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On the super edge-magic deficiency of some families related to ladder graphs

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Abstract

A graph G is called *edge-magic* if there exists a bijective function $\phi : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G)|\}$ such that $\phi(x) + \phi(xy) + \phi(y)$ is a constant $c(\phi)$ for every edge $xy \in E(G)$; here $c(\phi)$ is called the valence of ϕ. A graph G is said to be *super edge-magic* if $\phi(V(G)) = \{1, 2, \ldots, |V(G)|\}$. The *super edge-magic deficiency*, denoted by $\mu_s(G)$, is the minimum nonnegative integer n such that $G \cup nK_1$ has a super edge-magic labeling; if such an integer does not exist we define $\mu_s(G)$ to be $+\infty$. In this paper we study the super edge-magic deficiency of some families of graphs related to ladder graphs.

1 Introduction

In this paper, we consider only finite, simple and undirected graphs. We denote the vertex set and edge set of a graph G by $V(G)$ and $E(G)$ respectively, where $|V(G)| = p$ and $|E(G)| = q$. An edge-magic labeling of a graph G is a bijection $\phi : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p+q\}$, such that $\phi(x) + \phi(xy) + \phi(y) = c(\phi)$ for every edge $xy \in E(G)$. The value $c(\phi)$ is called a magic constant or valence, and a graph with an edge-magic labeling is called *edge-magic*. An edge-magic labeling ϕ is called super edge-magic if $\phi(V(G)) = \{1, 2, \ldots, p\}$.

In [11], Kotzig and Rosa proved that for any graph G there exists an edge-magic graph H such that $H \cong G \cup nK_1$ for some nonnegative integer n. This fact leads to the concept of edge-magic deficiency of a graph G, which is the minimum nonnegative integer n such that $G \cup nK_1$ is edge-magic; it is denoted by $\mu(G)$. In particular,

$$\mu(G) = \min\{n \geq 0 : G \cup nK_1 \text{ is edge-magic}\}.$$

In the same paper, Kotzig and Rosa gave an upper bound for the edge-magic deficiency of a graph G with n vertices, $\mu(G) \leq F_{n+2} - 2 - n - \frac{1}{2}n(n-1)$, where F_n is the nth Fibonacci number. Motivated by Kotzig and Rosa's concept of edge-magic deficiency, Figueroa-Centeno et al. [7] defined a similar concept for super edge-magic labelings. The super edge-magic deficiency of a graph G, denoted by $\mu_s(G)$, is the minimum nonnegative integer n such that $G \cup nK_1$ has a super edge-magic labeling, or is denoted $+\infty$ if no such n exists.

Let $M(G) = \{n \geq 0 : G \cup nK_1 \text{ is a super edge-magic graph}\}$; then

$$\mu_s(G) = \begin{cases} \min \ M(G), & \text{if } M(G) \neq \phi; \\ +\infty, & \text{if } M(G) = \phi. \end{cases}$$

As a consequence of the above two definitions, we have that for every graph G, $\mu(G) \leq \mu_s(G)$.

In [7, 8], Figueroa-Centeno et al. provided the exact values for the super edge-magic deficiencies of several classes of graphs, such as cycles, complete graphs and complete bipartite graphs $K_{2,n}$. They also proved that all forests have finite deficiency. They proved that

$$\mu_s(C_n) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ 1, & \text{if } n \equiv 0 \pmod 4 \\ +\infty, & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Acharya and Hegde in [1] introduced independently the concept of strong indexable graphs which turn out to be equivalent to the concept of super edge-magic labelings. In [12], Ngurah et al. proved some upper bounds for the super edge-magic deficiency of fans, double fans and wheels. More results concerning super edge-magic and super edge-magic labelings of some graphs can be found in [2, 3, 4, 10, 13, 14], and a complete survey in [9].

