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Sums of squares and negative correlation for spanning forests of series parallel graphs

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Abstract

We provide new evidence that spanning forests of graphs satisfy the same negative correlation properties as spanning trees, derived from Lord Rayleigh's monotonicity property for electrical networks. The main result of this paper is that the Rayleigh difference for the spanning forest generating polynomial of a series parallel graph can be expressed as a certain positive sum of monomials times squares of polynomials. We also show that every regular matroid is independent-set-Rayleigh if and only if every basis-Rayleigh binary matroid is also independent-set-Rayleigh.

1 Introduction

The well-known theorem of Fortuin, Ginibre, and Kasteleyn in [4] gives a locally verifiable sufficient condition for identifying positively associated measures. Unlike its well explored counterpart, no such condition is known for negatively associated measures, as defined by Kahn in [7], nor is there one for the following special case. Consider a measure on a collection of sets so that the probability of an event $\mathcal{A}$ occurring is $P(\mathcal{A})$. We say the measure is *negatively correlated* if

$$P(\{X : e, f \in X\}) \;\leq\; P(\{X : e \in X\})P(\{X : f \in X\}) \tag{1.1}$$

for every pair of distinct elements e and f.

We are concerned with a certain family of measures that are positive for bases, independent sets and spanning sets of matroids, represented here by the letters B, I and S, respectively. Let $\mathcal{M}$ be a matroid and let $(y_g : g \in E)$ be a weighting of its ground set so that the polynomial

$$Z = \sum_{X \in \mathsf{Z}(\mathcal{M})} \mathbf{y}^X,$$

is a sum over Z-sets where $\mathsf{Z} \in \{\mathsf{B}, \mathsf{I}, \mathsf{S}\}$ and $\mathbf{y}^X = \left(\prod_{x \in X} y_x\right)$.

Let Z_e indicate the partial derivative $\dfrac{\partial Z}{\partial y_e}$. For a given positive evaluation of all the y_g, suppose the term $\mathbf{y}^X$ is selected with probability $P(\{X\}) := \frac{\mathbf{y}^X}{Z}$. The monomials $y_e Z_e$ are precisely those of Z which contain y_e, so that $\frac{y_e Z_e}{Z} = P(\{X : e \in X\})$. The difference

$$\Delta Z\{e, f\} := Z_e Z_f - Z Z_{ef} \tag{1.2}$$

is called the Z-*Rayleigh difference* and it is non-negative for every positive evaluation of the y_gs if and only if (1.1) holds for the corresponding measures. If $\Delta Z\{e, f\} \geq 0$ for every pair of distinct edges e and f and every positive evaluation of the y_gs, then $\mathcal{M}$ is Z-Rayleigh.

Graphs are B-Rayleigh as a result of Kirchhoff's laws for electrical resistor networks ([8]) and the intuitive property, due to Lord Rayleigh, that increasing the conductance of any resistor in the circuit does not decrease the conductance between any two nodes. The spanning forest analogue was circulated by Kahn in the early 1990s ([7]).

Conjecture 1.1 ([10], [5], [11], [14], [6]) *Graphs are I-Rayleigh.*

Semple and Welsh show in [11] that the conjecture holds for series parallel extensions of K_4. Cocks and Wagner prove independently ([2],[14]) that two-sums of I-Rayleigh graphs are I-Rayleigh. Grimmett and Winkler show in [5] that graphs on at most eight vertices and nine vertices with at most 18 edges have a non-negative I-Rayleigh difference when the variables are evaluated at 1. Cocks and Erickson ([2], [3]) prove independently that if all graphs satisfy this last condition, then they are all I-Rayleigh as well.

Wagner conjectures privately that the I-Rayleigh difference for graphs can be expressed as a certain positive sum of monomials times squares of polynomials. Our main result is that the special form holds for series-parallel graphs. We introduce Wagner's conjecture after defining notation.

Let $\mathcal{G} := (V, E)$ be a graph whose spanning forests are generated by

$$G := \sum_{X \in \mathsf{I}(\mathcal{G})} \mathbf{y}^X,$$

which is the sum over all acyclic subsets X of E, where $\mathbf{y} := (y_g : g \in E)$ are indeterminates. Similarly, H and K generate the spanning forests of graphs $\mathcal{H}$ and $\mathcal{K}$. Braces and commas are dropped from small sets of elements, as in efg instead of $\{e, f, g\}$.

Definition 1.2 (S-sets, A-sets) *Let $\mathscr{S}$ be the collection of those sets $S \subseteq E - ef$ such that $S \cup ef$ is contained in a simple cycle of $\mathcal{G}$. For each S in $\mathscr{S}$, let $\mathscr{A}(S)$ be the collection of those spanning forests A such that $A \subseteq E - ef$ and $S \cup ef \subseteq C \subseteq A \cup ef$ for a simple cycle, C, and $A \cup ef$ contains no other cycle.*

Use a subscripted G wherever the graph $\mathcal{G}$ needs to be specified, as in $\mathscr{A}_G(S)$. We refer to the elements of $\mathscr{S}$ and $\mathscr{A}(S)$ as S-sets and A-sets, respectively. Throughout the rest of this paper, given an S-set S and one of its A-sets A, the cycle C is the cycle described in the above definition unless otherwise noted. For each S and C, the sign $c(S,e,f,C) \in \{-1,+1\}$, and it is written $c(S,C)$ when e and f are understood.

