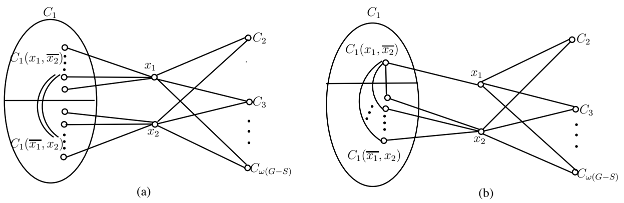

Figure 2.

Let F be a card with an endvertex. Then F must be obtained from G by deleting a vertex v from S or v from C_1 (this happens when $C_1(x_1, \overline{x_2}) = \{v\}$ or $C_1(\overline{x_1}, x_2) = \{v\}$) (Figure 2(b)).

If v is in S, then the neighbours of v in G can only be:

(i) all the endvertices of F, and
(ii) all the vertices that are not adjacent to the unique neighbour of all the endvertices of F.

If v is in C_1, then the neighbours of v in G can only be:

(i) all the endvertices of F, and
(ii) all the vertices (in fact only one such vertex) which are not adjacent to the unique neighbour of all the endvertices of F.

Hence, for either possibility of v, the neighbours of v in F are:

(i) all the endvertices of F, and
(ii) all the vertices that are not adjacent to the unique neighbour of all the endvertices of F.

Therefore, the set of all neighbours of v is uniquely identifiable in the card F and hence G is reconstructible. ∎

Theorem 4. *All triangle-free graphs G with $\kappa(G) = 2$ such that* $\operatorname{diam}(G) = \operatorname{diam}(\overline{G}) = 3$ *are reconstructible.*

Proof. *Recognition:* This follows from Kelly's Lemma and Lemmas 1 and 3.

We first analyze the structure of G before proving that G is weakly reconstructible.

Suppose that, for a vertex cut S of size two in G, if all the components of $G-S$ were trivial, then diam(G) would be at most two, giving a contradiction. Hence, for any vertex cut $S=\{x_1,x_2\}$ of G, at least one component of $G-S$ is nontrivial. Also,

(**P1**) for $1 \le k \le \omega(G-S)$, every vertex in $C_k(x_i,x_j)$ is adjacent in G only to a vertex in $C_k(\overline{x_i},\overline{x_j}) \cup S$, and

(**P2**) $d_{\overline{G}}(x_i,x_j)=3$ or $d_{\overline{G}}(u,x_i)=3$ for some $u \in V(G)-S$ (since $C_i \sim C_j$ in $\overline{G}$).

We consider two cases as below.

Case 1. Suppose that $C_p(\overline{x_i},\overline{x_j})=\phi$ for all $p=1,2,\dots,\omega(G-S)$.

Then, if $C_p(x_i,x_j)\neq\phi$ for some p, then $(P1)$ implies $C_p \cong K_1$. (1)

We now claim that $x_ix_j \in E(G)$. Suppose, to the contrary, that $x_ix_j \notin E(G)$. Then $(P2)$ implies $d_{\overline{G}}(u,x_i)=3$ for some $u\in V(G)-S$, and hence $u \in C_q(x_i,\overline{x_j}) \cup C_q(x_i,x_j)$ for some q. If u were in $C_q(x_i,\overline{x_j})$, then in $\overline{G}$, x_i would be adjacent to x_j, and u would be adjacent to x_j. Therefore $d_{\overline{G}}(u,x_i)$ would be 2, giving a contradiction. Also, if u were in $C_q(x_i,x_j)$, then since diam$(G)=3$, $G-S$ would have a component C_r (say) different from C_q such that $C_r(\overline{x_i},x_j)\neq\phi$. But then, in $\overline{G}$, $x_i \sim C_r(\overline{x_i},x_j)$ and $u \sim C_k$ for all $k\neq q$ (by (1)) and therefore $d_{\overline{G}}(u,x_i)$ would be 2, again a contradiction. Hence the claim, and therefore, for all p, $C_p(x_i,x_j)=\phi$ and each C_p is partitioned into $C_p(x_i,\overline{x_j})$ and $C_p(\overline{x_i},x_j)$ (Figure 3). So every vertex in $C_p(x_i,\overline{x_j})$ can only be adjacent to a vertex in $C_p(\overline{x_i},x_j)$ and to the vertex x_i. Hence the degree of every vertex of G that is in $C_p(x_i,\overline{x_j})$ does not exceed the value $|C_p(\overline{x_i},x_j)|+1$. That is $\deg_G v \le |C_p(\overline{x_i},x_j)|+1 \le \deg_G x_j - 1$ for all $v\in C_p(x_i,\overline{x_j})$. Similarly, $\deg_G v \le |C_p(x_i,\overline{x_j})|+1 \le \deg_G x_i -1$ for all $v \in C_p(\overline{x_i},x_j)$. Thus, in this case,

