AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **51** (2011), Pages 89–108

Ramsey numbers for small graphs versus small disconnected graphs

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Abstract

The Ramsey number $r(G, H)$ is determined for all disconnected (isolate-free) graphs H of order six and all graphs G of order at most five, except the three cases $(G, H) \in \{(K_5 - 2K_2, 2K_3), (K_5 - e, 2K_3), (K_5, 2K_3)\}$ where bounds with difference 1 are established. Moreover, general results are obtained for some small disconnected graphs H and any graph G.

1 Introduction

The Ramsey number $r(G, H)$ has been studied intensively for small graphs G and H. A detailed survey of known results is given in [20]. Especially, the values of $r(G, H)$ have been determined for all graphs G and H of order at most five except for the three cases where $H = K_5$ and $G \in \{K_5 - 2K_2,\ K_5 - e,\ K_5\}$ (see [1, 2, 4, 5, 6, 7, 9, 10, 17, 21, 23]). If H is a connected graph of order six, then $r(G, H)$ is known for all $G \subseteq K_4 - e$ (see [8, 11, 12, 15, 16, 18, 19, 22]), whereas only partial results have been obtained in case of $G = K_4$. Here we will study $r(G, H)$ for all disconnected graphs

H of order six and all graphs G of order at most five. Moreover, general results are obtained for $H \in \{P_3 \cup K_2, K_3 \cup K_2, 3K_2\}$ and any graph G.

Some specialized notation will be used. The number of vertices of a graph G is denoted by $p(G)$. When considering a 2-coloring χ of the edges of K_n, we refer to colors red and green, and say that χ is a (G, H)-coloring if it contains neither a red subgraph G nor a green subgraph H. We use V to denote the vertex set of K_n, and define functions $d_r(v)$ and $d_g(v)$, for $v \in V$, to be the numbers of red and green edges incident to v. For $U \subseteq V(K_n)$, the subgraph induced by U is denoted by $[U]$. In case of $U = \{u_1, u_2, \ldots, u_k\}$ we write $[u_1, u_2, \ldots, u_k]$ instead of $[\{u_1, u_2, \ldots, u_k\}]$. Moreover, for disjoint subsets $U_1, U_2 \subset V(K_n)$, $q_r(U_1, U_2)$ denotes the number of red edges from U_1 to U_2, and $q_g(U_1, U_2)$ is defined similarly.

2 General Results

To establish bounds and exact values of $r(G, H)$ for disconnected graphs H, the following theorem will be useful.

Theorem 2.1 *Let F, F_1, F_2, and G be isolate-free graphs, where F_1 and F_2 have disjoint vertex sets. Moreover, let $m = \max\{r(G, F_1), r(G, F_2)\}$. Then*

$$m \leq r(G, F_1 \cup F_2) \leq m + \min\{p(F_1), p(F_2)\}, \tag{1}$$

$$r(G, F) + 1 \leq r(G, 2F) \leq r(G, F) + p(F), \tag{2}$$

$$r(G, F_1 \cup F_2) = r(G, F_1) \quad \textit{if} \quad r(G, F_1) \geq r(G, F_2) + p(F_1). \tag{3}$$

Proof. (1): Obviously, $r(G, F_1 \cup F_2) \geq m$. To prove the second inequality, let χ be a 2-coloring of K_n where $n = m + \min\{p(F_1), p(F_2)\}$. We may assume that $p(F_1) \geq p(F_2)$. Suppose there is no red subgraph G in χ. Then we find a green subgraph F_2 in χ. Delete the vertices of a green F_2. It remains a 2-coloring of K_m where $m \geq r(F_1, G)$. Since a red subgraph G shall not occur, the coloring of K_m must contain a green F_1 yielding a green $F_1 \cup F_2$ in χ.

(2): The right inequality follows from (1). To prove the left inequality, consider a (G, F)-coloring of K_n where $n = r(G, F) - 1$. Add a vertex v and join it green to all vertices of K_n. We obtain a 2-coloring of K_{n+1} without red subgraph G and green subgraph $2F$ since G does not contain isolates and every green subgraph F must contain the vertex v. Thus, $r(G, 2F) \geq n + 2 = r(G, F) + 1$.

(3): By (1) we obtain that $r(G, F_1 \cup F_2) \geq m = r(G, F_1)$. To establish equality we have to prove that $r(G, F_1 \cup F_2) \leq m$. Consider any 2-coloring of K_m. Suppose that no red subgraph G occurs. Then $m = r(G, F_1)$ forces a green subgraph F_1. Delete the vertices of a green F_1. The remaining 2-coloring of K_n, where $n = m - p(F_1) \geq r(G, F_2)$, must contain a green subgraph F_2, and this yields a green $F_1 \cup F_2$. ■

The only isolate-free disconnected graph H with $p(H) \leq 4$ is $H = 2K_2$, and for any isolate-free graph G with p vertices it is well-known (see [4]) that

$$r(G, 2K_2) = \begin{cases} p+2 & \text{if } G = K_p, \\ p+1 & \text{otherwise.} \end{cases} \tag{4}$$

In case of $p(H) = 5$ there are two isolate-free disconnected graphs H, namely $H = P_3 \cup K_2$ and $H = K_3 \cup K_2$, and $r(G, H)$ will be considered for these graphs H in the following two theorems.

Theorem 2.2 *Let G be an isolate-free graph with $p \geq 4$ vertices and let $\beta_1(\overline{G})$ be the edge independence number of the complement $\overline{G}$ of G. Then*

$$r(G, P_3 \cup K_2) = \begin{cases} 2p - 2\beta_1(\overline{G}) - 1 & \text{if } \beta_1(\overline{G}) \leq \frac{p-3}{2}, \\ p+2 & \text{if } \beta_1(\overline{G}) > \frac{p-3}{2} \text{ and } K_3 \not\subseteq \overline{G}, \\ p+1 & \text{otherwise.} \end{cases}$$

Proof. We will use $r(G, K_2) = p$ and (see [4])

$$r(G, P_3) = \begin{cases} p & \text{if } \overline{G} \text{ has a 1-factor,} \\ 2p - 2\beta_1(\overline{G}) - 1 & \text{otherwise.} \end{cases} \tag{5}$$

Case I: $\beta_1(\overline{G}) \leq \frac{p-3}{2}$. Then $\overline{G}$ does not contain a 1-factor, and this implies $r(G, P_3) = 2p - 2\beta_1(\overline{G}) - 1$.

First let $\beta_1(\overline{G}) < \frac{p-3}{2}$. Then $r(G, P_3) \geq p + 3 = r(G, K_2) + p(P_3)$, and Theorem 2.1(3) implies $r(G, P_3 \cup K_2) = r(G, P_3)$.

Now let $\beta_1(\overline{G}) = \frac{p-3}{2}$. Then $\beta_1(\overline{G}) \geq 1$ because of $p \geq 4$, and $G \subseteq K_p - e$. By Theorem 2.1(1) we obtain $r(G, P_3 \cup K_2) \geq r(G, P_3) = p + 2$. To prove that $r(G, P_3 \cup K_2) \leq p+2$, consider any 2-coloring of K_{p+2}. Assume that no red subgraph G occurs. Then there must be a green P_3, i.e. a vertex v with $d_g(v) \geq 2$.

- If $d_g(v) \geq 4$, a green edge not incident to v has to occur since otherwise we find a red $K_{p+1} \supset G$. But this yields a green $P_3 \cup K_2$.
- If $d_g(v) = 3$ and no green $P_3 \cup K_2$ shall occur, then the $p-2$ vertices joined red to v together with v induce a red K_{p-1}. Moreover, no green edge from a green neighbor of v to one of the red neighbors of v occurs. But this yields a red $K_p - e \supseteq G$.
- If $d_g(v) = 2$ and no green $P_3 \cup K_2$ shall occur, the $p-1$ red neighbors of v yield a red K_{p-1} and together with v a red $K_p \supseteq G$.

Case II: $\beta_1(\overline{G}) > \frac{p-3}{2}$. This implies $r(G, P_3) \leq p+1$.

- First let $K_3 \not\subseteq \overline{G}$. We obtain a $(G, P_3 \cup K_2)$-coloring of K_{p+1} by coloring the edges of a subgraph K_4 green and all remaining edges red. This implies $r(G, P_3 \cup K_2) \geq p+2$. By a case distinction as in Case I it can be shown that $r(G, P_3 \cup K_2) \leq p+2$, and equality is established.