The *Ladder graph*, denoted by L_n, where $L_n \cong P_n \times P_2$, is the graph with vertex set $\{u_i, v_i : 1 \le i \le n\}$ and edge set

$$\{u_i u_{i+1}, v_i v_{i+1} : 1 \le i \le n-1\} \cup \{u_i v_i : 1 \le i \le n\}.$$

The *diagonal Ladder*, denoted by DL_n, is the graph obtained from the ladder by adding two diagonals to each rectangle. Thus the vertex set of DL_n is $\{v_i, u_i : 1 \le i \le n\}$, and the edge set of DL_n is

$$\{v_i v_{i+1}, u_i u_{i+1}, v_i u_{i+1}, u_i v_{i+1} : 1 \le i \le n-1\} \cup \{v_i u_i : 1 \le i \le n\}.$$

The *Mongolian tent*, denoted by Mt_n, is the graph obtained from the ladder graph L_n by adding a new vertex u joining each u_i, for $1 \le i \le n$, with u. Thus the vertex set of the Mongolian tent is

$$V(Mt_n) = \{u_i, v_i : 1 \le i \le n\} \cup \{u\}$$

and the edge set is

$$E(Mt_n) = \{u_i u_{i+1}, v_i v_{i+1} : 1 \le i \le n-1\} \cup \{u_i v_i, u u_i : 1 \le i \le n\}.$$

The *triangular chain*, denoted by TC_n, is the graph obtained from a non-trivial path P_n, where $P_n = v_1, v_2, \ldots, v_{2n}$, by adding new vertices $u_1, u_2, \ldots, u_n$ joining each u_i with v_{2i-1} and v_{2i}, for $1 \le i \le n$. Thus the vertex set of TC_n is

$$V(TC_n) = \{v_i : 1 \le i \le 2n\} \cup \{u_i : 1 \le i \le n\}$$

and the edge set of TC_n is

$$E(TC_n) = \{v_i v_{i+1} : 1 \le i \le 2n-1\} \cup \{u_i v_{2i-1}, u_i v_{2i} : 1 \le i \le n\}.$$

In this paper, we discuss the super edge-magic deficiency of these graphs. When proving the main results, we frequently use two lemmas.

Lemma 1. *[6] A graph G with p vertices and q edges is super edge-magic if and only if there exists a bijective function $\phi : V(G) \to \{1, 2, \ldots, p\}$ such that the set $S = \{\phi(x) + \phi(y) \mid xy \in E(G)\}$ consists of q consecutive integers. In such a case, ϕ extends to a super edge-magic labeling of G.*

Lemma 2. *[5] If a graph G with p vertices and q edges is super edge-magic, then $q \le 2p - 3$.*

2 Main Results

Figuereoa-Centeno et al. ([6], Theorem 10) stated that the ladder L_n is super edge-magic, where n is odd and the magic constant is $\frac{11n+1}{2}$. It is easy to see that L_2 is not super edge-magic. Figuereoa-Centeno et al. [6] found a super edge-magic labeling for $n = 4$ and $n = 6$. They suspected that super edge-magic labelings might be found for large even values of n. In the next theorem we show that an upper bound for the super edge-magic deficiency of the ladder is 1 if n is even.

Theorem 1. *For n even, the super edge-magic deficiency of the ladder graph L_n is $\mu_s(L_n) \leq 1$.*

Proof. Let $L_n^* \cong L_n \cup K_1$. The vertex set of L_n^* is

$$V(L_n^*) = \{v_i, u_i : 1 \leq i \leq n\} \cup \{z\}$$

and the edge set of L_n^* is

$$E(L_n^*) = \{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_i : 1 \leq i \leq n\}.$$

To prove that $\mu_s(L_n) \leq 1$, we define the labeling $\phi : V(L_n^*) \to \{1, 2, \ldots, |V(L_n)| + 1\}$ of L_n^* as follows:

$$\phi(v_i) = \begin{cases} i, & \text{if } 1 \leq i \leq \frac{n}{2} \text{ and } i \equiv 1 \pmod 2 \\ \frac{4i+n+2}{2}, & \text{if } 1 \leq i \leq \frac{n}{2} \text{ and } i \equiv 0 \pmod 2 \\ \frac{4i-n}{2}, & \text{if } \frac{n}{2}+1 \leq i \leq n \text{ and } i \equiv 1 \pmod 2 \\ n+1+i, & \text{if } \frac{n}{2}+1 \leq i \leq n \text{ and } i \equiv 0 \pmod 2 \end{cases}$$

$$\phi(u_i) = \begin{cases} i, & \text{if } 1 \leq i \leq \frac{n}{2} \text{ and } i \equiv 0 \pmod 2 \\ \frac{4i+n+2}{2}, & \text{if } 1 \leq i \leq \frac{n}{2} \text{ and } i \equiv 1 \pmod 2 \\ \frac{4i-n}{2}, & \text{if } \frac{n}{2}+1 \leq i \leq n \text{ and } i \equiv 0 \pmod 2 \\ n+1+i, & \text{if } \frac{n}{2}+1 \leq i \leq n \text{ and } i \equiv 1 \pmod 2. \end{cases}$$

The isolated vertex z is labeled as $\phi(z) = \frac{n+2}{2}$.