$$\deg_G v < max\{\deg_G x_i, \deg_G x_j\} \quad \text{for all} \quad v\in V(G)-S. \tag{2}$$

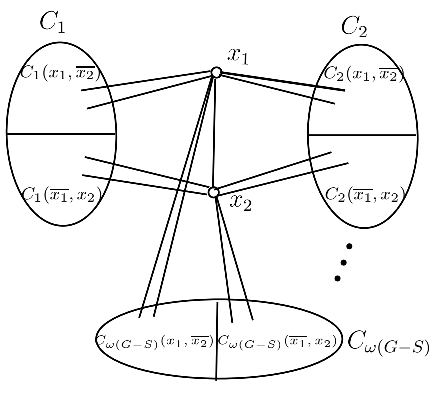

Figure 3. The graph G in Case 1.

We denote the set of all vertices which are at distance i from a vertex v in G by $N_i(v)$ and denote the set $N_i(v) \cup \{v\}$ by $N_i[v]$. Then $N_1(v) = N(v)$. Clearly, in the subgraph $G - x_i$,

$$N_1(v) \cup N_3(v) = \bigcup_{k=1}^{\omega(G-S)} C_k(x_i, \overline{x_j}) \cup \{x_j\} \quad \text{for} \quad v \in \bigcup_{k=1}^{\omega(G-S)} C_k(\overline{x_i}, x_j) \tag{3}$$

$$\text{and} \quad N_2[v] = \bigcup_{k=1}^{\omega(G-S)} C_k(x_i, \overline{x_j}) \cup \{v\} \quad \text{for} \quad v \in S. \tag{4}$$

Case 2. Suppose that $C_p(\overline{x_i}, \overline{x_j}) \neq \phi$ for some p.

Then $C_p(\overline{x_i}, \overline{x_j}) \neq \phi$ for exactly one value of p, since $\operatorname{diam}(G) = 3$. We take, for convenience, that $C_1(\overline{x_i}, \overline{x_j}) \neq \phi$ and $C_p(\overline{x_i}, \overline{x_j}) = \phi$ for all $p \neq 1$. Using (P1), we get the components C_p ($p \neq 1$) for which $C_p(x_i, x_j) \neq \phi$ are isomorphic to K_1. However, we shall prove that $C_p \cong K_1$ for all $p \neq 1$. It suffices to prove that $C_p(x_i, x_j) \neq \phi$ for all $p \neq 1$. Assume, to the contrary, that $C_p(x_i, x_j) = \phi$ for some $p \neq 1$. Then since $C_p(\overline{x_i}, \overline{x_j}) = \phi$, it follows that $C_p(x_i, \overline{x_j}) \cup C_p(\overline{x_i}, x_j) \neq \phi$. Suppose that $C_p(x_i, \overline{x_j}) \neq \phi$; let $v \in C_p(x_i, \overline{x_j})$. Then since $\kappa(G) = 2$, the vertex v must be adjacent to a vertex v' in $C_p(\overline{x_i}, x_j)$ and therefore $C_p(\overline{x_i}, x_j)$ is nonempty. From (P2) and the fact $C_1(\overline{x_i}, \overline{x_j}) \neq \phi$, there exists a component C_l of $G - S$ such that $d_{\overline{G}}(u, x_i) = 3$ for some $u \in C_l$. This implies $u \not\sim x_i$ in $\overline{G}$. Consequently, in G, the vertex u must be in $C_l(x_i, \overline{x_j}) \cup C_l(x_i, x_j)$ where $\{i, j\} = \{1, 2\}$. In G,