Conjecture 1.3 (Wagner, private communication, Sum of Squares) *Let $\mathcal{G}$ be a graph with distinct edges e and f. Then for some choice of signs $c(S,C)$,*

$$\Delta G\{e,f\} = \sum_{S\in\mathscr{S}} \mathbf{y}^S \left(\sum_{A\in\mathscr{A}(S)} c(S,C)\mathbf{y}^{A-S} \right)^2 . \tag{1.3}$$

When $\mathcal{G}$ and e and f satisfy the above we say $\Delta G\{e,f\}$ *is SOS*. If $\mathcal{G}$ satisfies the above for every pair of distinct edges e and f we say $\mathcal{G}$ is SOS.

It is immediate that if the I-Rayleigh difference can be expressed in the form (1.3), then it is non-negative for any positive evaluation. Thus, if Conjecture 1.3 holds for each pair of distinct edges, the graph is I-Rayleigh. The conjecture has been verified for the complete graph K_7, the cube and the Möbius ladder on eight vertices (Wagner, private communication). Other similarly sized graphs for which correct signs have not yet been found, exhibit discrepancies on the order of tens out of tens of thousands of terms.

The conjecture provides graphical insight into negative correlation of spanning forests, and may lead to a proof for all graphs. In the mean time, we have the following result:

Theorem 1.4 *If $\mathcal{G}$ is a series-parallel graph, then $\mathcal{G}$ satisfies (1.3) for some choice of signs $c(S,e,f,C)$.*

In Section 2 we prove that if (1.3) holds for graphs $\mathcal{G}$ and $\mathcal{H}$, then it holds for minors and direct sums of these and in Section 3.1 we present evidence that it also holds by taking two-sums.

Regular matroids are closely related to graphic matroids through decomposition. In Section 4 we prove the following relationship between regular and binary matroids.

Theorem 1.5 *The following are equivalent.*

(i) Regular matroids are I-Rayleigh.

(ii) Every B-Rayleigh binary matroid is also I-Rayleigh.

2 Minors and direct sums of SOS graphs

Recall that G_g is the partial derivative $\dfrac{\partial G}{\partial y_g}$. When g is not a loop, G_g describes the spanning forests of $\mathcal{G}$ contract g. The analogue for deletion, denoted G^g, is the evaluation at $y_g = 0$. If g is a loop, then it is not contained in any spanning forest, so it does not affect our calculations.

Lemma 2.1 *Let $\mathcal{G}$ be a graph with distinct edges e, f and g. If $\Delta G\{e, f\}$ is SOS then so are $\Delta G^g\{e, f\}$ and $\Delta G_g\{e, f\}$.*

Proof. From Section 4.4 of [14], $\Delta G^g\{e, f\} = \lim_{y_g \to 0} \Delta G\{e, f\}$. To show that this satisfies the sum-of-squares form for $\Delta G^g\{e, f\}$ use

$$\lim_{y_g \longrightarrow 0} \Delta G\{e, f\} = \sum_{\substack{S \in \mathscr{S}_G \\ g \notin S}} \mathbf{y}^S \left(\sum_{\substack{A \in \mathscr{A}_G(S) \\ g \notin A}} c_G(S, C) \mathbf{y}^{A-S} \right)^2 . \tag{2.1}$$

A cycle containing a set X is called an *X-cycle.* An S-set of $\mathcal{G} \setminus g$ is contained in an ef-cycle of $\mathcal{G} \setminus g$. Clearly $\mathscr{S}_{G \setminus g} \subseteq \{S : S \in \mathscr{S}_G, g \notin S\}$, the set indexing the outer sum of (2.1). On the other hand, given a set $\tilde{S}$ in $\{S : S \in \mathscr{S}_G, g \notin S\} - \mathscr{S}_{G \setminus g}$, there are no ef-cycles containing $\tilde{S}$ and not g. Thus, there are no A-sets for $\tilde{S}$ which do not contain g and the inner sum of (2.1) for these is empty. Therefore, together, the sets indexing the sums in (2.1) are the S-sets and A-sets of $\mathcal{G} \setminus g$.

The proof for $\Delta G_g\{e, f\}$ is slightly trickier due to the fact that when g is contracted, two cycles may be created from one. Using $\lim_{y_g \to \infty} y_g^{-2} \Delta G\{e, f\} = \Delta G_g\{e, f\}$ ([14], Section 4.4), terms of $\Delta G\{e, f\}$ without y_g^2 disappear, so we are left with

$$\lim_{y_g \longrightarrow \infty} y_g^{-2} \Delta G\{e, f\} = \sum_{\substack{S \in \mathscr{S}_G \\ g \notin S}} \mathbf{y}^S \left(\sum_{\substack{A \in \mathscr{A}_G(S) \\ g \in A}} c_G(S, C) \mathbf{y}^{A-(S \cup g)} \right)^2 . \tag{2.2}$$

Observe that g is not a chord of C because there is no other cycle in $A \cup ef$. Thus, if every cycle containing S has g as a chord, there are no A-sets in $\mathscr{A}(S)$ containing g. Therefore we are summing over S-sets not containing g for which there is a cycle C containing S and g is not a chord of C. This is equal to $\mathscr{S}_{G/g}$.