The set of all edge-sums generated by the above formula forms a $|E(L_n^*)|$ consecutive integer sequence $\frac{n+8}{2}, \frac{n+10}{2}, \ldots, \frac{7n+2}{2}$. Therefore by using Lemma 1, ϕ can be extended to a super edge-magic labeling of L_n^* with magic constant $\frac{11n+6}{2}$. Since there is one isolated vertex it follows that $\mu_s(L_n) \leq 1$. □

In the next theorem we find the super edge-magic deficiency of diagonal ladders.

Theorem 2. *The super edge-magic deficiency for the diagonal ladder graph DL_n is $\mu_s(DL_n) = \lfloor \frac{n}{2} \rfloor$.*

Proof. Let $G \cong DL_n \cup \lfloor \frac{n}{2} \rfloor K_1$. The vertex set of G is

$$\{v_i, u_i : 1 \leq i \leq n\} \cup \{z_i : 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$$

and the edge set of G is

$$\{v_i v_{i+1}, u_i u_{i+1}, v_i u_{i+1}, u_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_i u_i : 1 \leq i \leq n\}.$$

By Lemma 2, we know that the size of any super edge-magic graph is bounded above by two times its order minus 3. Now we know that for the diagonal ladder

$p = 2n$ and $q = 5n - 4$, which implies that we need $x \in \mathbb{N}$ such that $5n - 4 \leq 2(2n + x) - 3$, or $(n - 1)/2 \leq x$. This means that if n is odd, then we need at least $(n - 1)/2 = \lfloor \frac{n}{2} \rfloor$ isolated vertices. If n is even we need at least $(n - 1)/2$ isolated vertices. Hence for even n we need at least $\frac{n}{2}$ vertices.

Consequently, $\mu_s(DL_n) \geq \lfloor \frac{n}{2} \rfloor$. The upper bound of the super edge-magic deficiency of DL_n is $\lfloor \frac{n}{2} \rfloor$, as we state the labeling ϕ of the vertices of G in the following way:

$$\phi(u_i) = \begin{cases} \frac{5i-4}{2} & \text{if } 1 \leq i \leq n \text{ and } i \equiv 0 \pmod 2; \\ \frac{5i-1}{2} & \text{if } 1 \leq i \leq n \text{ and } i \equiv 1 \pmod 2; \end{cases}$$

$$\phi(v_i) = \begin{cases} \frac{5i}{2} & \text{if } 1 \leq i \leq n \text{ and } i \equiv 0 \pmod 2; \\ \frac{5i-3}{2} & \text{if } 1 \leq i \leq n \text{ and } i \equiv 1 \pmod 2. \end{cases}$$

The isolated vertices z_i are labeled as $\phi(z_i) = 5i - 1$ for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

The set of all edge-sums generated by the above formula forms a consecutive integer sequence. Therefore by using Lemma 1, It can be checked that ϕ extends to a super edge-magic labeling of $DL_n \cup \lfloor \frac{n}{2} \rfloor K_1$ with the magic constant $7n - 1 + \lfloor \frac{n}{2} \rfloor$. Hence $\mu_s(DL_n) = \lfloor \frac{n}{2} \rfloor$.

□

The following theorem gives an upper bound for the super edge-magic deficiency of the Mongolian tent graph.

Theorem 3. *The super edge-magic deficiency of the Mongolian tent graph Mt_n, for n odd, is bounded above by $\mu_s(Mt_n) \leq \frac{n-3}{2}$.*

Proof. Recall that the vertex set of Mt_n is $\{v_i, u_i : 1 \leq i \leq n\} \cup \{u\}$, and the edge set of Mt_n is

$$\{v_i v_{i+1}, u_i u_{i+1} : 1 \leq i \leq n - 1\} \cup \{uu_i, u_i v_i : 1 \leq i \leq n\}.$$

