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On a conjecture on total domination in claw-free cubic graphs: proof and new upper bound

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Abstract

In 2008, Favaron and Henning proved that if G is a connected claw-free cubic graph of order $n \geq 10$, then the total domination number $\gamma_t(G)$ of G is at most $\frac{5}{11}n$, and they conjectured that in fact $\gamma_t(G)$ is at most $\frac{4}{9}n$ (see [O. Favaron and M.A. Henning, *Discrete Math.* 308 (2008), 3491–3507] and [M.A. Henning, *Discrete Math.* 309 (2009), 32–63]). In this paper, in a first step, we prove this conjecture and show that the bound is reached for exactly two graphs of order 18. In a second step, we prove that if G is a connected claw-free cubic graph of order $n \geq 20$, then $\gamma_t(G) \leq \frac{10}{23}n$, and we show that this second bound is not reached. Henning and Southey (see [*Discrete Math.* 310 (2010), 2984–2999] also proved the initial conjecture, but in a less natural way. Moreover, they gave two graphs for which the bound is reached without proving that there are no others. An open problem is proposed in the last section.

1 Introduction and notation

We work on simple graphs G (no loops or multiple edges). The set $V(G)$ is the *vertex set* of G and $v(G)$ is its *order*, that is, the number of the vertices of G. A *neighbor* of a vertex x of G is a vertex y such that xy is an edge of G (we say then that x and y are *adjacent*). The number of the neighbors of x is the *degree* $d_G(x)$ of x. The graph G is *regular* of degree d if every vertex of G has degree d. When $d = 3$, G is a *cubic graph*. A *subgraph* of G is a graph H whose vertices are vertices of G and whose edges are edges of G. An *induced subgraph* of G is a subgraph H of G such that every edge xy of G, with x and y in $V(H)$, is an edge of H. For a subset S of $V(G)$, $G[S]$ is the induced subgraph whose vertex set is S (we say also that $G[S]$ is the subgraph of G induced by the vertices of S). A *spanning subgraph* of G is a subgraph H of G with $V(H) = V(G)$. For a subgraph H of G and a vertex x of H, a neighbor of x which is not a vertex of H is an H-*external* neighbor of x. When it is clear from the context, we omit H. We say that a subgraph G_1 of G is adjacent to a subgraph G_2 of G if there exists an edge between a vertex of G_1 and a vertex of G_2. If $G_1, \ldots, G_k$ are subgraphs of G, we say that G is a *vertex-disjoint union* of the subgraphs G_i, $1 \leq i \leq k$, if $V(G_1), \ldots, V(G_k)$ is a partition of $V(G)$.

For $n \geq 2$, the *complete graph* K_n is the graph of order n in which any two distinct vertices are adjacent. The *claw* $K_{1,3}$ is the graph of order 4 drawn in Fig. 1. A *claw-free graph* is a $K_{1,3}$-free graph G. This means that for every vertex x of G, the subgraph induced by any three neighbors of x, has at least one edge.

The graph G is *connected* if for every partition of the vertex set $V(G)$ into two non-empty sets X and Y, there is an edge xy with x in X and y in Y. This means that for every non-spanning subgraph H of G, there exists a vertex of H having an H-external neighbor.

Two simple graphs G and G' are *isomorphic* if there exists a bijection θ from $V(G)$ into $V(G')$ such that for any two vertices x and y of G, x and y are adjacent in G if and only if $\theta(x)$ and $\theta(y)$ are adjacent in G'.

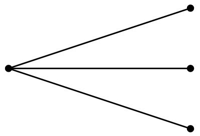


Fig. 1: The claw $K_{1,3}$

A *total dominating set* (TDS) of a graph G with no isolated vertices is a subset S of $V(G)$ such that every vertex of G is adjacent to a vertex of S. The *total domination number* $\gamma_t(G)$ of G is the minimum cardinality of a TDS of G. Total domination in graphs was introduced by Cockaine, Dawes and Hedetniemi (see[1]) and there is now a wide literature on this subject. Many of the results are in the books of Haynes, Hedetniemi and Slater ([2] and [3]).

In 2008, Favaron and Henning proved that if G is a connected claw-free cubic graph of order $n \geq 10$, then the total domination number $\gamma_t(G)$ of G is at most $\frac{5}{11}n$ and they conjectured that in fact $\gamma_t(G)$ is at most $\frac{4}{9}n$ (see [4] and [5]). In our paper, we prove this conjecture and then, by a deeper analysis, we improve the proposed bound. More precisely:

Theorem 1.1 **a)** *For every connected claw-free cubic graph G of order $n \geq 10$, we have $\gamma_t(G) \leq 4n/9$.*
b) *For every connected claw-free cubic graph G of order $n \geq 20$, we have $\gamma_t(G) \leq \frac{10}{23}n$.*

In fact Theorem 1.1.a was independently proved by Henning and Southey (see [6]), but our proof is completely different. In addition, our method allows us to prove that the bound is reached for exactly two connected claw-free cubic graphs (Henning and Southey pointed out these two graphs without proving that there are no others), but especially it is the starting point for proving Theorem 1.1.b.

In Section 2, we give some intermediate results, in Sections 3 and 4 we prove Theorem 1.1, and in Section 5 we study the sharpness of the bounds and an open problem will be given. We specify that for all the figures, the edges in bold are the edges of the graph induced by a TDS.

2 Intermediate results

Recall that a *diamond* is a graph of order 4, consisting of two triangles sharing a common edge (see Fig. 2).

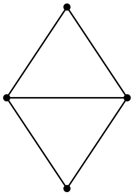


Fig. 2: A diamond D

Now G is a connected claw-free cubic graph of order $n \geq 10$. By a diamond of G, we mean a diamond D which is a subgraph of G. The *free vertices* of D are the two vertices of D of degree two in D. The two other vertices are the *non-free* vertices of D. We observe that since G is connected, the two free vertices of D are not adjacent (in other words a diamond of G is an induced subgraph of G). We also observe that these two free vertices cannot have a common neighbor not in D. Indeed, suppose the opposite and then, let x be a common neighbor, not in D, of the free vertices a and b of D. Since all the vertices of D are of degree 3 in G, x has a neighbor y not in D. Since G is a claw-free graph, there exist at least two vertices of $\{a, b, y\}$ which are adjacent. But then, at least one of the vertices a and b would be of degree greater than 3, which is not possible. Lastly, we observe that every edge of D is incident with a non-free vertex of D. By a *proper triangle*, we mean a triangle T of G which is not a subgraph of a diamond of G. This means that any two vertices of T have exactly one common neighbor, namely the third vertex of T.