It remains to be shown that for an S-set S of $\mathscr{S}_{G/g}$, the inner sum of (2.2) is indexed by the desired A-sets. Let A' be an element of $\mathscr{A}_{G/g}(S)$ and let $A := A' \cup g$. By definition $A' \cup e$ and $A' \cup f$ are forests of $\mathcal{G}/g$ and therefore $A \cup e$ and $A \cup f$ are forests of $\mathcal{G}$. Furthermore there is a unique cycle C such that $S \cup ef \subseteq C \subseteq A \cup ef$ containing S, so $A \in \mathscr{A}_G(S)$ and $g \in A$.

Conversely, suppose $A \in \mathscr{A}_G(S)$ and $g \in A$. Clearly $A - g \in \mathscr{A}_{G/g}(S)$, since g cannot be a chord of C. Therefore the S-sets and A-sets of $\mathcal{G}/g$ are exactly those sets indexed by 2.2. $\square$

Let the *direct sum of graphs* $\mathcal{H}$ and $\mathcal{K}$ be a graph $\mathcal{G}$ whose spanning forests are generated by HK. There may be several graphs which satisfy this, however their graphic matroid is the (matroid) direct sum of the graphic matroids of $\mathcal{H}$ and $\mathcal{K}$.

One such graph $\mathcal{G}$ may be obtained by identifying a vertex of $\mathcal{H}$ with a vertex of $\mathcal{K}$. A spanning forest of $\mathcal{G}$ is the disjoint union of a spanning forest of $\mathcal{H}$ and another of $\mathcal{K}$.

Proposition 2.2 *If $\mathcal{H}$ and $\mathcal{K}$ are SOS graphs and $\mathcal{G}$ is their direct sum, then $\mathcal{G}$ is SOS.*

Proof. If $e \in E(\mathcal{H})$ and $f \in E(\mathcal{K})$, then there are no cycles through e and f and hence no S-sets. In this case $\Delta G\{e,f\} = 0$.

Let e and f be distinct edges in $E(\mathcal{H})$. Since $G = HK$, it is easy to show that $\Delta G\{e,f\} = K^2 \Delta H\{e,f\}$, so that

$$\Delta G\{e,f\} = \sum_{S \in \mathscr{S}_H} \mathbf{y}^S \left(\sum_{A \in \mathscr{A}_H(S)} c_H(S,C) \mathbf{y}^{A-S} K \right)^2. \tag{2.3}$$

The S-sets of $\mathcal{G}$ are equal to those of $\mathcal{H}$, since an ef-cycle of $\mathcal{G}$ cannot contain an edge of $\mathcal{K}$. Let $\mathscr{X}$ and $\mathscr{Y}$ be collections of sets and define $\mathscr{X} \vee \mathscr{Y} := \{X \cup Y : X \in \mathscr{X}, Y \in \mathscr{Y}\}$. Now, if S is an S-set of $\mathcal{G}$, then $\mathscr{A}_G(S) = \mathscr{A}_H(S) \vee \mathbf{I}(\mathcal{K})$, as required.

Since $\mathcal{H}$ and $\mathcal{K}$ are both SOS, the case where $\{e,f\} \subseteq E(\mathcal{K})$ is dealt with by symmetry. □

3 Series-Parallel Graphs

Let $\mathcal{H}$ and $\mathcal{K}$ be graphs. The two-sum, defined in [12], of $\mathcal{H}$ and $\mathcal{K}$ along a common edge g is denoted $\mathcal{H} \oplus_g \mathcal{K}$. In general there are up to two, non-isomorphic ways of two-summing along g. In spite of this fact the spanning forests of two-sums are unique, so we do not make this distinction.

Denote the complete graph on three vertices by K_3 and let a superscript * indicate matroid dual. The graph $(K_3)^*$ consists of three mutually parallel edges. Define a *parallel extension* of $\mathcal{G}$ to be $\mathcal{G} \oplus_g (K_3)^*$ for some edge g. Similarly $\mathcal{G} \oplus_g K_3$ is called a *series extension*. A graph $\mathcal{H}$ is a *series-parallel extension* of $\mathcal{G}$ if it can be obtained by a sequence of series and parallel extensions, starting with $\mathcal{G}$. A graph is called *series-parallel* if it is a minor of a series-parallel extension of K_3 or $(K_3)^*$.