- (**P3**) if u were in $C_l(x_i, \overline{x_j}) \cup C_l(x_i, x_j)$ for $l \neq 1$, then since $u \sim C_1(\overline{x_i}, \overline{x_j})$ and $x_i \sim C_1(\overline{x_i}, \overline{x_j})$ in $\overline{G}$, $d_{\overline{G}}(u, x_i)$ would be 2, a contradiction.
- (**P4**) Similarly, if u were in $C_1(x_i, \overline{x_j}) \cup C_1(x_i, x_j)$, then since $x_i \sim C_p(\overline{x_i}, x_j)$ (for all p) and $u \sim C_p(x_i, \overline{x_j}) \cup C_p(\overline{x_i}, x_j)$ (for all $p \neq 1$) in $\overline{G}$, $d_{\overline{G}}(u, x_i)$ would be 2, again a contradiction.

Consequently, $C_p(\overline{x_i}, x_j) \cup C_p(x_i, \overline{x_j}) = \phi$ and $C_p \cong K_1$ for all $p \neq 1$ (Figure 4).

Using (P2) and the fact $C_p(x_i, x_j) \neq \phi$ for all $p \neq 1$, we get $d_{\overline{G}}(u, x_i) = 3$. This, together with (P3), imply that $u \in C_1(x_i, \overline{x_j}) \cup C_1(x_i, x_j)$. Suppose that if u were in $C_1(x_i, \overline{x_j})$, then, in $\overline{G}$, we would have $x_1 \sim x_2$ and $x_j \sim C_1(x_i, \overline{x_j})$. Hence $d_{\overline{G}}(u, x_i)$ would be 2, a contradiction. Therefore $u \in C_1(x_i, x_j)$; let U be the set comprising all such u. Then the subset U of $C_1(x_i, x_j)$ is nonempty and $d_{\overline{G}}(u, x_i) = 3$ for all $u \in U$. Consequently, every vertex u of G that is in U is adjacent only to the vertices of G that are in $C_1(\overline{x_i}, \overline{x_j}) \cup S$ and so such vertex u has equal degree in G. Now we prove, for any vertex w in $V(G) - (U \cup S)$, that

$$\deg_G w < \max\{\deg_G x_i, \deg_G x_j, \deg_G u\}. \tag{5}$$

where $u \in U$. Let w be a vertex in $V(G) - (U \cup S)$. We can assume that w is in $V(C_1)$ (otherwise $\deg_G w = 2$ and so (5) holds immediately). Then $w \in C_1(x_i, \overline{x_j}) \cup C_1(\overline{x_i}, \overline{x_j}) \cup C_1(x_i, x_j)$. If w is in $C_1(x_i, \overline{x_j})$, then $\deg_G w \leq |C_1(\overline{x_i}, \overline{x_j})| +$

$1 = \deg_G u - 1$, where $u \in U$, and therefore (5) holds. If w is in $C_1(\overline{x_i}, \overline{x_j})$, then, in G, the vertex w can only be adjacent to the vertices in $C_1(x_i, \overline{x_j}) \cup C_1(x_i, x_j)$ and $\deg_G w \leq |C_1(x_i, \overline{x_j})| + |C_1(x_i, x_j)| < \deg_G x_i$ and hence (5) holds in this case. Finally, if w is in $C_1(x_i, x_j)$, then, in G, the vertex w can only be adjacent to the vertices in $C_1(\overline{x_i}, \overline{x_j}) \cup S$. Since the vertex w is not in U, it follows that w must be nonadjacent to at least one vertex in $C_1(\overline{x_i}, \overline{x_j})$ to which $u \in U$ is adjacent to all. Therefore $\deg_G w \leq \deg_G u - 1$. Thus, the inequality (5) holds for all vertices w in $V(G) - (U \cup S)$.