A *piece* P of G is either a diamond of G or a proper triangle of G. The first result is:

Lemma 2.1 *Two distinct pieces of G are vertex-disjoint.*

Proof. Let P and P' be two distinct pieces of G. Suppose that P and P' are not vertex-disjoint, and then let a be a vertex of P and P'. Since a has degree at least 2 in P and P', it follows that a has a neighbor b which is a vertex of P and P'. This means that ab is an edge of P and P'. We distinguish three cases:

Case 1: P and P' are proper triangles.
Since $V(P) \neq V(P')$ and since a proper triangle cannot be a subgraph of a diamond of G, clearly this is not possible.

Case 2: P and P' are diamonds of G.
Observe first that a vertex of $V(P) \cap V(P')$ cannot be a non-free vertex of P and P' (otherwise we would have $V(P) = V(P')$ and then P and P' would be identical. We may assume that a is a free vertex of P, and then b is a non-free vertex of P. This implies that b is a free vertex of P', and then a is a non-free vertex of P'. Let d be the other non-free vertex of P. Since a is a vertex of P, d is neighbor of a. Since a is a non-free vertex of P', we deduce that d is also a vertex of P', and then bd is an edge of P and P'. Since d is non-free in P, it follows that d is free in P'. But then, the vertices b and d are free and adjacent in P', which is not possible.

Case 3: P and P' are of distinct types.
We may suppose that P is a diamond and that P' is a proper triangle of G. Since a and b are adjacent vertices of P, at least one of these vertices, say a is a non-free vertex of P. Then the third vertex c of P' is also a vertex of P (because c is a neighbor of a which is non-free in P). We deduce that the triangle P' is a subgraph of P, which is not possible.

In conclusion, we may assert that P and P' are vertex-disjoint. ■

Lemma 2.2 *Every vertex x of G is contained in a unique piece of G.*

Proof. Since G is claw-free, x is contained in a triangle and then in a piece of G. The uniqueness follows from Lemma 2.1. ■

Lemmas 2.1 and 2.2 imply that G is vertex-disjoint union of its pieces.

For $m \geq 2$, a *chain* of pieces of G of length $m-1$ consists of m vertex-disjoint pieces $P_1, \dots, P_m$, such that for every i, $1 \leq i \leq m-1$, there exists an edge e_i between P_i and P_{i+1}. The *chain-graph* $H = P_1 \dots P_m$ of G is the subgraph of G whose vertex set is the union of the vertex sets of the pieces P_i, $1 \leq i \leq m$, and whose edges are the edges of the pieces P_i, $1 \leq i \leq m$ and the edges e_i, $1 \leq i \leq m-1$. We observe that the edges e_i are independent. We further observe that a chain-graph is not necessarily an induced subgraph of G.

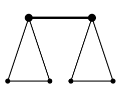
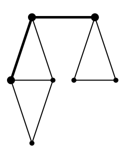
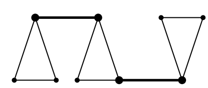
The *left piece* of H is the piece P_1 and the *right piece* of G is the piece P_m. We specify that the terms "left" and "right" are relative to the labeling $H = P_1 \dots P_m$ and also to the drawing of H (from left to right by beginning with P_1). An *end-vertex* of H is a vertex of P_1 or P_m. A *free left vertex* of H is a vertex of the left piece, of degree 2 in H. A *free right vertex* of H is a vertex of the right piece, of degree 2 in H. A *free end-vertex* of H is a free left or a free right piece of H. An *internal piece* of H is a piece P_i with $2 \leq i \leq m-1$. Observe that a chain-graph of length 1 has no internal pieces. A *free internal vertex* of H is a vertex of an internal piece, of degree

2 in H. Observe that an internal diamond has no free internal vertices and that an internal proper triangle has exactly one free internal vertex.

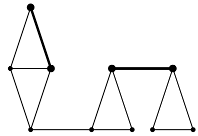
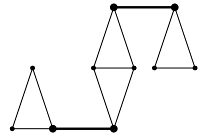
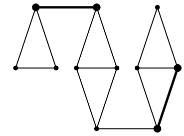
We put $\bar{H} = P_m \dots P_1$, and clearly $\bar{H}$ is a chain-graph. In fact H and $\bar{H}$ are identical but differently labeled and drawn. If $H = P_1 \dots P_j$ and $H' = P_1' \dots P_k'$ are vertex-disjoint chain-graphs, and if a free right vertex of H is adjacent to a free left vertex of H', we may define the chain-graph $HH' = P_1 \dots P_j P_1' \dots P_k'$.

We consider now the set $\mathcal{H}$ of the chain-graphs $H_1, \dots, H_{13}$, labeled and drawn in Fig. 3. The total domination numbers of the graphs of $\mathcal{H}$ are:

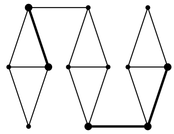
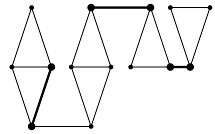
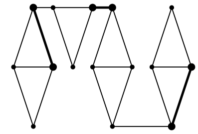
$$\begin{aligned}
\gamma_t(H_1) &= 2 = \frac{1}{3}v(H_1), & \gamma_t(H_2) &= 3 = \tfrac{3}{7}v(H_2), & \gamma_t(H_3) &= 4 = \frac{4}{9}v(H_3),\\
\gamma_t(H_4) &= 4 = \frac{2}{5}v(H_4), & \gamma_t(H_5) &= 4 = \tfrac{2}{5}v(H_5), & \gamma_t(H_6) &= 4 = \frac{4}{11}v(H_6),\\
\gamma_t(H_7) &= 5 = \frac{5}{12}v(H_7), & \gamma_t(H_8) &= 6 = \tfrac{3}{7}v(H_8), & \gamma_t(H_9) &= 6 = \frac{2}{5}v(H_9),\\
\gamma_t(H_{10}) &= 6 = \frac{2}{5}v(H_{10}), & \gamma_t(H_{11}) &= 6 = \tfrac{3}{8}v(H_{11}), & \gamma_t(H_{12}) &= 8 = \frac{2}{5}v(H_{12}),\\
\gamma_t(H_{13}) &= 8 = \frac{8}{19}v(H_{13}).
\end{aligned}$$

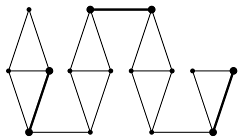
Graph $H_1 = TT$ Graph $H_2 = DT$ Graph $H_3 = TTT$

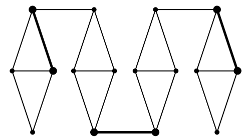
Graph $H_4 = DTT$ Graph $H_5 = TDT$ Graph $H_6 = TDD$