- The remaining case is $K_3 \subseteq \overline{G}$. This implies $G \subseteq K_p - K_3$. Consider a red K_{p-1} and join an additional vertex green to all vertices of the red K_{p-1}. This yields a $(G, P_3 \cup K_2)$-coloring of K_p implying that $r(G, P_3 \cup K_2) \geq p+1$. To prove that $r(G, P_3 \cup K_2) \leq p+1$ consider any 2-coloring of K_{p+1}. Assume that no red subgraph G occurs. Because of $r(G, P_3) \leq p+1$ we find a green P_3, i.e. a vertex v with two green neighbors u and w. If no green $P_3 \cup K_2$ shall occur, the remaining $p-2$ vertices induce a red K_{p-2}. If there are two vertices of the red K_{p-2} joined green to u or w, a green $P_3 \cup K_2$ is established. If at most one vertex of the red K_{p-2} is joined green to u or w, then the red K_{p-2} together with u and w yields a red $K_p - K_3 \supseteq G$. ■

Theorem 2.3 *Let G be an isolate-free graph with $p \geq 4$ vertices. Then*

$$r(G, K_3 \cup K_2) = \begin{cases} r(G, K_3) & \text{if } r(G, K_3) \geq p+3, \\ p+2 & \text{otherwise.} \end{cases}$$

Proof. In case of $r(G, K_3) \geq p+3$ the desired result follows from Theorem 2.1(3) since $p+3 = r(G, K_2) + p(K_3)$.

The remaining case is $r(G, K_3) \leq p+2$. To prove that $r(G, K_3 \cup K_2) \geq p+2$ take a K_{p+1} where a subgraph K_{p-1} is colored red and all other edges are colored green. To prove that $r(G, K_3 \cup K_2) \leq p+2$ we will first establish that in case of $r(G, K_3) \leq p+2$ the graph G has to be disconnected. It is well known (see [4]) that

$$r(G, F) \geq (\chi_{chr}(F) - 1)(c(G) - 1) + 1$$

where $\chi_{chr}(F)$ denotes the chromatic number of F and $c(G)$ the cardinality of the largest connected component of G. Applying $\chi_{chr}(K_3) = 3$ and $r(G, K_3) \leq p+2$ we obtain that $2c(G) - 1 \leq p+2$. This implies $c(G) \leq \frac{p+3}{2} < p$ for $p \geq 4$. Thus, G has to be disconnected and must contain a component with at most $\lfloor \frac{p}{2} \rfloor$ vertices. This yields a vertex not adjacent to $\lceil \frac{p}{2} \rceil$ vertices in G, and $G \subseteq K_p - K_{1,\lceil \frac{p}{2} \rceil}$.

Now consider any 2-coloring χ of K_n where $n = p+2$. Assume that no red subgraph G occurs in χ. Because of $r(G, K_3) \leq p+2$ we then find a green K_3 in χ. Let $V = \{v_1, v_2, v_3\}$ be the vertex set of a green K_3, and let U be the set of the remaining $p-1$ vertices. If two vertices in U are joined green, a green $K_3 \cup K_2$ occurs. Otherwise U induces a red K_{p-1}. Assume first that a vertex $u \in U$ is joined green to at least two of the vertices in V, say v_1 and v_2. Then any green edge from v_3 to $U \setminus \{u\}$

yields a green $K_3 \cup K_2$ and only red edges yield a red $K_p - e \supseteq G$. The remaining case is that every vertex of U is joined green to at most one of the vertices v_1, v_2 and v_3. Then one of these three vertices is joined green to at most $\lfloor \frac{p-1}{3} \rfloor$ vertices in U. But this yields a red $K_p - K_{1,\lfloor \frac{p-1}{3} \rfloor} \supseteq G$. ■

In the following theorem $r(G, H)$ is determined for one of the disconnected graphs H with $p(H) = 6$, namely $H = 3K_2$, and any graph G.

Theorem 2.4 *Let G be an isolate-free graph with p vertices. Then*

$$r(G, 3K_2) = \begin{cases} p+4 & \textit{if } \ G = K_p, \\ p+3 & \textit{if } \ G \neq K_p \ \textit{ and } \ K_3 \not\subseteq \overline{G}, \\ p+2 & \textit{otherwise.} \end{cases}$$

Proof. First let $G = K_p$. We obtain $r(K_p, 3K_2) \leq \max\{r(K_p, 2K_2), r(K_p, K_2)\}+2 = p + 4$ from Theorem 2.1(1) and (4). To prove that $r(K_p, 3K_2) \geq p + 4$ take a K_{p+3} where a subgraph K_5 is colored green and all other edges are colored red.

Now let $G \neq K_p$ and $K_3 \not\subseteq \overline{G}$. Theorem 2.1(1) yields $r(G, 3K_2) \leq p + 3$. To prove that $r(K_p, 3K_2) \geq p + 3$ take a K_{p+2} where a subgraph K_5 is colored green and all other edges are colored red.

The remaining case is $G \neq K_p$ and $K_3 \subseteq \overline{G}$. A $(G, 3K_2)$-coloring of K_{p+1} is obtained if the vertices of a green K_2 are joined green to all vertices of a red K_{p-1}. Thus, $r(G, 3K_2) \geq p + 2$. To prove $r(G, 3K_2) \leq p + 2$ consider any 2-coloring of K_{p+2}. Assume that no red subgraph G occurs. Since $r(G, 2K_2) = p + 1$ we find a green $2K_2$. Let U be the set of the remaining $p - 2$ vertices. If no green $3K_2$ shall occur, U induces a red K_{p-2}. From at least three of the four vertices of the green $2K_2$ there must be at least one green edge to U since otherwise we would find a red $K_p - e \supseteq G$. Thus, there are two vertices of the green $2K_2$ which are joined green, and each of them produces at least one green neighbor in U. But this yields either a green $3K_2$ or a red $K_p - K_3 \supseteq G$. ■

3 Results for Small Graphs

In this section we will consider all isolate-free graphs G and H where $3 \leq p(G) \leq 5$ and $p(H) = 6$. The case $p(G) = 2$, i.e. $G = K_2$, is omitted since $r(K_2, H) = p(H)$ is obvious. Exact values or bounds for $r(G, H)$ are given in Table 1. The graphs G are labelled $G_{i,j}$ where i denotes the number of vertices of $G_{i,j}$. All graphs $G_{i,j}$ not defined in Table 1 are given in Figure 1.

The values of $r(G, H)$ in Table 1 concerning the rows where $G \in \{P_3, 2K_2, P_3 \cup K_2, K_3 \cup K_2\}$ follow from (5), (4), Theorem 2.2, and Theorem 2.3, and $r(G, 3K_2)$ is

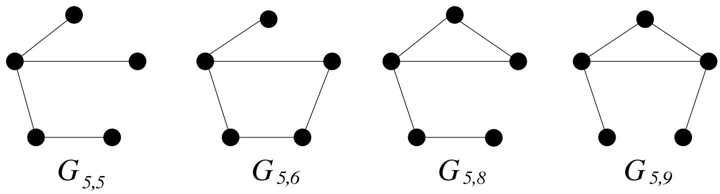

Figure 1: The graphs $G_{i,j}$ not defined in Table 1

obtained from Theorem 2.4. The remaining cases will be considered in the following sections. The values for $G \in \{C_4, C_5\}$ have already been determined in [12] and [14]. In case of $H \in \{3K_2, 2P_3, 2K_3\}$ the graph H consists of multiple copies of a graph H'. General results for this case can be found in [3].

3.1 $H = F \cup K_2$

Since $r(G, K_2) = p(G)$ and $p(F) = 4$ for $F \in \{P_4, K_{1,3}, C_4, K_{1,3}+e, K_4-e, K_4\}$ from Theorem 2.1(3) we obtain

$$r(G, F \cup K_2) = r(G, F) \text{ if } r(G, F) \geq p(G) + 4.$$

Thus, the values of $r(G, F)$ imply the values of $r(G, F \cup K_2)$ in case of $G = K_3$ and all other values of $r(G, F \cup K_2) \geq 9$ in Table 1. Additionaly, note that the values of $r(G, F \cup K_2)$ are already known for $G \in \{P_3, 2K_2, P_3 \cup K_2, K_3 \cup K_2\}$.

Next we consider the two remaining cases where $r(G, F \cup K_2) = 6$, namely $G = K_{1,3}$ and $F \in \{P_4, C_4\}$. From $r(K_{1,3}, 3K_2) = 6$ we deduce $r(K_{1,3}, F \cup K_2) \geq 6$. The following lemma establishes equality.