We set out to prove Theorem 1.4, that every series-parallel graph is SOS. By Lemma 2.1 we need not consider proper minors of series-parallel extensions of K_3 or $(K_3)^*$. Let $\mathcal{G} := \mathcal{H} \oplus_g \mathcal{K}$ for SOS graphs $\mathcal{H}$ and $\mathcal{K}$ and let e and f be distinct edges of $\mathcal{G}$. We prove that if $\mathcal{H}$ and $\mathcal{K}$ are SOS and if $(H^g - H_g)H_g$ and $(K^g - K_g)K_g$ each satisfy a similar sum-of-squares identity, then $\mathcal{G}$ is SOS and $(G^h - G_h)G_h$ satisfies the same identity for every edge $h \in E(\mathcal{G})$. The above mentioned identity is proved for all series-parallel graphs in Lemma 3.5.

There are three cases with respect to the locations of e and f. Either $e \in E(\mathcal{H}) - g$ and $f \in E(\mathcal{K}) - g$ or they are both in $\mathcal{H}$ or in $\mathcal{K}$. The first holds for two-sums without any assumptions on $(H^g - H_g)H_g$ or $(K^g - K_g)K_g$.

We use the following result of Wagner in Lemmas 3.2 and 3.6.

Lemma 3.1 (Wagner, [14].) *Let $\mathcal{G} = \mathcal{H} \oplus_g \mathcal{K}$.*

(a) If $e \in E(\mathcal{H}) - g$ and $f \in E(\mathcal{K}) - g$, then

$$\Delta G\{e,f\} = \Delta H\{e,g\}\Delta K\{g,f\}.$$

(b) If e and f are distinct edges in $E(\mathcal{H}) - g$, then

$$\Delta G\left\{e,f\right\} = (K_g)^2 \Delta H\left\{e,f\right\},$$

where $y_g = K^g/K_g - 1$.

Proof. See the proof of Theorem 5.8 in [14]. □

Lemma 3.2 *Let $\mathcal{G} = \mathcal{H} \oplus_g \mathcal{K}$. If $e \in E(\mathcal{H}) - g$ and $f \in E(\mathcal{K}) - g$, then $\Delta G\left\{e,f\right\}$ is SOS.*

Proof. By Lemma 3.1(a), $\Delta G\{e,f\} = \Delta H\{e,g\}\Delta K\{g,f\}$. Since $\mathcal{H}$ and $\mathcal{K}$ are SOS,

$$\begin{aligned}&\Delta H\{e,g\}\Delta K\{g,f\}\\ &= \sum_{\substack{S_H \in \mathscr{S}_H \\ S_K \in \mathscr{S}_K}} \mathbf{y}^{S_H \cup S_K} \left(\sum_{\substack{A_H \in \mathscr{A}_H(S_H) \\ A_K \in \mathscr{A}_K(S_K)}} c_G(S_G, C_G)\mathbf{y}^{(A_H \cup A_K)-(S_H \cup S_K)} \right)^2 \end{aligned} \tag{3.1}$$

where we set $c_G(S_G, C_G) := c_H(S_H, C_H)c_K(S_K, C_K)$.

Notice that a cycle is an ef-cycle of $\mathcal{G}$ if and only if it is the symmetric difference of an eg-cycle in $\mathcal{H}$ and a gf-cycle in $\mathcal{K}$. It is straightforward to show that 3.1 is the sum of squares we are expecting by showing that the outer and inner sums index the S-sets and A-sets of $\mathcal{G}$, respectively. □

The proof of the case where $\{e,f\} \subseteq E(\mathcal{H}) - g$ reduces to proving a sum-of-squares form for $(K^g - K_g)K_g$. For any graph $\mathcal{G}$ and an edge e let

$$\Phi G\{e\} := (G^e - G_e)G_e.$$

The proof of (3.2) in the following lemma is straight forward and similar to the one in Section 4.4 of [14], and the proof of (3.4) can be adapted from that of Lemma 3.1(b).

Lemma 3.3 *Let $\mathcal{G}$ be a graph with distinct edges e and f. Then*

$$\Phi G\{e\} = \Phi G^f\{e\} + y_f \Psi G\{e|f\} + y_f^2 \Phi G_f\{e\}, \tag{3.2}$$

where

$$\Psi G\{e|f\} = G^e_f G^f_e + G^{ef} G_{ef} - 2G^f_e G_{ef}. \tag{3.3}$$

If $\mathcal{G} = \mathcal{H} \oplus_g \mathcal{K}$ and $e \in E(\mathcal{H})$, then by setting $y_g := K^g/K_g - 1$,

$$\Phi G\{e\} = (K_g)^2 \Phi H\{e\}. \tag{3.4}$$

To express the sum-of-squares form for $\Phi G\{e\}$ we need some notation similar to that defined for Conjecture 1.3. Note, however, that Q-sets are required to be non-empty, unlike S-sets. The significance of this becomes clear later.