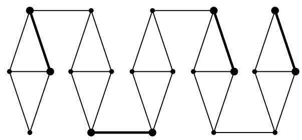
Graph $H_7 = DDD$ Graph $H_8 = DDTT$ Graph $H_9 = DTDD$



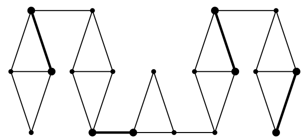
Graph $H_{10} = DDDT$



Graph $H_{11} = DDDD$



Graph $H_{12} = DDDDD$



Graph $H_{13} = DDTDD$

Fig. 3: The chain-graphs of the set $\mathcal{H}$

We observe that for every chain-graph H of $\mathcal{H}$, we have $\gamma_t(H) \leq \dfrac{4}{9}v(H)$, with equality for H_3. We also observe that the graphs of $\mathcal{H}$ containing a free internal vertex are those of the set $\mathcal{H}' = \{H_3, H_4, H_8, H_9, H_{13}\}$ and that each of these graphs contains exactly one free internal vertex. We put $\bar{\mathcal{H}} = \{\bar{H};\ H \in \mathcal{H}\}$.

A chain-graph of type H_i (of type $\bar{H}_i$) is a chain-graph having, to within isomorphisms, the same labeling as H_i (as $\bar{H}_i$). For example, a chain-graph of type H_2 is a chain-graph D_1T_1 where D_1 is a diamond of G and T_1 is a proper triangle of G; a chain-graph of type $\bar{H}_6$ is a chain-graph $D_1D_2T_1$ where D_1 and D_2 are diamonds and T_1 is a proper triangle. A subgraph of type in $\mathcal{H}$ is a subgraph G' of G, isomorphic to a graph of $\mathcal{H}$. If the type H_i is specified, we say that G' is of type H_i.

When we consider a subgraph G' of type H_i, we always represent it by a chain-graph of type H_i, and then speaking about left vertex, right vertex or internal vertex of G' has a sense.

If $G_1, \ldots, G_k$ are vertex-disjoint chain-graphs with G_i of type H'_{p_i} in $\mathcal{H}$ or in $\bar{\mathcal{H}}$ for $1 \leq i \leq k$, and if $G' = G_1 \ldots G_k$ is a chain-graph of G, we say that this chain-graph is of type $H'_{p_1} \ldots H'_{p_k}$. An important remark is that G' is a vertex-disjoint union of subgraphs of type in $\mathcal{H}$. For example, a chain-graph $G' = T_1T_2D_1T_3D_2D_3$ may be considered as a chain-graph of type $\bar{H}_4H_6$ but it may also be considered as a chain-graph of type H_1H_9. Then G' is a vertex-disjoint union of a subgraph of type H_4 and of a subgraph of type H_6, but it is also a vertex-disjoint union of a subgraph of type H_1 and of a subgraph of type H_9.

Let m be the maximum number of vertices of G covered by a vertex-disjoint union of subgraphs of G of type in $\mathcal{H}$. Since G is connected, it is easy to see that G contains a subgraph of type in $\{H_1, H_2, H_7\}$, and consequently $m > 0$. For a set S of vertex-disjoint subgraphs of G of type in $\mathcal{H}$ and covering m vertices of G, we put $W(S) = \bigcup_{H \in S} V(H)$. Now $F(S)$ is the set of the subgraphs of S of type in $\mathcal{H}'$ and $f(S)$ is its cardinality. We denote by $f_3(S)$ the number of the subgraphs of S of type H_3.

Among the sets of vertex-disjoint subgraphs of G of type in $\mathcal{H}$ and covering m vertices of G, we consider sets S with a minimum number of subgraphs of G of type in $\mathcal{H}'$ (**Condition C1**), and among the sets verifying C1, we choose a set $S = \{H'_j;\ 1 \leq j \leq r\}$ with a minimum number of subgraphs of type H_3 (**condition C2**). It is clear that if $m < n$, then $V(G) \setminus W(S)$ is a vertex-disjoint union of proper triangles and diamonds of G. We state:

Lemma 2.3

- **a)** *Each vertex of a proper triangle T of $V(G) \setminus W(S)$ has exactly one neighbor in $W(S)$ and this neighbor is a free internal vertex of a graph of $F(S)$.*
- **b)** *Let D be a diamond of $V(G) \setminus W(S)$. At least one of its free vertices has a neighbor in $W(S)$, and a neighbor in $W(S)$ of a free vertex of D is either a free right vertex of a subgraph of S of type H_2, or a free internal vertex of a subgraph of S of type in $\{H_4, H_9, H_{13}\}$.*

Proof. a) Let x be a free vertex of the triangle T. Now x has exactly one neighbor z which is not in T. Suppose that z is not in $W(S)$. Then z is contained in a piece P of $V(G) \setminus W(S)$, vertex-disjoint with T. Since the subgraph PT is of type in $\mathcal{H}$, $S \cup \{PT\}$ would be a set of vertex-disjoint subgraphs of type in $\mathcal{H}$ covering $m + 6$ or $m + 7$ vertices of G, which, by maximality of m, is not possible. So z is a vertex of $W(S)$.

Suppose that z is not a free internal vertex of a subgraph of $F(S)$. Then z is either a left vertex of a subgraph H'_j of S and this yields the chain-graph TH'_j, or a right vertex of a subgraph H'_j of S and this yields the chain-graph H'_jT. A simple verification shows that a chain-graph TH or HT, with H a chain-graph of type in $\mathcal{H}$, is either a chain graph of type in $\mathcal{H} \cup \bar{\mathcal{H}}$ or a chain-graph G_1G_2 with chain-graphs G_1 and G_2 of type in $\mathcal{H} \cup \bar{\mathcal{H}}$. This implies that TH or HT is either a subgraph of type in $\mathcal{H}$, or a vertex-disjoint union of two subgraphs of type in $\mathcal{H}$. Then the set $W(S) \cup V(T)$ would be covered by a vertex-disjoint union of subgraphs of type in $\mathcal{H}$, which, by maximality of m, is not possible. Consequently z is a free internal vertex of a subgraph of $F(S)$, and the result is proved.

b) Let a and c be the free vertices of D. Suppose that a and c have no neighbors in $W(S)$. Since G does not contain a K_4, vertex a has a neighbor a_1 not in D but in $V(G) \setminus W(S)$. This neighbor a_1 is contained in a piece P_1 of $V(G) \setminus W(S)$ vertex-disjoint with D (by Lemma 1.1). This yields a chain-graph DP_1. Since a chain-graph DT is of type in $\mathcal{H}$, by maximality of m, P_1 is a diamond, and a_1 is a free vertex of P_1. Since c is adjacent neither with a_1, nor with the other free vertex of P_1 (because G is a connected cubic graph of order $n > 8$), it follows that c has a neighbor c_1, not in D and not in P_1 but in $V(G) \setminus W(S)$. The vertex c_1 is contained in a piece P_2 of $V(G) \setminus W(S)$, vertex-disjoint with D and P_1. This yields the subgraph P_2DP_1, which is of type in $\mathcal{H}$. This means that the set $W(S) \cup V(P_2) \cup V(D) \cup V(P_1)$ would be covered by a vertex-disjoint union of subgraphs of type in $\mathcal{H}$, which, by maximality of m, is not possible. Consequently, at least one of the free vertices of D has a neighbor in $W(S)$.