Lemma 3.1

$$r(K_{1,3}, C_4 \cup K_2) \leq 6.$$

Proof. Let χ be a 2-coloring of K_6 without a red subgraph $K_{1,3}$. Then, because of $r(K_{1,3}, C_4) = 6$, a green subgraph C_4 must occur. Let $U = \{u_1, u_2, u_3, u_4\}$ be the vertex set of a green C_4. Let the edges $u_1u_2, u_2u_3, u_3u_4, u_4u_1$ be green and let v_1 and v_2 be the two remaining vertices of K_6. If the edge v_1v_2 is green, a green $C_4 \cup K_2$ is obtained. Otherwise, as no red $K_{1,3}$ shall occur, there must be at least three green edges from v_1 as well as from v_2 to U. Again we achieve a green subgraph $C_4 \cup K_2$. ∎

Now consider the remaining cases in Table 1 where $r(G, F \cup K_2) = 7$. We already know that $r(2K_2, F \cup K_2) = 7$ establishing $r(G, F \cup K_2) \geq 7$ for all G where $2K_2 \subseteq G$.

$G \setminus H$	$3K_2$	$P_4 \cup K_2$	$K_{1,3} \cup K_2$	$C_4 \cup K_2$	$G_{4,5} \cup K_2$	$G_{4,6} \cup K_2$	$K_4 \cup K_2$	$2P_3$	$K_3 \cup P_3$	$2K_3$
$G_{3,1} = P_3$	6	6	6	6	6	6	7	6	6	6
$G_{3,2} = K_3$	7	7	7	7	7	7	9	8	8	8
$G_{4,1} = 2K_2$	7	7	7	7	7	7	7	7	7	7
$G_{4,2} = P_4$	7	7	7	7	7	7	10	7	7	8
$G_{4,3} = K_{1,3}$	6	6	7	6	7	7	10	6	7	8
$G_{4,4} = C_4$	7	7	7	7	7	7	10	7	7	8
$G_{4,5} = K_{1,3}+e$	7	7	7	7	7	7	10	8	8	8
$G_{4,6} = K_4 - e$	7	7	7	7	7	10	11	8	8	8
$G_{4,7} = K_4$	8	10	10	10	10	11	18	10	10	11
$G_{5,1} = P_3 \cup K_2$	7	7	7	7	7	7	8	7	7	8
$G_{5,2} = K_3 \cup K_2$	8	8	8	8	8	8	9	8	8	8
$G_{5,3} = P_5$	7	7	7	7	9	9	13	7	9	10
$G_{5,4} = K_{1,4}$	7	7	7	7	9	9	13	6	9	10
$G_{5,5}$	7	7	7	7	9	9	13	7	9	10
$G_{5,6}$	7	7	7	7	9	9	13	7	9	10
$G_{5,7} = K_{2,3}$	7	7	8	8	9	10	14	7	9	10
$G_{5,8}$	8	8	8	8	9	9	13	8	9	10
$G_{5,9}$	7	7	7	7	9	9	13	8	9	10
$G_{5,10} = K_{1,4}+e$	7	7	7	7	9	9	13	8	9	10
$G_{5,11} = C_5$	8	8	8	8	9	9	13	8	9	10
$G_{5,12} = K_5 - C_4$	8	8	8	8	9	9	13	8	9	10
$G_{5,13} = K_5 - G_{4,5}$	7	7	7	7	9	10	13	8	9	10
$G_{5,14} = K_5 - G_{5,5}$	8	8	8	8	9	10	13	8	9	10
$G_{5,15} = K_5 - P_5$	8	8	8	8	9	9	13	8	9	10
$G_{5,16} = K_5 - P_4$	8	8	8	8	9	10	13	8	9	10
$G_{5,17} = K_5 - G_{5,1}$	8	8	8	8	9	10	14	8	9	10
$G_{5,18} = K_5 - K_3$	7	7	8	9	9	11	14	8	9	10
$G_{5,19} = K_5 - K_{1,3}$	8	10	10	10	10	11	18	10	10	11
$G_{5,20} = K_5 - P_3$	8	10	10	10	10	11	18	10	10	11
$G_{5,21} = K_5 - 2K_2$	8	8	9	9	11	11	17	8	11	12/13
$G_{5,22} = K_5 - e$	8	10	10	11	11	13	19	10	11	12/13
$G_{5,23} = K_5$	9	13	13	14	14	16	25	12	14	15/16

Table 1: Ramsey numbers $r(G, H)$ for small graphs

The cases $G = G_{4,3} = K_{1,3}$ and $G = G_{5,4} = K_{1,4}$ are left. A $(K_{1,3}, K_{1,3} \cup K_2)$-coloring of K_6 is obtained, if the edges of two vertex-disjoint subgraphs K_3 are colored red and all other edges are colored green. This implies $r(K_{1,3}, F \cup K_2) \geq 7$ for $F \in \{K_{1,3}, K_{1,3} + e, K_4 - e\}$. Using that $r(K_{1,4}, F) = 7$ for $F \in \{P_4, K_{1,3}, C_4\}$ we obtain $r(K_{1,4}, F \cup K_2) \geq 7$. To prove equality we have to show that $r(G, F \cup K_2) \leq 7$. Especially, note that all graphs G in question are subgraphs of $G_{5,18} = K_5 - K_3$ for $F = P_4$, subgraphs of $G_{5,13}$ for $F \in \{K_{1,3}, C_4\}$, subgraphs of $K_4 - e$ for $F = K_{1,3} + e$, and subgraphs of C_4 or $K_{1,3} + e$ for $F = K_4 - e$. Thus, the following lemma yields the desired results.

Lemma 3.2 *Let* $M = \{(K_5 - K_3, P_4), (G_{5,13}, K_{1,3}), (G_{5,13}, C_4), (K_4 - e, K_{1,3} + e), (K_{1,3} + e, K_4 - e), (C_4, K_4 - e)\}$. *Then, for all* $(G, F) \in M$,

$$r(G, F \cup K_2) \leq 7.$$

Proof. Assume that a $(G, F \cup K_2)$-coloring χ of K_7 exists for some $(G, F) \in M$. Because of $r(G, F) = 7$ there must be a green subgraph F in χ. Let $U = \{u_1, u_2, u_3, u_4\}$ be the vertex set of a green subgraph F and $V = \{v_1, v_2, v_3\}$ the set of the three remaining vertices of K_7. Moreover, no green $F \cup K_2$ implies that $[V]$ is a red K_3.

Case I: $(G, F) = (K_5 - K_3, P_4)$. Let the edges u_1u_2, u_2u_3, and u_3u_4 be green. Only red edges from u_1 and u_4 to V yield a red subgraph $K_5 - K_3$ and we obtain a contradiction. Thus, we may assume that u_1v_1 is green. Then no green $P_4 \cup K_2$ forces only red edges from u_4 and from u_2 to v_2 and v_3. Again we obtain a red $K_5 - K_3$.

Case II: $(G, F) = (G_{5,13}, K_{1,3})$. Let the edges u_1u_2, u_1u_3, and u_1u_4 be green. If one of the edges in $[u_2, u_3, u_4]$ is green, say u_2u_3, and no green $K_{1,3} \cup K_2$ shall occur, there must be at least two red edges from u_4 to V, say to v_1 and v_2, and at least one red edge from u_1 to v_1 or v_2 producing a red subgraph $G_{5,13}$. The remaining case is that $[u_2, u_3, u_4]$ is a red K_3. At least two red edges from every vertex in $\{u_2, u_3, u_4\}$ to V yield a red $G_{5,13}$. Thus we may assume that from u_2 there are at least two green edges to V, say to v_1 and v_2. Then no green $K_{1,3} \cup K_2$ forces that v_3u_3 and v_3u_4 have to be red. If one of the edges from v_1 and v_2 to u_3 and u_4 is red, a red $G_{5,13}$ occurs. If all these edges are green, u_2v_3 has to be red yielding a red $G_{5,13}$ once again.

Case III: $(G, F) = (G_{5,13}, C_4)$. Let the edges u_1u_2, u_2u_3, u_3u_4, and u_4u_1 be green. First assume that two edges from u_1 to V, say u_1v_1 and u_1v_2, are red. No green $C_4 \cup K_2$ implies that one of the edges from u_2 and u_3 to v_1 and v_2 is red leading to a red $G_{5,13}$. The remaining case is that there are at least two green edges from every vertex in U to V. Thus, two vertices $x, y \in U$ must have two common green neighbors in V, say v_1 and v_2. No green $C_4 \cup K_2$ forces that x and y are not adjacent on the green C_4 in $[U]$. We may assume that $x = u_1$ and $y = u_3$. But then a green $C_4 \cup K_2$ cannot be avoided since one of the edges in $\{u_2, u_4\} \times \{v_1, v_2\}$ must be green.

Case IV: $(G, F) = (K_4 - e, K_{1,3} + e)$. Let the edges u_1u_2, u_1u_3, u_1u_4, and u_2u_3 be green. Since no red $K_4 - e$ shall occur, there must be at least two green edges from every vertex in U to V. Thus, we may assume that the edges u_2v_1, u_2v_2, and u_3v_1 are green. But this produces a green $(K_{1,3}+e)\cup K_2$ in χ.