Definition 3.4 (Q-sets, B-sets) *Let $\mathcal{G}$ be a graph. Let $\mathscr{Q}$ be the collection of those sets Q such that $\varnothing \subset Q \subseteq E-e$ and $Q \cup e$ is contained in a simple cycle of $\mathcal{G}$. For each Q in $\mathscr{Q}$ let $\mathscr{B}(Q)$ be the collection of those spanning forests B such that $B \subseteq E-e$ and $Q \cup e \subseteq D \subseteq B \cup e$ for a simple cycle, D.*

We refer to elements of $\mathscr{Q}$ and $\mathscr{B}(Q)$ as Q-sets and B-sets, respectively. Given a Q-set Q and one of its B-sets B, the cycle D is the cycle described above. For each Q and D, the sign $d(Q, e, D) \in \{-1, +1\}$, and it is written $d(Q, D)$ when e is understood.

To avoid ambiguity, the qualification SOS, becomes Δ-SOS. If a graph $\mathcal{G}$ and an edge e satisfy the conclusion of the following lemma we say $\Phi G\{e\}$ is Φ-SOS. If $\Phi G\{e\}$ is Φ-SOS for every edge e, then $\mathcal{G}$ is Φ-SOS.

Lemma 3.5 *Let $\mathcal{G}$ be a series-parallel graph with an edge e. With the above notation*

$$\Phi G\{e\} = \sum_{Q \in \mathscr{Q}} \mathbf{y}^Q \left(\sum_{B \in \mathscr{B}(Q)} d(Q, D) \mathbf{y}^{B-Q} \right)^2 \tag{3.5}$$

for some choice of signs $d(Q, D)$.

Proof. Recall that series-parallel graphs are minors of series parallel extensions of K_3 and $(K_3)^*$.

Let $E(K_3) := \{e, f, g\}$ so that $\Phi K_3\{e\} = (1 + y_f + y_g + y_f y_g)(1 + y_f + y_g) - (1 + y_f + y_g)^2 = y_f (y_g)^2 + y_g (y_f)^2 + y_f y_g$ and $\Phi (K_3)^* \{e\} = (1 + y_f + y_g)(1) - (1)^2 = y_f + y_g$. Thus both K_3 and $(K_3)^*$ are Φ-SOS. Furthermore, small modifications of Lemma 2.1 and Proposition 2.2 serve to prove that the Φ-sum-of-squares form holds by taking minors and direct sums. Therefore, by induction, it is enough to show that two-sums of Φ-SOS graphs are Φ-SOS.

Let $\mathcal{H}$ and $\mathcal{K}$ be Φ-SOS graphs such that $e \in E(\mathcal{H}) - g$. By Lemma 3.3 we have $\Phi G\{e\} = (K_g)^2 \Phi H \{e\}$ and by the inductive hypothesis,

$$(K_g)^2 \Phi H \{e\} = (K_g)^2 \sum_{Q \in \mathscr{Q}_H} \mathbf{y}^Q \left(\sum_{B \in \mathscr{B}_H(Q)} d_H(Q, D) \mathbf{y}^{B-Q} \right)^2, \tag{3.6}$$

in which $y_g = K^g/K_g - 1$.

A Q-set of (3.5) is contained in $\mathcal{H}$ or it is not. Table 1, which is divided according to this, shows the bijections between index sets of (3.5) and (3.6), highlighting the way they factor over the two-sum. The case where $g \notin Q_H$ uses the fact that $d_H(Q, D)$ does not depend on $B - D$, so we are able to group some B-sets of $\mathcal{H}$ (see Table 1). The case where $g \in Q_H$ gives

$$\Phi K\{g\} \sum_{Q : g \in Q} \mathbf{y}^{Q-g} \left(\sum_{B \in \mathscr{B}_H(Q)} d_H(Q, D)\mathbf{y}^{B-Q} \right)^2$$

and it corresponds to Q-sets of $\mathcal{G}$ with edges in both factors. For this reason Q-sets cannot be empty. See Figure 1 and Table 1. □

notes	(3.6), $(K_g)^2\Phi H\{e\}$		(3.5), $\Phi G\{e\}$
	$\mathcal{H}$-part	$\mathcal{K}$-part	$\mathcal{G}$-part
Q-sets	$Q_H : g \notin Q_H$	none	$Q_G : Q_G \cap E(\mathcal{K}) = \varnothing$
			below, $B_H \in \mathscr{B}_H(Q_G)$
B-sets	$B_H - g : g \in D$	$K^g - K_g$	$\{B_H - g : g \in D\}$ $\vee (\mathbf{I}(\mathcal{K} \setminus g) - \mathbf{I}(\mathcal{K}/g))$
group terms	$B_H - g : g \in B_H - D$	$K^g - K_g$	$\{B_H : g \notin B_H,$ $g \notin \mathrm{cl}(B_H)\} \vee \mathbf{I}(\mathcal{K} \setminus g)$
	$B_H : g \notin B_H,\ g \notin \mathrm{cl}(B_H)$	K_g	
	$B_H : g \notin B,\ g \in \mathrm{cl}(B_H)$	K_g	$\{B_H : g \notin B_H,\ g \in \mathrm{cl}(B_H)\} \vee \mathbf{I}(\mathcal{K}/g)$
Q-sets	$Q_H - g : g \in Q_H$	$\mathscr{Q}_K$	$Q_G : Q_G \cap E(\mathcal{K}) \neq \varnothing$
B-sets	$B_H - g : B_H \in \mathscr{B}_H(Q_H)$	$B_K(Q_K)$	$B_G \in \mathscr{B}_G(Q_G)$

Table 1: Explicit bijections on the index sets of (3.5) and (3.6). The $\mathcal{K}$-part column accounts for y_g and the $(K_g)^2$ factor. Write $g \in \mathrm{cl}(X)$ if and only if g completes a cycle in the set of edges X and recall the notation $\mathscr{X} \vee \mathscr{Y} := \{X \cup Y : X \in \mathscr{X}, Y \in \mathscr{Y}\}$. Finally, sets have been labelled naturally so that $Q_H \in \mathscr{Q}_H$, etc.