This being settled, let a be a free vertex of D having a neighbor a_1 in $W(S)$. As in the proof of 2.3.a, by verification, we prove that a subgraph DH or HD with $H \in \mathcal{H}$, $H \neq H_2$, is either a subgraph of type in $\mathcal{H}$, or a vertex-disjoint union of two subgraphs of type in $\mathcal{H}$. This implies that a_1 cannot be a free end-vertex of a subgraph of S not of type H_2. Since a subgraph DH, with H subgraph of type H_2 is of type H_6, it follows that a_1 cannot be a left vertex of a subgraph of type H_2. We claim that a_1 cannot be the free internal vertex of a subgraph of S of type H_3. Indeed, suppose that a_1 is the free internal vertex of a subgraph $G' = T_1T_2T_3$ of S. Then DT_2T_3 is a subgraph of G of type $H_4 \in \mathcal{H}$. Clearly, $W_1 = (W(S) \setminus V(T_1)) \cup V(D)$ is covered by a vertex-disjoint union of subgraphs of type in $\mathcal{H}$, and since W_1 contains $m+1$ vertices, this is not possible. With a similar reasoning, we deduce that a_1 cannot be the internal free vertex of a subgraph of S of type H_8. So the assertion is proved. ■

This lemma implies that every end-vertex of a subgraph of S, distinct from H_2 has all its neighbors in $W(S)$. We continue with:

Lemma 2.4 *Every chain-graph of $\mathcal{H}'$ is either of type G_1T or of type G_1DD with G_1 in $\mathcal{H} \cup \bar{\mathcal{H}}$ but not in $\mathcal{H}' \cup \bar{\mathcal{H}}'$.*

Proof. By simple verification, the result is immediate. ■

Lemma 2.5 *Let H be a chain-graph of $\mathcal{H}$ distinct from H_1.*
a) *If H is distinct from H_6, then a chain-graph TH or DDH is either of type G_2, with G_2 in $\mathcal{H} \cup \bar{\mathcal{H}}$ but not in $\mathcal{H}' \cup \bar{\mathcal{H}}'$, or of type G_2G_3, with G_2 and G_3 in $\mathcal{H} \cup \bar{\mathcal{H}}$ but not in $\mathcal{H}' \cup \bar{\mathcal{H}}'$.*
b) *If H is distinct from H_2, then a chain graph $T\bar{H}$ or $DD\bar{H}$ is either of type G_2, with G_2 in $\mathcal{H} \cup \bar{\mathcal{H}}$ but not in $\mathcal{H}' \cup \bar{\mathcal{H}}'$, or of type G_2G_3, with G_2 and G_3 in $\mathcal{H} \cup \bar{\mathcal{H}}$ but not in $\mathcal{H}' \cup \bar{\mathcal{H}}'$.*

Proof. **a)** With the help of Fig. 3, we set: a chain-graph TH_2 is of type H_5; a chain-graph TH_3 is of type H_1H_1; a chain-graph TH_4 is of type $\bar{H}_2H_1$; a chain-graph TH_5 is of type H_1H_2; a chain-graph TH_7 is of type $\bar{H}_{10}$; a chain-graph TH_8 is of type H_6H_1; a chain-graph TH_9 is of type $\bar{H}_2H_6$; a chain-graph TH_{10} is of type H_6H_2; a chain-graph TH_{11} is of type $\bar{H}_2H_7$; a chain-graph TH_{12} is of type H_6H_7; a chain-graph TH_{13} is of type H_6H_6.

We also set: a chain-graph DDH_2 is of type H_{10}; a chain-graph DDH_3 is of type $\bar{H}_6H_1$; a chain-graph DDH_4 is of type H_7H_1; a chain-graph DDH_5 is of type $\bar{H}_6H_2$; a chain-graph DDH_7 is of type H_{12}; a chain-graph DDH_8 is of type $H_{11}H_1$; a chain-graph DDH_9 is of type H_7H_6; a chain-graph DDH_{10} is of type $H_{11}H_2$; a chain-graph DDH_{11} is of type H_7H_7; a chain-graph DDH_{12} is of type $H_{11}H_7$; a chain-graph DDH_{13} is of type $H_{11}H_6$. This proves the assertion.

b) Again with the help of Fig. 3, we set: a chain-graph $T\bar{H}_3$ is of type H_1H_1; a chain-graph $T\bar{H}_4$ is of type $H_1\bar{H}_2$; a chain-graph $T\bar{H}_5$ is of type H_1H_2; a chain-graph $T\bar{H}_6$ is of type $\bar{H}_2H_2$; a chain-graph $T\bar{H}_7$ is of type $\bar{H}_{10}$; a chain-graph $T\bar{H}_8$ is of type H_1H_6; a chain-graph $T\bar{H}_9$ is of type $H_6\bar{H}_2$; a chain-graph $T\bar{H}_{10}$ is of type H_1H_7; a chain-graph $T\bar{H}_{11}$ is of type $\bar{H}_2H_7$; a chain-graph $T\bar{H}_{12}$ is of type H_6H_7; a chain-graph $T\bar{H}_{13}$ is of type H_6H_6.