Case V: $(G,F)=(K_{1,3}+e,K_4-e)$. Any red edge between U and V yields a red $K_{1,3}+e$, and only green edges imply a green $(K_4-e)\cup K_2$.

Case VI: $(G,F)=(C_4,K_4-e)$. Let the edges u_1u_2, u_1u_3, u_1u_4, u_2u_3, and u_3u_4 be green. If any vertex in U produces two red neighbors in V, then we obtain a red C_4. Therefore every vertex in U must have at least two green neighbors in V, and some vertex in V, say v_1, has at least three green neighbors in U. Thus, v_1u_1 and v_1u_4 may be assumed green. Avoiding a green $(K_4-e)\cup K_2$, u_2v_2 and u_2v_3 have to be red yielding a red C_4. ■

Finally we have to consider the remaining cases in Table 1 where $r(G,F\cup K_2)=8$. Except for $(G,F)\in\{(K_{2,3},K_{1,3}),(K_{2,3},C_4),(C_5,K_{1,3}),(K_5-K_3,K_{1,3})\}$ we obtain $r(G,F\cup K_2)\geq 8$ using $r(G_{5,2},F\cup K_2)=8$ and $G_{5,2}=K_3\cup K_2\subseteq G$ or $r(G,3K_2)=8$ and $3K_2\subseteq F\cup K_2$. From $r(K_{2,3},C_4)=8$ we deduce $r(K_{2,3},C_4\cup K_2)\geq 8$. A 2-coloring of K_7 where the edges of a subgraph $K_{3,3}$ are colored green and all other edges are colored red neither contains a green subgraph $K_{1,3}\cup K_2$ nor a red subgraph $K_{2,3}$, C_5, or K_5-K_3. This implies $r(G,K_{1,3}\cup K_2)\geq 8$ for $G\in\{K_{2,3},C_5,K_5-K_3\}$. To prove equality it suffices to consider the cases in the following lemma.

Lemma 3.3 *Let* $M=\{(K_5-2K_2,P_4),(K_5-P_4,K_{1,3}),(K_5-(P_3\cup K_2),K_{1,3}),(K_5-K_3,K_{1,3}),(K_5-P_4,C_4),(K_5-(P_3\cup K_2),C_4)\}$. *Then, for all* $(G,F)\in M$,

$$r(G,F\cup K_2)\leq 8.$$

Proof. Suppose that a $(G,F\cup K_2)$-coloring χ of K_8 exists where $(G,F)\in M$. Because of $r(G,F)\leq 8$ there must be a green subgraph F in χ. Let $U=\{u_1,u_2,u_3,u_4\}$ be the vertex set of a green subgraph F and let $V=\{v_1,v_2,v_3,v_4\}$ be the set of the four remaining vertices. No green $F\cup K_2$ forces $[V]$ to be a red K_4.

Case I: $F=P_4$. Let the edges u_1u_2, u_2u_3, and u_3u_4 be green. Then $q_g(u_1,V)\geq 2$ and $q_g(u_2,V)\geq 2$ is forbidden since no green $P_4\cup K_2$ shall occur. The remaining case is $q_r(u_1,V)\geq 3$ or $q_r(u_2,V)\geq 3$ yielding a red $K_5-e\supset K_5-2K_2$, a contradiction.

Case II: $F=K_{1,3}$. Let the edges u_1u_2, u_1u_3, and u_1u_4 be green. No green $K_{1,3}\cup K_2$ forces at least two red edges from u_2 to V. But this yields a red K_5-P_3 containing a red K_5-P_4, a red $K_5-(P_3\cup K_2)$, and a red K_5-K_3, a contradiction.

Case III: $F=C_4$. Let the edges u_1u_2, u_2u_3, u_3u_4, and u_4u_1 be green. Then $q_r(x,V)\geq 2$ for some $x\in U$ is forbidden, because otherwise a red K_5-P_3 would occur yielding a contradiction. The remaining case is $q_g(x,V)\geq 3$ for every $x\in U$. But here u_1 and u_2 must have two common green neighbors in V, and these four vertices and the edge u_3u_4 yield a green $C_4\cup K_2$. ■

3.2 $H = 2P_3$

From Theorem 2.1(2) we obtain

$$r(G, P_3) + 1 \le r(G, 2P_3) \le r(G, P_3) + 3. \tag{6}$$

The values given for $r(G, 2P_3)$ in Table 1 match $r(G, P_3)+1$ for $G \in \{K_{1,3}, K_{1,4}, K_5 - K_3\}$, $r(G, P_3) + 2$ for $G \in \{P_3 \cup K_2, P_5, G_{5,5}, G_{5,6}, K_{2,3}\}$, and $r(G, P_3) + 3$ otherwise.

In case of $r(G, 2P_3) = r(G, P_3) + 1$ it suffices to prove $r(G, 2P_3) \le r(G, P_3) + 1$ due to (6). Because of $K_{1,3} \subseteq K_{1,4}$ we only have to show that $r(K_{1,4}, 2P_3) \le 6$ and $r(K_5 - K_3, 2P_3) \le 8$ (see Lemma 3.5).

In case of $r(G, 2P_3) = r(G, P_3) + 2$ we already know that $r(P_3 \cup K_2, 2P_3) = 7$. This implies the lower bound 7 for the remaining G because $P_3 \cup K_2 \subseteq G$. Since $G \subseteq K_{2,3}$ for all G in question, equality follows from $r(K_{2,3}, 2P_3) \le 7$ (see Lemma 3.5).

In case of $r(G, 2P_3) = r(G, P_3) + 3$ the values of $r(G, 2P_3)$ are already known for $G \in \{P_3, 2K_2, K_3 \cup K_2\}$. Because of (6) for the remaining G we only have to show that $r(G, P_3)+3$ is a lower bound. For $G = P_4$ and $G = C_4$ the lower bound 7 follows from $r(2K_2, 2P_3) = 7$. A 2-coloring of K_7 where the green subgraph is isomorphic to $K_5 \cup K_2$ neither contains a green $2P_3$ nor a red C_5, and it implies $r(C_5, 2P_3) \ge 8$. In all other cases we obtain the desired lower bound considering a maximal complete subgraph of G and the following lemma, which presents a class of graphs where the upper bound $r(G, P_3) + 3$ from (6) is attained.

Lemma 3.4

$$r(K_n, 2P_3) = 2n + 2 \ \ \textit{for} \ \ n \ge 2.$$

Proof. Note that $r(K_n, P_3) = 2n - 1$ follows from (5). Thus, $r(K_n, 2P_3) \le 2n+2$ by (6). To establish $2n + 2$ as lower bound take a 2-coloring of K_{2n+1} where the green subgraph is isomorphic to $K_5 \cup (n - 2)K_2$. ■

Lemma 3.5

$$r(K_{1,4}, 2P_3) \le 6, \quad r(K_{2,3}, 2P_3) \le 7, \quad r(K_5 - K_3, 2P_3) \le 8.$$

Proof. Assume that a $(G, 2P_3)$-coloring χ of K_n exists where $n = 6$ in case of $G = K_{1,4}$, $n = 7$ in case of $G = K_{2,3}$, and $n = 8$ in case of $G = K_5 - K_3$.

Let ℓ be the maximal length of a green path in χ and let P be a green path of length ℓ. Because of $r(K_{1,4}, P_3) = r(K_{2,3}, P_3) = 5$ and $r(K_5 - K_3, P_3) = 7$ a green P_3 must occur in χ implying $\ell \ge 2$. Moreover, no green $2P_3$ forces $\ell \le 4$. Let $U = \{u_0, u_1, \dots, u_\ell\}$ be the vertex set of P where the edges $u_i u_{i+1}$ are green for

$i = 0, \ldots, \ell - 1$. Moreover, let $V = \{v_1, \ldots, v_{n-\ell-1}\}$ be the set of the remaining vertices of K_n. Especially, note that all edges from u_0 and u_ℓ to V have to be red.

Case I: $\ell = 4$. In this case also the edges from u_1 and u_3 to V have to be red, avoiding a green $2P_3$. This yields a red $K_{1,4}$ in case of $n = 6$. For $n = 7$ we obtain a red $K_{2,3}$ because $|V| = 2$. In case of $n = 8$ no green $2P_3$ forces red edges in $[V]$. Thus, let v_1v_2 be red. But then $[v_1, v_2, u_0, u_1, u_3]$ contains a red $K_5 - K_3$.