Let n be an odd nonnegative integer. According to Lemma 1 it is sufficient to prove that there exists a vertex labeling with the property that the edge-sums under this labeling are consecutive integers. It is easy to see that the labeling $\phi : V(Mt_n \cup \frac{n-3}{2} K_1) \rightarrow \{1, 2, \ldots, |V(Mt_n)| + \frac{n-3}{2}\}$ has the desired property, for $n \equiv 1 \pmod 2$. Here, we label $Mt_n \cup \frac{n-3}{2} K_1$ where $V(\frac{n-3}{2} K_1) = \{z_i : 1 \leq i \leq \frac{n-3}{2}\}$.

$$\phi(v_i) = \begin{cases} \frac{i+1}{2} & \text{if } 1 \leq i \leq n \text{ and } n \equiv 1 \pmod 2; \\ \frac{n+i+1}{2} & \text{if } 1 \leq i \leq n \text{ and } n \equiv 0 \pmod 2; \end{cases}$$

$$\phi(u_i) = \begin{cases} \frac{3n+i}{2} & \text{if } 1 \leq i \leq n \text{ and } n \equiv 1 \pmod 2; \\ \frac{2n+i}{2} & \text{if } 1 \leq i \leq n \text{ and } n \equiv 0 \pmod 2; \end{cases}$$

$$\phi(u) = \frac{5n-1}{2}.$$

The isolated vertices z_i under the labeling ϕ are labeled $\phi(z_i) = 2n + i$, for $1 \leq i \leq \frac{n-3}{2}$.

It is easy to see that the edge-sums form a q consecutive integer sequence

$$\frac{n+5}{2}, \frac{n+5}{2} + 1, \ldots, \frac{9n-1}{2}.$$

This shows that $\mu_s(Mt_n) \leq (n-3)/2$, which completes the proof. □

Open Problem 1. *Find a better upper bound for the super edge-magic deficiency of Mt_n or else prove that a better upper bound does not exist.*

Theorem 4. *Let TC_n be a triangular chain graph. Then $H \cong TC_n \cup \lfloor \frac{n}{2} \rfloor K_1$ admits a super edge-magic labeling, $\mu_s(H) = 0$.*

Proof. The vertex set and edge set of H are defined as follows:

$$V(H) = \{v_i : 1 \leq i \leq 2n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{z_i : 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\};$$

$$E(H) = \{v_i v_{i+1} : 1 \leq i \leq 2n-1\} \cup \{u_i v_{2i-1}, u_i v_{2i} : 1 \leq i \leq n\}.$$

We define the labeling $\phi : V(H) \to \{1, 2, \ldots, |H|\}$ of H as follows:
For $1 \leq i \leq n$, let $\phi(v_{2i-1}) = i$ and $\phi(v_{2i}) = n + i$.

For n even,

$$\phi(u_i) = \begin{cases} 3n + i, & \text{if } 1 \leq i \leq \frac{n}{2} \\ 2n - 1 + i, & \text{if } \frac{n}{2} + 1 \leq i \leq n. \end{cases}$$

The isolated vertices z_i are labeled as

$$\phi(z_i) = \begin{cases} 2n + i, & \text{if } 1 \leq i \leq \frac{n}{2} - 1 \\ 3n, & \text{if } i = \frac{n}{2}. \end{cases}$$

For n odd,

$$\phi(u_i) = \begin{cases} 3n + i, & \text{if } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor \\ 2n + i, & \text{if } \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n. \end{cases}$$

The isolated vertices z_i are labeled as $\phi(z_i) = 2n + i$, for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

It is easy to see that the edge-sums form a q consecutive integers sequence $n + 2, n + 3, \ldots, 5n$. Therefore by using Lemma 1, ϕ can be extended to a super edge-magic labeling. Hence $H \simeq TC_n \cup \lfloor \frac{n}{2} \rfloor$ admits a super edge-magic labeling with valence $8n + 1 + \lfloor \frac{n}{2} \rfloor$. □

Open Problem 2. *Find a better upper bound of the super edge-magic deficiency of TC_n or else prove that a better upper bound does not exist.*

3 Closing remarks

We have found upper bounds for the super edge-magic deficiency of ladder graphs, Mongolian tents and triangular chains, as well as the exact value for the super edge-magic deficiency of the diagonal ladder graph. It would be interesting to find the super edge-magic deficiency of the Mongolian tent for n even. We encourage researchers to try to determine the super edge-magic deficiency of graphs for further research. In fact it is a very challenging problem, in general, to find the super edge-magic deficiency of families of graphs.

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