We also set: a chain-graph $DD\bar{H}_3$ is of type $\bar{H}_6H_1$; a chain-graph $DD\bar{H}_4$ is of type $\bar{H}_6\bar{H}_2$; a chain-graph $DD\bar{H}_5$ is of type $\bar{H}_6H_2$; a chain-graph $DD\bar{H}_6$ is of type H_7H_2; a chain-graph $DD\bar{H}_7$ is of type H_{12}; a chain-graph $DD\bar{H}_8$ is of type $\bar{H}_6H_6$; a chain-graph $DD\bar{H}_9$ is of type $H_{11}\bar{H}_2$; a chain-graph $DD\bar{H}_{10}$ is of type $\bar{H}_6H_7$; a chain-graph $DD\bar{H}_{11}$ is of type H_7H_7; a chain-graph $DD\bar{H}_{12}$ is of type $H_{11}H_7$; a chain-graph $DD\bar{H}_{13}$ is of type $H_{11}H_6$. This proves the second assertion. ■

Lemma 2.6 *Let x be a right vertex of a subgraph G' of $F(S)$ with an external neighbor y.*
a) *If G' is of type H_3, y is either in a subgraph of type H_1 of S or the internal vertex of a subgraph of $F(S)$.*
b) *If G' is not of type H_3, y is either in a subgraph of type H_1 of S, or a right vertex of a subgraph of type H_2 of S, or a left vertex of a subgraph of type H_6 of S, or the internal vertex of a subgraph of $F(S)$.*

Proof. For the moment, G' is of type not necessarily distinct from H_3. By Lemma 2.4, G' is of type G_1H' with G_1 in $\mathcal{H} \cup \bar{\mathcal{H}}$ but not in $\mathcal{H}' \cup \bar{\mathcal{H}}'$ and with $H' \in \{T, DD\}$. It was seen that y is in $W(S)$. Suppose first that y is a free left vertex of a subgraph H of S of type distinct from H_1 and H_6. This yields the chain-graph $G'H$. By Lemma 2.5.a, $H'H$ is either of type G_2, with G_2 in $\mathcal{H} \cup \bar{\mathcal{H}}$ but not in $\mathcal{H}' \cup \bar{\mathcal{H}}'$, or of type G_2G_3, with G_2 and G_3 in $\mathcal{H} \cup \bar{\mathcal{H}}$ but not in $\mathcal{H}' \cup \bar{\mathcal{H}}'$. It follows that $G'H$ is either of type G_1G_2, with G_1 and G_2 in $\mathcal{H} \cup \bar{\mathcal{H}}$ but not in $\mathcal{H}' \cup \bar{\mathcal{H}}'$, or of type $G_1G_2G_3$, with G_1, G_2 and G_3 in $\mathcal{H} \cup \bar{\mathcal{H}}$ but not in $\mathcal{H}' \cup \bar{\mathcal{H}}'$. This implies that $G'H$ is vertex-disjoint union of two or three subgraphs of type in $\mathcal{H}$ but not in $\mathcal{H}'$. If we replace G' and H by these two or three subgraphs, we obtain a set S' of vertex-disjoint subgraphs of type in $\mathcal{H}$, covering m vertices and with either $f(S') = f(S) - 1$ or $f(S') = f(S) - 2$, which, by minimality of $f(S)$, is not possible. So y cannot be a free left vertex of a subgraph of S not of type H_1 or H_6.

Suppose now that y is a free right vertex of a subgraph H of S of type distinct from H_1 and H_2. This yields the chain-graph $G'\bar{H}$. By Lemma 2.5.b, $H'\bar{H}$ is either of type G_2, with G_2 in $\mathcal{H} \cup \bar{\mathcal{H}}$ but not in $\mathcal{H}' \cup \bar{\mathcal{H}}'$, or of type G_2G_3, with G_2 and G_3 in $\mathcal{H} \cup \bar{\mathcal{H}}$ but not in $\mathcal{H}' \cup \bar{\mathcal{H}}'$. It follows that $G'\bar{H}$ is either of type G_1G_2, with G_1 and G_2 in in $\mathcal{H} \cup \bar{\mathcal{H}}$ but not in $\mathcal{H}' \cup \bar{\mathcal{H}}'$, or of type $G_1G_2G_3$, with G_1, G_2 and G_3 in $\mathcal{H} \cup \bar{\mathcal{H}}$ but not in $\mathcal{H}' \cup \bar{\mathcal{H}}'$. This implies that $G'\bar{H}$ is the vertex-disjoint union of two or three subgraphs of type in $\mathcal{H}$ but not in $\mathcal{H}'$. If we replace G' and H by these two or three subgraphs, we obtain a set S' of vertex-disjoint subgraphs of type in $\mathcal{H}$, covering m vertices and with either $f(S') = f(S) - 1$ or $f(S') = f(S) - 2$, which, by minimality of $f(S)$, is not possible. So y cannot be a free right vertex of a subgraph of S not of type H_1 or H_2.

Consider now the case where G' is of type H_3, and suppose first that y is a free left vertex of a subgraph H of S of type H_6. Now $G'H$ is a chain-graph of type $H_1\bar{H}_8$, and therefore $G'H$ is vertex-disjoint union of two subgraphs, one of type H_1 and the other of type H_8 (which is in $\mathcal{H}'$ but not of type H_3). If we replace G' and H by these two subgraphs, we obtain a set S' of vertex-disjoint subgraphs of type in $\mathcal{H}$ covering m vertices, with $f(S') = f(S)$ and $f_3(S') = f_3(S) - 1$, which, by minimality of $f_3(S)$, is not possible. So y cannot be a free left vertex of a subgraph of S of type H_6. Suppose now that y is a free right vertex of a subgraph H of S of type H_2. Now $G'\bar{H}$ is of type $H_1\bar{H}_4$, and therefore $G'\bar{H}$ is the vertex-disjoint union of a subgraph of type H_1 and of a subgraph of type H_4 (which is in $\mathcal{H}'$ but not of type H_3). If we replace G' and H by these two subgraphs, we obtain a set S' of vertex-disjoint subgraphs of type in $\mathcal{H}$ covering m vertices, with $f(S') = f(S)$ and $f_3(S') = f_3(S) - 1$, which, by minimality of $f_3(S)$, is not possible. So y cannot be a free right vertex of a subgraph of S of type H_2. All this proves the assertions a) and b). ∎

We state now:

Lemma 2.7 *Let $F_1 = G_1T_1G'_1$ and $F_2 = G_2T_2G'_2$ be subgraphs of $F(S)$ (T_1 and T_2 being the internal triangles).*

a) *A chain-graph TT_1G_1 is a subgraph of type in $\mathcal{H}'$.*

b) *A chain-graph G'_1T is a subgraph of type in $\mathcal{H}$.*

c) *If G'_1G_1 is a chain-graph, then the subgraph G'_1G_1 is of type in H.*

d) *A chain-graph G'_1H, where H is a chain-graph of type in $\{H_1, \bar{H}_2, H_6\}$, is a subgraph of type in $\mathcal{H}'$.*

e) *If $G'_1T_2G_2$ is a chain-graph, then the subgraph $G'_1T_2G_2$ is of type in $\mathcal{H}'$.*