Case II: $\ell = 3$. In case of $n = 6$ no red $K_{1,4}$ implies that one of the edges from v_1 to u_1 or u_2, say to u_2, must be green. Then $\ell = 3$ forces that v_1u_1 and v_1v_2 are red yielding a red $K_{1,4}$. In case of $n = 7$ a red $K_{2,3}$ must occur because $|V| = 3$. In case of $n = 8$ the edge u_0u_3 has to be green as no red $K_5 - K_3$ shall occur. Then $\ell = 3$ forces that all edges from U to V are red. But this leads to a red $K_5 - K_3$, since $[V]$ must contain red edges.

Case III: $\ell = 2$. For $n = 6$ no green $2P_3$ forces two red edges in $[V]$. Let x be the common vertex of these edges. But then the red edges from x to u_0 and u_2 complete a red $K_{1,4}$. In case of $n = 7$ the red edges from u_0 and u_2 to V lead to a red $K_{2,3}$. For $n = 8$ no green $2P_3$ implies a red K_3 in $[V]$ because $r(K_3, P_3) = 5$. But then the red edges from u_0 and u_2 to the red K_3 yield a red $K_5 - e \supset K_5 - K_3$. ■

3.3 $H = K_3 \cup P_3$

From Theorem 2.1 we obtain

$$r(G, K_3) \leq r(G, K_3 \cup P_3) \leq r(G, K_3) + 3 \tag{7}$$

and

$$r(G, K_3 \cup P_3) = r(G, K_3) \text{ if } r(G, K_3) \geq r(G, P_3) + 3. \tag{8}$$

The values given for $r(G, K_3 \cup P_3)$ in Table 1 match $r(G, K_3) + 2$ for $G \in \{K_3, 2K_2, K_3 \cup K_2\}$, $r(G, K_3) + 1$ for $G \in \{P_3, K_{1,3} + e, K_4 - e, K_4, P_3 \cup K_2, K_5 - K_{1,3}, K_5 - P_3\}$, and $r(G, K_3)$ otherwise.

In case of $r(G, K_3 \cup P_3) = r(G, K_3) + 2$ the values for $G = 2K_2$ and $G = K_3 \cup K_2$ are already known and $r(K_3, K_3 \cup P_3) = 8$ follows from $8 = r(K_3, 2P_3) \leq r(K_3, K_3 \cup P_3) \leq r(K_3 \cup K_2, K_3 \cup P_3) = 8$.

In case of $r(G, K_3 \cup P_3) = r(G, K_3) + 1$ the values for $G = P_3$ and $G = P_3 \cup K_2$ are already known. For the remaining G we obtain $r(G, K_3) + 1$ as lower bound applying $r(G, K_3 \cup P_3) \geq r(G, 2P_3)$. Equality follows from $r(K_4 - e, 2K_3) \leq 8$ (see Lemma 3.8) and $r(K_5 - P_3, K_3 \cup P_3) \leq 10$ (see Lemma 3.6).

In case of $r(G, K_3 \cup P_3) = r(G, K_3)$ the values of $r(G, K_3 \cup P_3)$ are a direct consequence of (8) except for $G \in \{K_{1,3}, K_5 - K_3\}$. In these two cases it suffices to establish $r(G, K_3)$ as an upper bound. This will be done in the following lemma.

Lemma 3.6

$$r(K_{1,3}, K_3 \cup P_3) \leq 7, \quad r(K_5 - K_3, K_3 \cup P_3) \leq 9, \quad r(K_5 - P_3, K_3 \cup P_3) \leq 10.$$

Proof. Assume that a $(G, K_3 \cup P_3)$-coloring χ of K_n exists where $n = 7$ in case of $G = K_{1,3}$, $n = 9$ in case of $G = K_5 - K_3$, and $n = 10$ in case of $G = K_5 - P_3$. There must be a green K_3 in χ because $r(K_{1,3}, K_3) = 7$ and $r(K_5 - K_3, K_3) = r(K_5 - P_3, K_3) = 9$. Let $U = \{u_1, u_2, u_3\}$ be the vertex set of a green K_3 and let $V = \{v_1, v_2, \dots, v_{n-3}\}$ be the set of the remaining vertices of K_n. Note that adjacent green edges in $[V]$ are forbidden, since otherwise a green $K_3 \cup P_3$ would occur.

Case I: $n = 7$. As no red $K_{1,3}$ shall occur in $[V]$, the green subgraph of $[V]$ must be isomorphic to $2K_2$. But then the edges from U to V either lead to a red $K_{1,3}$ or to a green $K_3 \cup P_3$.

Case II: $n = 9$. As no red $K_5 - K_3$ shall occur in $[V]$, the green subgraph of $[V]$ must be isomorphic to $3K_2$. But then at least four green edges from every vertex in U to V yield a green $K_3 \cup P_3$, and at least three red edges from any vertex in U to V yield a red $K_5 - K_3$.

Case III: $n = 10$. Since adjacent green edges in $[V]$ are forbidden, a red $K_5 - P_3$ occurs in $[V]$. ■

3.4 $H = 2K_3$

From Theorem 2.1(2) we obtain

$$r(G, K_3) + 1 \leq r(G, 2K_3) \leq r(G, K_3) + 3. \tag{9}$$

The values given for $r(G, 2K_3)$ in Table 1 match $r(G, K_3) + 2$ for $G \in \{K_3, 2K_2, K_4, P_3 \cup K_2, K_3 \cup K_2, K_5 - K_{1,3}, K_5 - P_3\}$ and $r(G, K_3) + 1$ otherwise, except the three cases $G \in \{K_5 - 2K_2, K_5 - e, K_5\}$ where the given bounds are $r(G, K_3) + 1 \leq r(G, 2K_3) \leq r(G, K_3) + 2$.

In case of $r(G, 2K_3) = r(G, K_3) + 2$ the values for $G \in \{2K_2, P_3 \cup K_2, K_3 \cup K_2\}$ are already known. For the remaining G we will first establish $r(G, 2K_3) \geq r(G, K_3) + 2$. The bound $r(K_3, 2K_3) \geq 8$ follows from $r(K_3, 2P_3) = 8$. The 2-coloring of K_{10} where the green subgraph is isomorphic to $K_5 \cup C_5$ neither contains a red K_4 nor a green $2K_3$. Thus, $K_4 \subset K_5 - K_{1,3} \subset K_5 - P_3$ yields $r(G, 2K_3) \geq 11$ for $G \in \{K_4, K_5 - K_{1,3}, K_5 - P_3\}$. In Lemma 3.8 it will be shown that $r(K_4 - e, 2K_3) \leq 8$ implying $r(K_3, 2K_3) \leq 8$. In Lemma 3.9 we will prove $r(K_5 - P_3, 2K_3) \leq 11$ implying $r(G, 2K_3) \leq 11$ for $G = K_4$ and $G = K_5 - K_{1,3}$.

In case of $r(G, 2K_3) = r(G, K_3) + 1$ the value for $G = P_3$ is already known. For the remaining G it suffices to prove $r(K_4 - e, 2K_3) \leq 8$ (see Lemma 3.8) and

$r(G_{5,i}, 2K_3) \leq 10$ where $i = 16, 17, 18$ considering (9) and $G \subseteq K_4 - e$ or $G \subseteq G_{5,i}$ for some $i \in \{16, 17, 18\}$ (see Lemma 3.13).

To show $r(G, K_3) + 1 \leq r(G, 2K_3) \leq r(G, K_3) + 2$ for $G \in \{K_5 - 2K_2, K_5 - e, K_5\}$ it is sufficient to establish $r(K_5 - e, 2K_3) \leq 13$ and $r(K_5, 2K_3) \leq 16$ (see Lemma 3.9).

To prove Lemma 3.8 and Lemma 3.9 the following lemma will be useful.