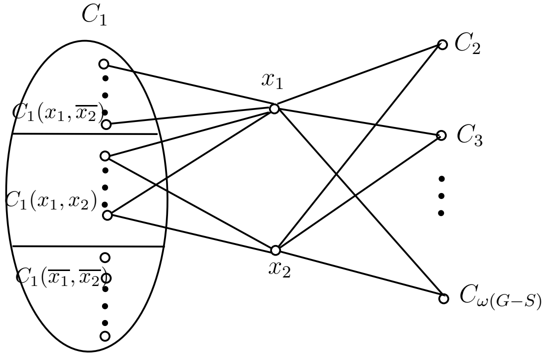

Figure 4. The graph G in Case 2.

Clearly in the subgraph $G - x_i$,

$$N_1(v) \cup N_3(v) = \bigcup_{k=1}^{\omega(G-S)} C_k(x_i, x_j) \cup C_1(x_i, \overline{x_j}) \quad \text{for} \quad v \in C_1(\overline{x_i}, \overline{x_j}) \cup \{x_j\}. \tag{6}$$

Similarly, in the subgraph $G - u$, where $u \in U$,

$$N_1(v) \cup N_3(v) = C_1(\overline{x_i}, \overline{x_j}) \cup S \quad \text{for} \quad v \in \bigcup_{k=1}^{\omega(G-S)} C_k(x_i, x_j) \cup C_1(x_i, \overline{x_j}) \tag{7}$$

$$\text{and} \quad N_2[v] = C_1(\overline{x_i}, \overline{x_j}) \cup S \quad \text{for} \quad v \in S. \tag{8}$$

We now prove that G is weakly reconstructible. Amongst the cards containing a cutvertex, choose a card $G - v_i$ such that the degree of the deleted vertex v_i is maximum. Then either v_i is in S or not in S. If the latter holds, then v_i must be in $C_1(x_1, x_2)$ by (5). We proceed by two recognizable cases as below.

Case A. The card $G - v_i$ has an endvertex, say u; let u' be the neighbour of u.

In this case, we consider three exhaustive cases in G. But in all the three cases, we prove that the set $N_1(u') \cup N_3(u')$ in $G - v_i$ is the neighbourhood of v_i in G and hence G is reconstructible. Suppose that, if $v_i \in S$ and $C_p(\overline{x_i}, \overline{x_j}) = \phi$ for all p, then Case 1 implies the vertex u must be in $C_k(x_i, \overline{x_j})$ and hence the vertex u' must be in $C_k(\overline{x_i}, x_j)$ for some k. Therefore, the set of all neighbours of v_i in G must be the set

$\bigcup_{k=1}^{\omega(G-S)} C_k(x_i, \overline{x_j}) \cup \{x_j\}$, which is clearly equal to the set $N_1(u') \cup N_3(u')$ in $G - v_i$ (by (3)). Also, if $v_i \in S$ and $C_p(\overline{x_i}, \overline{x_j}) \neq \phi$ for some p, then again Case 2 implies the vertex u must be in $C_k(x_i, x_j) \cup C_1(x_i, \overline{x_j})$ for some k. Therefore, the set of all neighbours of v_i in G must be the set $\bigcup_{k=1}^{\omega(G-S)} C_k(x_i, x_j) \cup C_1(x_i, \overline{x_j})$, which is nothing but the set $N_1(u') \cup N_3(u')$ in $G - v_i$ (by (6)). Similarly, if $v_i \in C_1(x_i, x_j)$, then the set of all neighbours of v_i in G is $S \cup C_1(\overline{x_i}, \overline{x_j})$, which is clearly equal to the set $N_1(u') \cup N_3(u')$ in $G - v_i$ (by (7)).

Case B. The card $G - v_i$ has no endvertex.