We use Lemma 3.5 to prove in a similar way, that the Δ-sum-of-squares conjecture holds over two-sums when $\{e, f\} \subseteq E(\mathcal{H})$ and $\mathcal{K}$ is Φ-SOS.

Lemma 3.6 *Let $\mathcal{G} := \mathcal{H} \oplus_g \mathcal{K}$ and let e and f be distinct edges in $E(\mathcal{H}) - g$. If $\mathcal{K}$ is Φ-SOS and $\mathcal{H}$ is Δ-SOS, then $\Delta G\{e, f\}$ satisfies the Δ-sum-of-squares form for some choice of signs $c_G(S, C)$.*

Proof. By Lemma 3.1(b), $\Delta G\{e, f\} = (K_g)^2 \Delta H\{e, f\}$, where $y_g = K^g/K_g - 1$. By assumption we have

$$(K_g)^2 \Delta H\{e, f\} = (K_g)^2 \sum_{S \in \mathscr{S}_H} \mathbf{y}^S \left(\sum_{A \in \mathscr{A}_H(S)} c_H(S, C)\mathbf{y}^{A-S} \right)^2 . \tag{3.7}$$

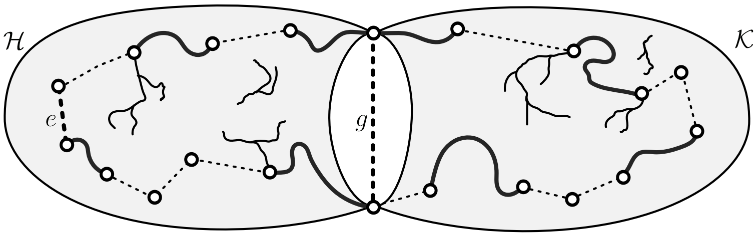

Figure 1: An element of $\mathscr{B}_G(Q_G)$ may look like this. Thin dashed lines represent edges of $Q_G = (Q_H - g) \cup Q_K$, thick solid lines complete a cycle D_G containing $Q_G \cup e$.

An S-set of (1.3) is contained in $\mathcal{H}$ or it is not. Table 2, which is divided according to this, shows the bijections between index sets of (1.3) and (3.7), highlighting the way they factor over the two-sum. The case where $g \notin S_H$ uses the fact that $c_H(S, C)$ does not depend on $A - C$, so we are able to group some A-sets of $\mathcal{H}$ (see Table 2).

The terms indexed by S-sets containing g are

$$\Phi K\{g\} \sum_{S:g\in S} \mathbf{y}^{S-g} \left(\sum_{A\in\mathscr{A}_H(S)} c_H(S,C)\mathbf{y}^{A-S} \right)^2 \tag{3.8}$$

and it corresponds to S-sets having edges in both $\mathcal{H}$ and $\mathcal{K}$. The parts in $\mathcal{K}$ are the Q-sets of $\Phi K\,\{g\}$. See Table 2 and again Figure 1 noting this time that f is on the cycle in $\mathcal{H}$. □

notes	(3.7), $(K_g)^2 \Delta H\,\{e,f\}$		(1.3), $\Delta G\,\{e,f\}$
	$\mathcal{H}$-part	$\mathcal{K}$-part	$\mathcal{G}$-part
S-sets	$S_H : g \notin S_H$	none	$S_G : S_G \cap E(\mathcal{K}) = \varnothing$
			below, $A_H \in \mathscr{A}_H(S_G)$
A-sets	$A_H - g : g \in C$	$K^g - K_g$	$\{A_H - g : g \in C\}$ $\vee (\mathbf{I}(\mathcal{K} \setminus g) - \mathbf{I}(\mathcal{K}/g))$
group terms	$A_H - g : g \in A_H - C$	$K^g - K_g$	$\{A_H : g \notin A_H,$ $g \notin \mathrm{cl}(A_H)\} \vee \mathbf{I}(\mathcal{K} \setminus g)$
	$A_H : g \notin A_H,\ g \notin \mathrm{cl}(A_H)$	K_g	
	$A_H : g \notin A_H,\ g \in \mathrm{cl}(A_H)$	K_g	$\{A_H : g \notin A_H,\ g \in \mathrm{cl}(A_H)\} \vee \mathbf{I}(\mathcal{K}/g)$
S-sets	$S_H - g : g \in S_H$	$\mathscr{Q}_K$	$S_G : S_G \cap E(\mathcal{K}) \neq \varnothing$
A-sets	$A_H - g : A_H \in \mathscr{A}_H(S_H)$	$\mathscr{B}_K(Q_K)$	$A_G \in \mathscr{A}_G(S_G)$

Table 2: Explicit bijections on the index sets of (1.3) and (3.7). See notes at Table 1.