Lemma 3.7 *If χ is a $(G, 2K_3)$-coloring of K_n where $n = r(G, K_3) + 2$ then χ must contain a green subgraph K_5.*

Proof. Because of $n > r(G, K_3)$ a green K_3 with vertex set $V_1 = \{u_1, u_2, u_3\}$ must occur in χ. Note that if any two vertices x and y of a green $K_3 = [x, y, z]$ are deleted, the remaining coloring of K_{n-2} still must contain a green K_3 since $n - 2 = r(G, K_3)$. This green K_3 must cover z since no green $2K_3$ shall occur. Thus, there must be a green K_3 with vertex set $V_2 = \{u_3, u_4, u_5\}$. Suppose that $[u_1, u_2, u_3, u_4, u_5]$ is not a green K_5. We may assume that the edge u_1u_5 is red. Now delete u_3 and u_4. Then there must be a green K_3 with vertex set V_3 and $u_5 \in V_3$. No green $2K_3$ and $u_1 \notin V_3$ imply $u_2 \in V_3$. Let u_6 be the third vertex in V_3. The edges u_1u_6, u_6u_4, and u_1u_4 have to be red. Now delete u_2 and u_5. Then there exists a green K_3 with vertex set V_4 and $u_3, u_6 \in V_4$. Moreover, $u_1, u_4 \notin V_4$. Let u_7 be the third vertex in V_4. The edges from u_7 to u_1, u_2, u_4, and u_5 have to be red since otherwise a green $2K_3$ would occur. Finally delete u_3 and u_6. Then a green K_3 exists with u_7 and none of its red neighbors u_1, u_2, u_4, and u_5. But this yields a green $2K_3$, a contradiction. ■

Lemma 3.8

$$r(K_4 - e, 2K_3) \leq 8.$$

Proof. Assume that a $(K_4 - e, 2K_3)$-coloring χ of K_8 exists. Since $r(K_4 - e, K_3) = 7$, there must be a green K_3 in χ. Moreover, χ also must contain a red K_3. Otherwise, by $r(K_3, K_3) + 2 = 8$ and Lemma 3.7 a green K_5 has to occur in χ. Let U be the vertex set of a green K_5 and let V be the set of the three remaining vertices. No green $2K_3$ forces at least one red edge in $[V]$ and $q_r(v, U) \geq 4$ for every $v \in V$. But then two vertices joined red in V and a common red neighbor in U yield a red K_3, a contradiction.

Let $U = \{u_1, u_2, u_3\}$ be the vertex set of a green K_3 and let $V = \{v_1, \ldots, v_5\}$ be the set of the five remaining vertices of K_8.

Case I: $[V]$ contains a red K_3. Let v_1, v_2, and v_3 be its vertices. The edge v_4v_5 has to be red since otherwise the edges between $\{v_1, v_2, v_3\}$ and $\{v_4, v_5\}$ would lead to a red $K_4 - e$ or a green $2K_3$.

- **I.1:** One of the vertices v_1, v_2, v_3, say v_1, is joined green to u_1, u_2, and u_3. At least one of the edges between $\{u_1, u_2\}$ and $\{v_4, v_5\}$ must be green. We may assume that u_1v_4 is green. Moreover, v_4v_2 or v_4v_3, say v_4v_2, must be green. Then u_1v_2 has to be red, u_1v_3 green, v_4v_3 red, and v_4v_1 green. From v_3 there must be at least one red edge to u_2 or u_3, say to u_2. Then u_2v_2 has to be green, u_2v_4 and v_2u_3 red, v_5u_2 and u_3v_3 green, and v_5v_2 and v_5v_1 red. But this produces a red $K_4 - e$ in $[v_1, v_2, v_3, v_5]$.

- **I.2:** There is at least one red edge from every vertex of the red K_3 to U. As a red $K_4 - e$ has to be avoided, we may assume that the edges v_1u_1, v_2u_2, and v_3u_3 are red and all other edges between U and $\{v_1, v_2, v_3\}$ are green. At least one edge between $\{u_1, u_2\}$ and $\{v_4, v_5\}$ has to be green, say u_1v_4. But then a green $2K_3$ occurs if one of the edges v_4v_2 or v_4v_3 is green, and a red $K_4 - e$ exists if both edges are red.

Case II: $[V]$ does not contain a red K_3. Since a green K_3 in $[V]$ would lead to a green $2K_3$, $[V]$ must consist of a red and a green C_5. Let the edges v_1v_2, v_2v_3, v_3v_4, v_4v_5, and v_5v_1 be red. We know that there must be a red K_3 in χ. Thus, we may assume that $[u_1, v_1, v_2]$ is a red K_3. No red $K_4 - e$ forces u_1v_3 and u_1v_5 to be green. At least one of the edges from u_2 to v_1 or v_2 has to be green and additionally at least one of the edges from u_3 to v_1 or v_2. We may assume that u_2v_1 is green. Then v_1u_3 has to be red, u_3v_2 green, u_2v_2, u_2v_4, and u_3v_4 red, and u_2v_3 and u_3v_5 green. But in that case $[u_3, v_2, v_5]$ and $[u_2, v_1, v_3]$ yield a green $2K_3$, a contradiction. ■

Lemma 3.9

$$r(K_5 - P_3, 2K_3) \leq 11, \quad r(K_5 - e, 2K_3) \leq 13, \quad r(K_5, 2K_3) \leq 16.$$

Proof. Assume that a $(G, 2K_3)$-coloring χ of K_n exists where $n = 11$ in case of $G = K_5 - P_3$, $n = 13$ in case of $G = K_5 - e$, and $n = 16$ in case of $G = K_5$. Since $n = r(G, K_3) + 2$, by Lemma 3.7 there must be a green K_5 in χ. Let U be the vertex set of a green K_5 and let V be the set of the $n - 5$ remaining vertices of K_n. Note that no green $2K_3$ implies at least four red edges from every vertex in V to U.

If $|V| \geq 6$, i.e. $n \geq 11$, a red K_3 must occur in $[V]$ because $r(K_3, K_3) = 6$ and a green K_3 is forbidden in $[V]$. From the three vertices of a red K_3 in $[V]$ there must be at least $3 \cdot 4 = 12$ red edges to U. Thus, two vertices in U are joined red to every vertex of the red K_3. This yields a red $K_5 - e$, and we obtain a contradiction in case of $G = K_5 - P_3$ or $G = K_5 - e$.

If $n = 16$, then $|V| = 11$, and a red K_4 must occur in $[V]$ because $r(K_4, K_3) = 9$. There must be at least $4 \cdot 4 = 16$ red edges from the vertices of a red K_4 in $[V]$ to U. Thus, one vertex in U is joined red to every vertex of the red K_4 producing a red K_5, and again we have obtained a contradiction. ■

The following three lemmas will be useful for proving Lemma 3.13.

Lemma 3.10 *Any 2-coloring of K_{10} containing a red K_4 and a green K_3 with disjoint vertex sets contains a red subgraph $K_5 - P_3$ or a green subgraph $2K_3$.*

Proof. Assume to the contrary that we have a $(K_5 - P_3, 2K_3)$-coloring of K_{10} containing a red K_4 with vertex set $U = \{u_1, u_2, u_3, u_4\}$ and a green K_3 with vertex set $V = \{v_1, v_2, v_3\}$. Let $W = \{w_1, w_2, w_3\}$ be the set of the three remaining vertices of K_{10}.

No red $K_5 - P_3$ forces at least three green edges from every vertex in $V \cup W$ to U. Thus, from V to U as well as from W to U there are at least 9 green edges. This yields a vertex in U, say u_1, joined green to v_1, v_2, and v_3. Moreover, a vertex in U has to be joined green to w_1, w_2, and w_3. This forces $[W]$ to be a red K_3 since otherwise a green $2K_3$ would occur. We may assume that the edges v_1u_2 and v_1u_3 are green. Since every vertex in W must be joined green to u_2 or u_3, no green $2K_3$ implies that all edges from v_1 to W are red yielding a red $K_4 = [v_1, w_1, w_2, w_3]$. Avoiding a red $K_5 - P_3$ we may assume that the edges v_2u_2, v_2w_1, and v_2w_2 are green. But then a green $2K_3$ occurs if one of the edges from u_2 to w_1 or w_2 is green, and otherwise we obtain a red $K_5 - P_3$. ■

Lemma 3.11 *Any 2-coloring of K_{10} containing a green subgraph K_5 contains a red subgraph $K_5 - K_3$ or a green subgraph $2K_3$.*

Proof. Assume to the contrary that we have a $(K_5 - K_3, 2K_3)$-coloring of K_{10} containing a green subgraph K_5. Let U be the vertex set of a green K_5 and let V be the set of the remaining vertices of K_{10}. As no green $2K_3$ shall occur, there must be vertices v_1 and v_2 in V where v_1v_2 is red. If $q_g(v_1, U) \geq 2$ or $q_g(v_2, U) \geq 2$, we obtain a green $2K_3$. The remaining case is that $q_r(v_1, U) \geq 4$ and $q_r(v_2, U) \geq 4$. But then v_1 and v_2 have three common red neighbors in U yielding a red subgraph $K_5 - K_3$. ■

Lemma 3.12 *The green subgraph of any (K_4, K_3)-coloring χ of K_7 is isomorphic to one of the graphs in Figure 2, where dotted edges may be either red or green.*

Proof. Since no red K_4 shall occur, the green subgraph must not be bipartite. This forces a green cycle of odd length in χ. Let C_ℓ be a green cycle of maximal odd length ℓ. No green K_3 demands $\ell = 5$ or $\ell = 7$. Let the vertices of K_7 be denoted by $v_0, v_1, \ldots, v_6$.