Proof. **a)** If F_1 is of type H_3, G_1 is a proper triangle, and then TT_1G_1 is a chain-graph of type $H_3 \in \mathcal{H}'$. If F_1 is of type H_4, G_1 is a diamond, and then TT_1G_1 is a chain-graph of type $\bar{H}_4 \in \bar{\mathcal{H}}'$. If F_1 is of type H_8, G_1 is a chain-graph DD and then TT_1G_1 is a chain-graph of type $\bar{H}_8 \in \bar{\mathcal{H}}'$. If F_1 is of type H_9, G_1 is a diamond, and then TT_1G_1 is a chain-graph of type $\bar{H}_4 \in \bar{\mathcal{H}}'$. If F_1 is of type H_{13}, G_1 is a chain-graph DD and then TT_1G_1 is a chain-graph of type $\bar{H}_8 \in \bar{\mathcal{H}}'$. These observations show that TT_1G_1 is a subgraph of type in $\mathcal{H}'$.

b) If F_1 is of type H_3, G'_1 is a proper triangle, and then G'_1T is a chain-graph of type $H_1 \in \mathcal{H}$. If F_1 is of type H_4, G'_1 is a proper triangle, and then G'_1T is a chain-graph of type $H_1 \in \mathcal{H}$. If F_1 is of type H_8, G'_1 is a proper triangle, and then G'_1T is a chain-graph of type $H_1 \in \mathcal{H}$. If F_1 is of type H_9, G'_1 is a DD, and then G'_1T is a chain-graph of type $\bar{H}_6 \in \bar{\mathcal{H}}$. If F_1 is of type H_{13}, G'_1 is a DD, and then G'_1T is a chain-graph of type $\bar{H}_6 \in \bar{\mathcal{H}}$. These observations show that G'_1T is a subgraph of type in $\mathcal{H}$.

c) If F_1 is of type H_3, G'_1 is a proper triangle, G_1 is a proper triangle and then G'_1G_1 is of type $H_1 \in \mathcal{H}$. If F_1 is of type H_4, G'_1 is a proper triangle, G_1 is a diamond and then G'_1G_1 is of type $\bar{H}_2 \in \bar{\mathcal{H}}$. If F_1 is of type H_8, G'_1 is a proper triangle, G_1 is a double diamond and then G'_1G_1 is of type $H_6 \in \mathcal{H}$. If F_1 is of type H_9, G'_1 is a double diamond, G_1 is a diamond and then G'_1G_1 is of type $H_7 \in \mathcal{H}$. If F_1 is of type H_{13}, G'_1 is a double diamond, G_1 is a double diamond and then G'_1G_1 is of type $H_{11} \in \mathcal{H}$. These observations show that G'_1G_1 is a subgraph of type in $\mathcal{H}$.

d) G'_1 is either a proper triangle or a double diamond. Suppose first that G'_1 is a proper triangle. Then a chain-graph G'_1H_1 is of type $H_3 \in \mathcal{H}'$. A chain-graph $G'_1\bar{H}_2$ is of type $\bar{H}_4 \in \bar{\mathcal{H}}'$. A chain-graph G'_1H_6 is of type $\bar{H}_8 \in \bar{\mathcal{H}}'$. Suppose now that G'_1 is a double diamond. Then a chain-graph G'_1H_1 is of type $H_8 \in \mathcal{H}'$. A chain-graph $G'_1\bar{H}_2$ is of type $\bar{H}_9 \in \bar{\mathcal{H}}'$. A chain-graph G'_1H_6 is of type $H_{13} \in \mathcal{H}'$. So, the assertion is proved.

e) It is easy to see that we have $G'_1 \in \{T, DD\}$ and $G_2 \in \{T, D, DD\}$. Then $G'_1T_2G_2$ is of type in

$$\{TTT, TTD, TTDD, DDTT, DDTD, DDTDD\} = \{H_3, \bar{H}_4, \bar{H}_8, H_8, \bar{H}_9, H_{13}\}.$$

This proves the assertion. ■

Let $D_{F(S)}$ be the digraph whose vertex set is $F(S)$ and whose arcs are the couples (H'_j, H'_k) such that there exists a free right vertex of H'_j having the free internal vertex of H'_k as neighbor. Observe that $D_{F(S)}$ could have loops. An important result is:

Lemma 2.8 *The set $V(G) \setminus W(S)$ does not contain proper triangles.*

Proof. Suppose the opposite, and then let T be a proper triangle of $V(G) \setminus W(S)$. Let x be a vertex of T. By Lemma 2.3.a, x is adjacent to the free internal vertex of a graph $G_1T_1G'_1$ of $F(S)$ (so, x is adjacent to the free vertex of the internal triangle T_1). Let $G_1T_1G'_1, \dots G_sT_sG'_s$ be a directed path of $D_{F(S)}$ starting from $G_1T_1G'_1$ and of maximum length (see Fig. 4 when $s \geq 6$). Observe that the maximum length could be 0.

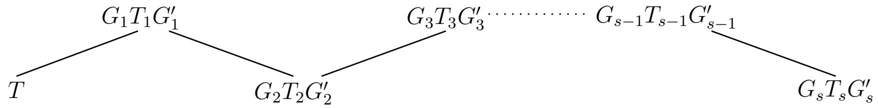


Fig. 4: A directed path of maximum length

Let y be a free right vertex of $G_sT_sG'_s$. By maximality of s, y is not adjacent to the free internal vertex of a graph of $D_{F(S)}$, distinct for the graphs $G_iT_iG'_i$, $1 \leq i \leq s$. Furthermore y is not adjacent to the free internal vertex of a graph $G_iT_iG'_i$, $1 \leq i \leq s$ (because all the vertices of the triangles T_i have three neighbors distinct from y. By Lemma 2.3.a, y is not adjacent to a vertex of T. In view of Lemma 2.6, four cases remain:

Case 1: y is adjacent to a left vertex of the subgraph G_s.
When $s \geq 3$, by Lemma 2.7, $TT_1G_1, G'_1T_2G_2, \dots, G'_{s-2}T_{s-1}G_{s-1}, G'_{s-1}T_s$ and G'_sG_s are vertex-disjoint subgraphs of type in $\mathcal{H}$, and their union is $V(T) \cup \bigcup_{1 \leq i \leq s} V(G_iT_iG'_i)$. Then $W(S) \cup V(T)$ would be a vertex-disjoint union of subgraphs of type in $\mathcal{H}$, which,

by maximality of m, is not possible.
When $s = 1$ (this means that the directed path is reduced to a vertex), TT_1 and G'_1G_1 are vertex-disjoint subgraphs of type in $\mathcal{H}$ and then $W(S) \cup V(T)$ would be a vertex-disjoint union of subgraphs of type in $\mathcal{H}$, which, by maximality of m, is not possible.
When $s = 2$, by Lemma 2.7 (a,b and c), TT_1G_1, G'_1T_2 and G'_2G_2 are vertex-disjoint subgraphs of type in $\mathcal{H}$ and then $W(S) \cup V(T)$ would be a vertex-disjoint union of subgraphs of type in $\mathcal{H}$, which, by maximality of m, is not possible.