We are finally in a position to prove Theorem 1.4 which states that series-parallel graphs are Δ-SOS.

Proof. (of Theorem 1.4) Let $\mathcal{G}$ be a series-parallel graph. Either $\mathcal{G}$ is obtained by a sequence of series-parallel extensions starting with K_3 or $(K_3)^*$, or $\mathcal{G}$ is a proper minor of one of these. By Lemma 2.1 we need only prove the theorem for the first case. It is straightforward to show that the base cases, K_3 and $(K_3)^*$ are Δ-SOS. Let $\mathcal{G} := \mathcal{H} \oplus_g \mathcal{K}$ where $\mathcal{K}$ is K_3 or $(K_3)^*$. We assume that $\mathcal{H}$ is Δ-SOS. If $e \in E(\mathcal{H})$ and $f \in E(\mathcal{K})$ then by Lemma 3.2, $\Delta G\{e, f\}$ is Δ-SOS. If $\{e, f\} \subseteq E(\mathcal{H})$ or $\{e, f\} \subseteq E(\mathcal{K})$ then Lemma 3.6 is applicable, since $\mathcal{H}$ and $\mathcal{K}$ are Φ-SOS by Lemma 3.5. Thus, $\mathcal{G}$ is Δ-SOS. $\square$

3.1 Two-sums of Δ-SOS graphs

One might have hoped to prove, more generally, that if $\mathcal{H}$ and $\mathcal{K}$ are Δ-SOS, then $\mathcal{H} \oplus_g \mathcal{K}$ is as well. The problem lies in the fact that we are not assuming $\Phi H\{g\}$ and $\Phi K\{g\}$ are Φ-SOS. To get around this we might try to bootstrap this assumption by showing that it follows from the induction hypothesis. In fact, this looks promising and it is given as the following conjecture.

Conjecture 3.7 *If $\mathcal{G}$ is Δ-SOS, then $\mathcal{G}$ is Φ-SOS.*

Let $\mathcal{G}$ be a graph with distinct edges e and f. It is easy to show that $\Delta G\{e, f\} = G_e^f G_f^e - G_{ef} G^{ef}$ by using the fact that $G = G^g + y_g G_g$ for any edge g. Thus, recalling Lemma 3.3,

$$\begin{aligned}\Psi G\{e|f\} =& G_f^e G_e^f + G^{ef} G_{ef} - 2G_e^f G_{ef} + (G^{ef} G_{ef} - G^{ef} G_{ef}) \\ =& \Delta G\{e, f\} + 2(G^{ef} G_{ef} - G_e^f G_{ef}). \end{aligned} \tag{3.9}$$

We want to show that $\mathcal{G}$ is Φ-SOS, whenever $\Phi G^f\{e\}$ and $\Phi G_f\{e\}$ are Φ-SOS and $\Delta G\{e, f\}$ is Δ-SOS by showing that the expansion (3.2) can be reduced to the desired Φ-sum-of-squares form of $\Phi\mathcal{G}\{e\}$, for some choice of signs $d_G(Q, D)$.

Dividing the Φ-sum-of-squares sum for $\mathcal{G}$ into the two usual cases where Q-sets do and do not contain f yields

$$\sum_{\substack{Q \in \mathscr{Q}_G \\ f \notin Q}} \mathbf{y}^Q \left(y_f \left(\sum_{B: f \in D} \wp(B) + \sum_{B: f \in B - D} \wp(B) \right) + \sum_{\substack{B: f \notin B \\ f \notin \mathrm{cl}(B)}} \wp(B) + \sum_{\substack{B: f \notin B \\ f \in \mathrm{cl}(B)}} \wp(B) \right)^2 \tag{3.10}$$

$$+ y_f \left(\sum_{\substack{Q \in \mathscr{Q} \\ f \in Q}} \mathbf{y}^{Q-f} \left(\sum_{B \in \mathscr{B}(Q)} d(Q, D) \mathbf{y}^{B-Q} \right)^2 \right), \tag{3.11}$$

where $\wp(B)$ stands for $d(Q, D)\mathbf{y}^{B-(Q \cup f)}$. We are left with comparing the coefficients of two polynomials in y_f, namely, (3.2) and (3.10)-(3.11). The degree 0 and 2 terms come from (3.10) and the degree 1 terms are a combination of (3.11) with the cross terms of (3.10).

The methods in this section are easily adapted to showing that the degree 0 and 2 terms are the sum-of-squares forms of $\Phi G^f\{e\}$ and $\Phi G_f\{e\}$, respectively, and that (3.11) is the sum-of-squares form of $\Delta G\{e,f\}$, which accounts for the first term in (3.9). We are left with showing that the cross terms of (3.10) are equal to $G^{ef}G_{ef} - G^f_e G_{ef}$. The proof of the following proposition is cumbersome and again similar to what we have seen.