First let $\ell = 7$ and let the edges $v_iv_{i+1 \pmod 7}$ be green for $i = 0, \ldots, 6$. Then the edges $v_iv_{i+2 \pmod 7}$ have to be red and additionally at least four of the edges $v_iv_{i+3 \pmod 7}$. Thus, the green subgraph of χ is isomorphic to H_1.

The remaining case is $\ell = 5$. Let the edges $v_iv_{i+1 \pmod 5}$ be green for $i = 0, \ldots, 4$. Then the edges $v_iv_{i+2 \pmod 5}$ have to be red. First assume that the edge v_5v_6 is green.

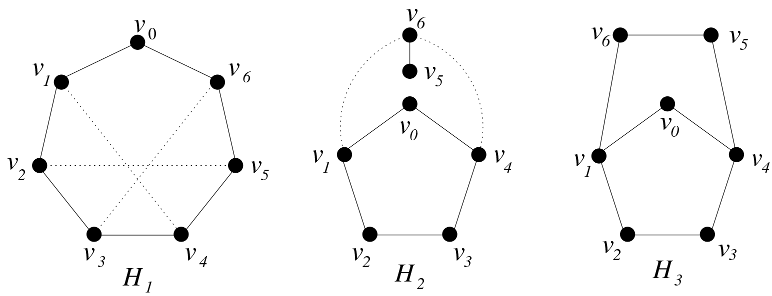

Figure 2: The possible green subgraphs of any (K_4, K_3)-coloring of K_7

No green K_3 and no green C_7 allow at most two green edges between $\{v_0, \ldots, v_4\}$ and $\{v_5, v_6\}$ leading to a green subgraph isomorphic to H_2 or H_3. It remains that v_5v_6 is red. We will prove that in this case we find another green cycle C_5 where the two vertices not belonging to this C_5 are joined green. No red K_4 forces that one of the edges from v_1 or v_4 to v_5 or v_6 is green. We may assume that v_1v_5 is green. This implies that v_5v_0 and v_5v_2 are red. Then one of the edges from v_6 to v_0 or v_2, say v_6v_2, has to be green. This demands that v_6v_3 is red and that v_5v_3 or v_6v_0 has to be green, yielding a green C_5 and a green edge between the two vertices not belonging to this C_5. As this case has already been considered, the proof is done. ■

Lemma 3.13

$$r(G, 2K_3) \leq 10 \quad \textit{for} \quad G \in \{G_{5,16}, G_{5,17}, G_{5,18}\}.$$

Proof. Assume to the contrary that a $(G, 2K_3)$-coloring χ of K_{10} exists for some $G \in \{G_{5,16}, G_{5,17}, G_{5,18}\}$. Then $r(G, K_3) = 9$ guarantees a green K_3 in χ. Let $U = \{u_1, u_2, u_3\}$ be the vertex set of a green K_3 and let $V = \{v_0, v_1, \ldots, v_6\}$ be the set of the seven remaining vertices of K_{10}. Since $G \subseteq K_5 - P_3$, a red K_4 in $[V]$ is forbidden by Lemma 3.10. Thus, the coloring of $[V]$ has to be a (K_4, K_3)-coloring. By Lemma 3.12 the green subgraph $[V]_g$ of $[V]$ is isomorphic to one of the graphs H_1, H_2, or H_3 in Figure 1.

Case I: $[V]_g$ is isomorphic to H_1.

- $G = G_{5,16} = K_5 - P_4$: No red subgraph G in $[V]$ forces the dotted edges to be green. Note that any red edge v_0u for some $u \in U$ implies that all edges from u to $\{v_2, v_3, v_4, v_5\}$ have to be green. Thus, at least two red edges from v_0 to U yield a green $2K_3$. The same holds if there are two green edges and one red edge from v_0 to U. The remaining case is that all edges from v_0 to U are green. Then only green edges from v_1 and v_6 to U imply a green $2K_3$. If one of the edges from v_1 or v_6 to U, say v_1u_1, is red and no red subgraph G shall occur, then the edges u_1v_3 and u_1v_5 have to be green. But this produces a green $2K_3$

if one of the edges u_1v_4 or u_1v_6 is green. Otherwise we find a red subgraph G in $[u_1, v_1, v_2, v_4, v_6]$.

- $G = G_{5,17} = K_5 - (P_3 \cup K_2)$: The edges v_1v_4 and v_3v_6 have to be green since otherwise we would obtain a red subgraph G in $[v_0, v_1, v_3, v_4, v_5]$ or in $[v_0, v_2, v_3, v_4, v_6]$. Consider now the two red subgraphs $K_4 - e$ in $[v_0, v_1, v_3, v_5]$ and $[v_0, v_2, v_4, v_6]$. If any edge uv_0 is red where $u \in U$ and a red G is avoided, then the edges uv_1 and uv_6 must be green. Moreover, one of the edges uv_2 or uv_4 and one of the edges uv_3 or uv_5 has to be green. This yields a green K_3 with u, v_1, and one of the vertices v_2 or v_4 and another green K_3 with u, v_6, and one of the vertices v_3 or v_5. Thus, we obtain a green $2K_3$ in case of $q_r(v_0, U) \geq 2$. Moreover, also in case of $q_r(v_0, U) = 1$, i.e. $q_g(v_0, U) = 2$, a green $2K_3$ must occur. The remaining case is $q_g(v_0, U) = 3$. If some edge uv where $u \in U$ and $v \in \{v_3, v_4\}$, say $v = v_3$, is green, then either a green $2K_3$ occurs or all the edges uv_2, uv_4, and uv_6 are red, and $[u, v_0, v_2, v_4, v_6]$ contains a red subgraph G. Thus, all edges from v_3 and v_4 to U have to be red. But then $q_r(v_1, U) \geq 2$ or $q_r(v_2, U) \geq 2$ yields a red G. In the remaining case $q_g(v_1, U), q_g(v_2, U) \geq 2$ the vertices v_1 and v_2 have a common green neighbor in U. This yields a green K_3, and the two other vertices in U together with v_0 lead to a green $2K_3$, a contradiction.

- $G = G_{5,18} = K_5 - K_3$: Consider the two red subgraphs $K_4 - e$ in $[v_0, v_1, v_3, v_5]$ and $[v_0, v_2, v_4, v_6]$. No red subgraph G implies $q_g(u, \{v_3, v_5\}) \geq 1$ and $q_g(u, \{v_2, v_4\}) \geq 1$ for every $u \in U$. Thus, $q_g(\{v_3, v_5\}, U), q_g(\{v_2, v_4\}, U) \geq 3$.

  (***i***) First let $q_g(v, U) \geq 2$ for some $v \in \{v_3, v_4\}$. We may assume that the edges v_3u_1 and v_3u_2 are green. Note that a green K_3 with u_1, v_3, and one of the vertices v_2 or v_4 and a green K_3 with u_2, v_3, and one of the vertices v_2 or v_4 must occur.

  Now assume that v_3u_3 is red. It follows that u_3v_5 is green, u_3v_4 and u_3v_6 red, u_3v_2 green, and u_3v_1 and v_2v_5 red. The edges v_1v_4 and v_3v_6 have to be green, the edges v_5u_1 and v_5u_2 have to be red. Only red edges from v_0 to u_1 and u_2 yield a red subgraph G in $[u_1, u_2, v_0, v_2, v_5]$. Thus, we may assume that v_0u_1 is green. This forces u_1v_1 and u_1v_6 to be red. Moreover, v_0u_3 has to be red since otherwise $[v_0, u_1, u_3]$ is a green K_3 leading to a green $2K_3$. No red G in $[v_1, v_3, v_5, u_1, u_2]$ forces v_1u_2 to be green and thereby u_2v_0 has to be red. No red G in $[v_0, v_2, v_4, u_2, u_3]$ implies that u_2v_4 must be green and thereby u_1v_2 has to be red. No red G in $[v_0, v_2, v_5, u_1, u_2]$ forces u_2v_2 to be green, and u_1v_4 must be red. But then $[v_2, v_4, v_6, u_1, u_3]$ contains a red subgraph G.

  It remains that v_3u_3 is green. We already know that $q_g(\{v_2, v_4\}, U) \geq 3$ implying $q_g(v_2, U) \geq 2$ or $q_g(v_4, U) \geq 2$. If $q_g(v_2, U) \geq 2$, we may assume that v_2u_1 and v_2u_2 are green. Then u_3v_4 has to be red and u_3v_2 green as $q_g(u_3, \{v_2, v_4\}) \geq 1$. But now $[v_2, v_3, u_1, u_2, u_3]$ is a green K_5, and Lemma 3.11 yields a contradiction. If $q_g(v_4, U) \geq 2$, we may assume that v_4u_1 and v_4u_2 are green. No green $2K_3$ implies that u_3v_2 is red, and this forces u_3v_4 to be green.