Case 2: y is adjacent to a vertex of a subgraph H of S of type H_1.
When $s \geq 2$, by Lemma 2.7, $TT_1G_1, G'_1T_2G_2, \ldots, G'_{s-1}T_sG_s$ and G'_sH are vertex-disjoint subgraphs of type in $\mathcal{H}$ and then $W(S) \cup V(T)$ would be a vertex-disjoint union of subgraphs of type in $\mathcal{H}$, which is not possible by maximality of m.
When $s = 1$, TT_1G_1 and G'_1H are vertex-disjoint subgraphs of type in $\mathcal{H}$, and then $W(S) \cup V(T)$ would be a vertex-disjoint union of subgraphs of type in $\mathcal{H}$, which, by maximality of m, is not possible.

Case 3: y is adjacent to a free right vertex of a subgraph H of S of type H_2.
When $s \geq 2$, by Lemma 2.7, $TT_1G_1, G'_1T_2G_2, \ldots, G'_{s-1}T_sG_s$ and $G'_s\bar{H}$ are vertex-disjoint subgraphs of type in $\mathcal{H}$ and then $W(S) \cup V(T)$ would be a vertex-disjoint union of subgraphs of type in $\mathcal{H}$, which, by maximality of m, is not possible.
When $s = 1$, TT_1G_1 and $G'_1\bar{H}$ are vertex-disjoint subgraphs of type in $\mathcal{H}$, and then $W(S) \cup V(T)$ would be a vertex-disjoint union of subgraphs of type in $\mathcal{H}$, which, by maximality of m, is not possible.

Case 4: y is adjacent to a free left vertex of a subgraph H of S of type H_6.
This is similar to the case 2.
In any case, we get a contradiction, and so, the result is proved. ■.

This lemma means that when $m < n$, $V(G) \setminus W(S)$ is a vertex-disjoint union of diamonds.

Lemma 2.9 *There exists a set S of vertex-disjoint subgraphs H'_j, $1 \leq j \leq r$ of type in $\mathcal{H}$, covering m vertices, verifying conditions C1 and C2, and such that for any diamond D of $V(G) \setminus W(S)$, the two D-external neighbors of its two free vertices are not in the same triangle of a subgraph of S of type H_2 (condition* **C3***).*

Proof. Consider a set S of vertex-disjoint subgraphs of type in $\mathcal{H}$, covering m vertices, verifying conditions C1 and C2, and such that $V(G) \setminus W(S)$ has a minimum number s of diamonds D whose two D-external neighbors of its two free vertices are in the same triangle of a subgraph of S of type H_2. Suppose that $s \geq 1$. Then there exists a diamond D of $V(G) \setminus W(S)$ such that the free vertices of D are adjacent to the two free vertices of the triangle T_1 of a subgraph D_1T_1 of S, of type H_2. Then $S' = (S \setminus \{D_1T_1\}) \cup \{DT_1\}$ is a set of vertex-disjoint subgraphs of type in $\mathcal{H}$, covering m vertices, verifying conditions C1 and C2, and there are $s - 1$ diamonds of $V(G) \setminus W(S)$ satisfying the required conditions. By minimality of s, this is not possible. So $s = 0$, and the result is proved. ■

3 Proof of Theorem 1.1.a

By hypothesis G is a connected claw-free cubic graph of order $n \geq 10$. We work on a set S of vertex-disjoint subgraphs H'_j, $1 \leq j \leq r$ of type in $\mathcal{H}$, covering m vertices, and verifying conditions C1, C2 and C3. We may suppose $m < n$ (for otherwise 1.1.a is proved).

Suppose that two diamonds D and D' are adjacent to the same triangle of a subgraph G' of type H_2. We denote then by $G'(D, D')$ the graph obtained in this way (Fig. 5), and we say that it is a graph of type H_{14}.

Suppose now that a diamond D is adjacent to the triangles of two subgraphs G' and G'_1 of type H_2. We denote then by $D(G', G'_1)$ the graph obtained in this way (Fig. 5), and we say that it is a graph of type H_{15}. We observe that for a graph $G'(D, D')$ we have $\gamma_t(G'(D, D')) = 6 = \frac{2}{5}v(G'(D, D'))$, and that for a graph $D(G', G'_1)$ we have $\gamma_t(D(G', G'_1)) = 8 = \frac{4}{9}v(D(G', G'_1))$.

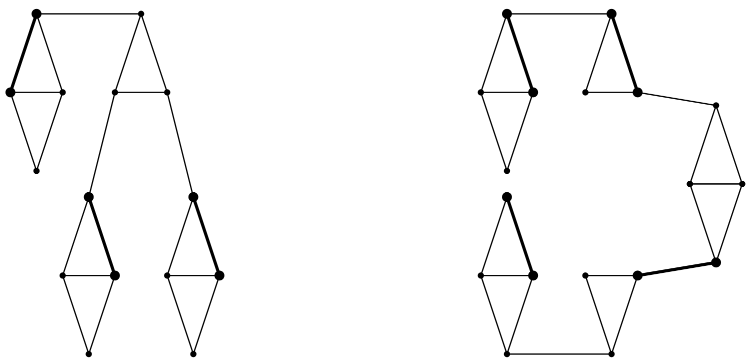


Fig. 5: Graphs $G'(D, D')$ of type H_{14} and $D(G', G'_1)$ of type H_{15}

For a diamond D having a free vertex adjacent to the free internal vertex of a subgraph G'' of type in $\{H_4, H_9, H_{13}\}$, we denote by $D \to G''$ the graph whose vertex set is $V(D) \cup V(G'')$ and whose edges are those of D and G'' and the edge between D and the internal triangle of G''. This yields three graphs which are drawn in Fig. 6. A graph $D \to H_4$ will be a graph of type H_{16}, a graph $D \to H_9$ will be a graph of type H_{17}, and a graph $D \to H_{13}$ will be a graph of type H_{18}. The total domination numbers of these graphs are $\gamma_t(D \to H_4) = 6 = \frac{3}{7}v(D \to H_4)$, $\gamma_t(D \to H_9) = 8 = \frac{8}{19}v(D \to H_9)$ and $\gamma_t(D \to H_{13}) = 10 = \frac{10}{23}v(D \to H_{13})$.