Proposition 3.8 *If for each pair e and f of distinct edges, $G^{ef}G_{ef} - G^f_e G_{ef}$ is equal to*

$$\sum_{\substack{Q\in\mathscr{Q}_G\\ f\notin Q}} \mathbf{y}^Q \left(\sum_{B:f\in D} \wp(B) + \sum_{B:f\in B-D} \wp(B) \right) \left(\sum_{\substack{B:f\notin B\\ f\notin cl(B)}} \wp(B) + \sum_{\substack{B:f\notin B\\ f\in cl(B)}} \wp(B) \right),$$

for a certain choice of signs, then Conjecture 3.7 is true.

4 Binary matroids

Graphic matroids are indeed interesting on their own, however, it is worth being reminded of their role in the decomposition of regular matroids. A few relevant facts are listed. For undefined terms see [1] and [12].

(1) A binary matroid is **B**-Rayleigh if and only if it has no S_8 minor ([1]).

(2) The affine geometry $\mathscr{AG}(3,2)$ is a splitter for the class of binary matroids containing no S_8 minor (Seymour unpublished, Appendix D, [9]).

(3) If a binary matroid contains neither $\mathcal{S}_8$ nor $\mathscr{AG}(3,2)$ as a minor, then it can be constructed from direct sums and two-sums of regular matroids, the fano matroid, F_7, and its dual, $(F_7)^*$ ([13]).

(4) From (3) it follows that a binary, three-connected matroid with no S_8 minor is regular or isomorphic to F_7, $(F_7)^*$ or $\mathscr{AG}(3,2)$.

(5) A three-connected regular matroid which is neither graphic nor co-graphic contains either R_{10} or R_{12} as a minor ([12]).

(6) Regular matroids decompose over direct sums, two-sums and three-sums into graphic and co-graphic matroids and R_{10} ([12]).

We derive Theorem 1.5.

Proof. (of Theorem 1.5) Assume (ii) and observe that by (4), regular matroids are a subclass of binary matroids with no S_8 and no $\mathscr{AG}(3,2)$ minor. Since (ii) implies that binary matroids with no S_8 minor are **I**-Rayleigh, (i) must be true.

Conversely, assume (i) and let $\mathcal{M}$ be a minor-minimal counter example. It is easy to show that if $\mathcal{M}$ is **I**-Rayleigh then it must be **B**-Rayleigh, so we may assume that

$\mathcal{M}$ is **B**-Rayleigh and not **I**-Rayleigh. By Theorem 5.8 of [14] and its minimality, $\mathcal{M}$ is three-connected. Now from (4), $\mathcal{M}$ is either F_7, $(F_7)^*$, $\mathscr{AG}(3,2)$ or it is regular. But F_7 and $(F_7)^*$ are minors of $\mathscr{AG}(3,2)$ which is **I**-Rayleigh ([11] page 12). Therefore $\mathcal{M}$ must be regular, but that contradicts (i), so (i) implies (ii). □

5 Concluding Remarks

We do not know whether or not regular matroids are **I**-Rayleigh, so we verify the fact for small matroids. In particular, regular matroids on up to nine elements are **I**-Rayleigh. Denote the graphic matroid of $\mathcal{G}$ by $M(\mathcal{G})$. A simple calculation in Maple shows that $(M(K_{3,3}))^*$ is **I**-Rayleigh by first subtracting $\Delta I((M(K_{3,3}))^*)\{e,f\} - \Delta B((M(K_{3,3}))^*)\{e,f\}$. The resulting difference has four negative terms, however, a small algebraic manipulation makes these disappear into squares so that any positive evaluation of $\Delta I((M(K_{3,3}))^*)\{e,f\} - \Delta B((M(K_{3,3}))^*)\{e,f\}$ is non-negative. Thus $\Delta I((M(K_{3,3}))^*)\{e,f\} \geq 0$ for every positive weighting. Let $K_{3,3}^+$ be $K_{3,3}$ plus an edge not parallel to any others. Three-connected regular matroids on at most 10 elements are either $(M(K_{3,3}))^*$, $(M(K_5))^*$, $(M(K_{3,3}^+))^*$, R_{10} or graphic on at most six vertices, which is the upper bound for such a graph. Wagner has verified that K_6 is Δ-SOS which shows that regular matroids on at most 9 elements are **I**-Rayleigh. Furthermore Cocks proves that $(M(K_5))^*$ is **I**-Rayleigh ([2]), so the only two obstructions to showing (i) for 10 elements are $(M(K_{3,3}^+))^*$ and R_{10}. Unfortunately for these last two, the method of subtracting $\Delta B\{e,f\}$ from $\Delta I\{e,f\}$ yields not four, but tens of negative terms.

Semple and Welsh also ask whether graphs are **S**-Rayleigh. This is equivalent to co-graphic matroids being **I**-Rayleigh and it is necessary for showing that regular matroids also possess the property. Is there a sum-of-squares form for the **S**-Rayleigh difference, analogous to that of Conjecture 1.3?

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