Here again we consider two exhaustive cases in G. But in both the cases, we prove that, if y is a cutvertex of $G - v_i$, then the set $N_2[y]$ in $G - v_i$ is the neighbourhood of v_i in G and hence G is reconstructible. Suppose that the vertex v_i is in S. Then Case 1 implies $C_i(\overline{x_1}, \overline{x_2}) = \phi$ in G for all i (otherwise Case 2 implies the card $G - v_i$ would contain an endvertex, a contradiction). Hence every cutvertex of G must be in S. Therefore, the set of all neighbours of v_i in G must be the set $\bigcup_{k=1}^{\omega(G-S)} C_k(x_i, \overline{x_j}) \cup \{x_j\}$, which is clearly equal to the set $N_2[y]$ in $G - v_i$ (by (4)). Similarly, if $v_i \in C_1(x_i, x_j)$, then again every cutvertex of G must be in S. Therefore, the set of all neighbours of v_i in G must be the set $C_1(\overline{x_i}, \overline{x_j}) \cup S$, which is nothing but the set $N_2[y]$ in $G - v_i$ (by (8)). This completes the proof of Theorem 4. ∎

3 Conclusion

We hope that similar techniques can be used to solve Problem C.

Problem C. Show that all triangle-free graphs G with $\kappa(G) \geq 3$ such that $\mathrm{diam}(G) = 2$ or $\mathrm{diam}(G) = \mathrm{diam}(\overline{G}) = 3$ are weakly reconstructible.

Since graphs containing a triangle are recognizable from the deck (using Kelly's Lemma), Lemmas 1 and 3 give that the two families of triangle-free graphs G with $\kappa(G) \geq 3$ in the hypothesis of Problem C are recognizable. Thus, to settle RC, it is enough to prove that all 2-connected graphs containing a triangle are reconstructible and Problem C. As many classes of 2-connected graphs have already shown to be reconstructible [3, 8], they may further narrow down the classes of graphs to be reconstructed to prove RC. These narrowed-down classes must contain counterexamples to RC if there exists one at all.

Acknowledgments

We are thankful to an anonymous referee for his/her many valuable comments which largely improved the style of the paper and the proof of Theorem 4.

References

[1] A. S. Asratian, T. M. J. Denley and R. Häggkvist, *Bipartite graphs and their applications*, Cambridge University Press, New York 1998.

[2] J. A. Bondy and R. L. Hemminger, Graph reconstruction—a survey, *J. Graph Theory* 1 (1977), 227–268.

[3] J. A. Bondy, A graph reconstructor's manual, in *Surveys in Combinatorics* (Proc. 13th British Combin. Conf.) London Math. Soc. Lecture Note Ser. 166 (1991), 221–252.

[4] S. K. Gupta, Pankaj Mangal and Vineet Paliwal, Some work towards the proof of the reconstruction conjecture, *Discrete Math.* 272 (2003), 291–296.

[5] F. Harary, *Graph Theory*, Addison-Wesley, Mass. 1969.

[6] P. J. Kelly, A congruence theorem for trees, *Pacific J. Math.* 7 (1957), 961–968.

[7] B. D. McKay, Small graphs are reconstructible, *Australas. J. Combin.* 15 (1997), 123–126.

[8] B. Manvel, Reconstruction of graphs—Progress and prospects, *Congr. Numer.* 63 (1988), 177–187.

[9] C. St. J. A. Nash Williams, The reconstruction problem, *Selected Topics in Graph Theory*, Academic Press, London (1978), 205–236.

[10] S. Ramachandran and S. Monikandan, Graph reconstruction conjecture: Reductions using complement, connectivity and distance, *Bull. Inst. Combin. Appl.* 56 (2009), 103–108.

[11] W. T. Tutte, All the King's horses, *Graph Theory and related topics* (Proc. Conf. in honor of W. T. Tutte's 60th birthday), eds. J. A. Bondy and U. S. R. Murty, Academic Press, New York (1976), 15–33.

[12] Yang Yongzhi, The reconstruction conjecture is true if all 2-connected graphs are reconstructible, *J. Graph Theory* 12 no. 2 (1988), 237–243.

(Received 1 Aug 2011; revised 9 Mar 2012)