But then $[v_3, v_4, u_1, u_2, u_3]$ is a green K_5 and again we obtain a contradiction using Lemma 3.11.

(***ii***) The remaining case is $q_g(v_2, U) \geq 2$ and $q_g(v_5, U) \geq 2$. We may assume that v_2u_1 and v_2u_2 are green. The edge v_2u_3 also has to be green because otherwise u_3v_4 and one of the edges u_3v_3 or u_3v_5 must be green yielding a green $2K_3$. By symmetry, the edges from v_5 to U also have to be green. But then any green edge from v_1 or v_6 to U yields a green $2K_3$, and only red edges between $\{v_1, v_6\}$ and U yield a red subgraph G.

Case II: $[V]_g$ is isomorphic to H_2. Since $[v_0, v_2, v_3, v_5, v_6]$ contains a red $K_5 - 2K_2$ and $G_{5,16}, G_{5,17} \subseteq K_5 - 2K_2$, only $G = G_{5,18} = K_5 - K_3$ is left. Note that $[v_0, v_1, v_3, v_5]$, $[v_1, v_2, v_4, v_5]$, $[v_0, v_2, v_3, v_5]$, $[v_1, v_3, v_4, v_5]$, $[v_0, v_2, v_4, v_5]$, $[v_0, v_2, v_3, v_6]$, $[v_0, v_2, v_5, v_6]$, and $[v_0, v_3, v_5, v_6]$ contain a red $K_4 - e$ each.

First let v_0u_1 be red. Then the edges from u_1 to v_2, v_3, v_5, and v_6 have to be green. No green $2K_3$ forces v_0u_2 or v_0u_3, say v_0u_2, to be red. But now all edges from u_2 to v_2, v_3, v_5, and v_6 must be green, and we obtain a green $2K_3$.

The remaining case is that all edges from v_0 to U are green. Only red edges from v_1 and v_4 to U yield a red subgraph G. Thus, we may assume that v_1u_1 is green. Then u_1v_2 has to be red, u_1v_5 green, and u_1v_6 red. Moreoer, one of the edges from v_5 to u_2 or u_3, say v_5u_3, has to be red. This forces u_3v_i to be green for $i = 1, 2, 3, 4$ yielding a green $2K_3$, a contradiction.

Case III: $[V]_g$ is isomorphic to H_3. Again $[v_0, v_2, v_3, v_5, v_6]$ contains a red $K_5 - 2K_2$ and only $G = G_{5,18} = K_5 - K_3$ is left. The red subgraph $K_4 - e$ in $[v_0, v_2, v_4, v_6]$ forces $q_g(u, \{v_2, v_6\}) \geq 1$ for every $u \in U$. Thus, since v_2 and v_6 are equivalent, we may assume that the edges v_2u_1 and v_2u_2 are green.

First let v_2u_3 also be green. The red subgraph $K_4 - e$ in $[v_0, v_1, v_3, v_5]$ forces $q_g(u, \{v_3, v_5\}) \geq 1$ for every $u \in U$. Then no green $2K_3$ implies that all edges from v_4 to U are red. No red subgraph G in $[v_1, v_4, u_1, u_2, u_3]$ demands $q_g(v_1, U) \geq 1$, and we may assume that v_1u_1 is green. But then we obtain a green $2K_3$ if one of the edges u_1v_0 or u_1v_6 is green, and otherwise $[u_1, v_0, v_2, v_3, v_6]$ contains a red subgraph G.

The remaining case is that v_2u_3 is red. No red G in $[u_3, v_0, v_2, v_4, v_6]$ implies that u_3v_6 is green. Now u_3v_1 and u_3v_5 have to be red, u_3v_0 and u_3v_3 green, as there is no red G in $[u_3, v_0, v_2, v_3, v_5]$ or in $[u_3, v_0, v_1, v_3, v_5]$, and u_3v_4 red. Furthermore, no red G forces $q_g(u_1, \{v_0, v_6\})$, $q_g(u_2, \{v_0, v_6\}) \geq 1$. This yields a green K_3 with u_1, u_3, and one of the vertices v_0 or v_6, and another green K_3 with u_2, u_3, and once again one of the vertices v_0 or v_6. No green $2K_3$ then implies $q_r(\{u_1, u_2\}, \{v_1, v_3\}) = 4$ yielding a red subgraph G in $[u_1, u_2, v_1, v_3, v_5]$, a contradiction. ■

Of course it would be nice to fill the three gaps in Table 1 where the exact values of $r(G, H)$ are unknown. But this seems to be somewhat out of reach without employing computer algorithms.

References

[1] A. Babak, S. P. Radziszowski and Kung-Kuen Tse, Computation of the Ramsey number $R(B_3, K_5)$, *Bull. Inst. Combin. Appl.* **41** (2004), 71–76.

[2] R. Bolze and H. Harborth, The Ramsey number $r(K_4 - e, K_5)$, *The Theory and Applications of Graphs*, (Kalamazoo, Mich., (1980)), Wiley & Sons, New York, (1981), 109–116.

[3] S. A. Burr, P. Erdős and J. H. Spencer, Ramsey Theorems for Multiple Copies of Graphs, *Trans. Amer. Math. Soc.* **209** (1975), 87–99.

[4] V. Chvátal and F. Harary, Generalized Ramsey theory for graphs III: Small off-diagonal numbers, *Pacific J. Math.* **41** (1972), 335–345.

[5] M. Clancy, Some small Ramsey numbers, *J. Graph Theory* **1** (1977), 89–91.

[6] C. Clapham, G. Exoo, H. Harborth, I. Mengersen and J. Sheehan, The Ramsey number of $K_5 - e$, *J. Graph Theory* **13** (1989), 7–15.

[7] G. Exoo, H. Harborth and I. Mengersen, The Ramsey number of K_4 versus $K_5 - e$, *Ars Combin.* **25 A** (1988), 277–286.

[8] R. J. Faudree, C. C. Rousseau and R. H. Schelp, All triangle-graph Ramsey numbers for connected graphs of order six, *J. Graph Theory* **4** (1980), 293–300.

[9] G. R. T. Hendry, Ramsey numbers for graphs with five vertices, *J. Graph Theory* **13** (1989), 245–248.

[10] G. R. T. Hendry, The Ramsey numbers $r(K_2 + \overline{K_3}, K_4)$ and $r(K_1 + C_4, K_4)$, *Utilitas Math.* **35** (1989), 40–54.

[11] M. Hoeth and I. Mengersen, Ramsey numbers for graphs of order four versus connected graphs of order six, *Utilitas Math.* **57** (2000), 3–19.

[12] C. J. Jayawardene and C. C. Rousseau, The Ramsey numbers for a quadrilateral vs. all graphs on six vertices, *J. Combin. Math. Combin. Comput.* **35** (2000), 71–87.

[13] C. J. Jayawardene and C. C. Rousseau, The Ramsey number for a cycle of length five vs. a complete graph of order six, *J. Graph Theory* **35** (2000), 99–108.

[14] C. J. Jayawardene and C. C. Rousseau, Ramsey numbers $r(C_5, G)$ for all graphs G of order six, *Ars Combin.* **57** (2000), 163–173.

[15] C. J. Jayawardene and C. C. Rousseau, The Ramsey numbers for a quadrilateral vs. all graphs on six vertices (Erratum), *J. Combin. Math. Combin. Comput.* **51** (2004), 221.

[16] R. Lortz and I. Mengersen, Ramsey numbers for special trees versus all connected graphs of order six, (in preparation).

[17] B. D. McKay and S. P. Radziszowski, $R(4,5) = 25$, *J. Graph Theory* **19** (1995), 309–322.

[18] J. McNamara, $r(K_4 - e, K_6) = 21$, (not published).

[19] J. McNamara and S. P. Radziszowski, The Ramsey numbers $R(K_4-e, K_6-e)$ and $R(K_4 - e, K_7 - e)$, *Congr. Numer.* **81** (1991), 89–96.

[20] S. P. Radziszowski, Small Ramsey Numbers, *Electronic J. Combin.* (2009), DS1.12.

[21] S. P. Radziszowski, J. Stinehour and Kung-Kuen Tse, Computation of the Ramsey number $R(W_5, K_5)$, *Bull. Inst. Combin. Appl.* **47** (2006), 53–57.

[22] C. C. Rousseau and C. J. Jayawardene, The Ramsey number for a quadrilateral vs. a complete graph on six vertices, *Congr. Numer.* **123** (1997), 97–108.

[23] Yuansheng Yang and G. R. T. Hendry, The Ramsey number $r(K_1 + C_4, K_5 - e)$, *J. Graph Theory* **19** (1995), 13–15.

(Received 14 Sep 2010; revised 15 May 2011)