Now, we suppose that S is such that the maximum number of vertex-disjoint subgraphs $G'(D, D')$, with D and D' in $V(G) \setminus W(S)$ and G' in S, of type H_2, is maximum (**condition C4**). We denote by p this maximum. If $p > 0$, let $G_1(D_1, D'_1), \ldots, G_p(D_p, D'_p)$ be p vertex-disjoint subgraphs of type H_{14} with D_i, D'_i in $V(G) \setminus W(S)$ and G_i in S, of type H_2.

Let q be the maximum number of vertex-disjoint subgraphs $D(G', G'_1)$, with D in

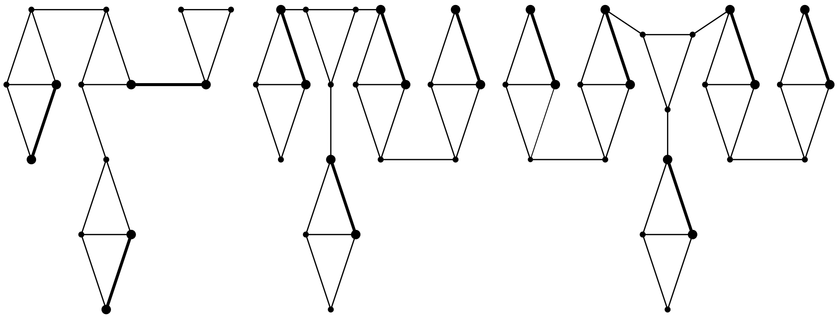


Fig. 6: Graphs $D \to H_4$, $D \to H_9$ and $D \to H_{13}$

$V(G) \setminus W(S)$, and G', G'_1 in S, of type H_2, and vertex-disjoint with the subgraphs $G_i(D_i, D'_i)$, $1 \le i \le p$. If $q > 0$, let $D''_i(G'_i, G''_i)$, $1 \le i \le q$ be a family of such subgraphs.

Let t be the maximum number of vertex-disjoint subgraphs $D \to G'$, with D in $V(G) \setminus W(S)$, G' in S of type in $\{H_4, H_9, H_{13}\}$, vertex-disjoint with the subgraphs $G_i(D_i, D'_i)$, $1 \le i \le p$, and with the subgraphs $D''_i(G'_i, G''_i)$, $1 \le i \le q$. If $t > 0$, let $D^3_i \to G^3_i$, $1 \le i \le t$ be a family of such graphs. We state:

Claim 3.1 *The vertex-disjoint subgraphs, defined above, use all the diamonds of $V(G) \setminus W(S)$.*

Proof. Suppose that there exists a diamond $\mathcal{D}$ of $V(G) \setminus W(S)$ which is not used. Suppose that there exists a free vertex of $\mathcal{D}$ adjacent to the free internal vertex of a subgraph H of S, of type in $\{H_4, H_9, H_{13}\}$. Since for $1 \le i \le t$, the free internal vertex of G^3_i has degree 3 in the subgraph $D^3_i \to G^3_i$, H is none of the subgraphs G^3_i, $1 \le i \le t$. But then, the $D^3_i \to G^3_i$, $1 \le i \le t$, and $\mathcal{D} \to H$, would form a family of $t+1$ subgraphs $D \to G'$, with D in $V(G) \setminus W(S)$, G' in S of type in $\{H_4, H_9, H_{13}\}$, vertex-disjoint with the subgraphs $G_i(D_i, D'_i)$, $1 \le i \le p$ and with the subgraphs $D''_i(G'_i, G''_i)$, $1 \le i \le q$, which is not possible. So each of the two free vertices of $\mathcal{D}$ is not adjacent to the free internal vertex of a subgraph of S of type in $\{H_4, H_9, H_{13}\}$. Then, by Lemma 2.3.b, there exists a free vertex of $\mathcal{D}$, say x, adjacent to a free right vertex of a subgraph H of S, of type H_2.

We claim that the $\mathcal{D}$-external neighbor z of the other free vertex y of $\mathcal{D}$, is in $W(S)$. Indeed, suppose the opposite. Then z is a free vertex of a diamond $\mathcal{D}'$ of $V(G) \setminus W(S)$, vertex-disjoint with $\mathcal{D}$. But then, since $\mathcal{D}'\mathcal{D}\bar{H}$ is a subgraph of type H_9, $S' = (S \setminus \{H\}) \cup \{\mathcal{D}'\mathcal{D}\bar{H}\}$ is a set of vertex-disjoint subgraphs of type in $\mathcal{H}$, covering $m+8$ vertices, which, by maximality of m, is not possible. So z is in $W(S)$ and then necessarily z is a free right vertex of a subgraph H' of S, of type H_2. Since S verifies condition C3 (see Lemma 2.9), the subgraphs H and H' are vertex-disjoint. Clearly, since for $1 \le i \le p$, all the vertices of the triangles of the subgraphs G_i are of degree 3 in $G_i(D_i, D'_i)$, H' and H are vertex-disjoint with the subgraphs G_i, $1 \le i \le p$. Then the subgraph $\mathcal{D}(H, H')$, of type H_{15} is vertex-disjoint with the subgraphs $G_i(D_i, D'_i)$, $1 \le i \le p$. If one of the subgraphs H and H', say H, is a subgraph G'_i or G''_i, $1 \le i \le q$, then $H(\mathcal{D}, D''_i)$ would be a graph of type H_{14}, vertex-disjoint with the subgraphs $G_i(D_i, D'_i)$, $1 \le i \le p$, which, by maximality of p, is not possible. So H' and H are vertex-disjoint with the graphs G'_i and G''_i, $1 \le i \le q$. But then the subgraph $\mathcal{D}(H, H')$, of type H_{15}, vertex-disjoint with the subgraphs $G_i(D_i, D'_i)$, $1 \le i \le p$, is also vertex-disjoint with the subgraphs $D''_i(G'_i, G''_i)$, $1 \le i \le q$, which is again impossible (by maximality of q). Consequently, all the diamonds of $V(G) \setminus W(S)$ are used, and the result is proved. ■

The cited subgraphs and the remaining subgraphs of the set S form a set S' of vertex-disjoint subgraphs of type in $\mathcal{H}_1 = \mathcal{H} \cup \{H_{14}, H_{15}, H_{16}, H_{17}, H_{18}\}$ and whose union is $V(G)$. Since for $H \in S'$ we have $\gamma_t(H) \le \frac{4}{9}v(H)$, it is easy to see that this implies $\gamma_t(G) \le \frac{4}{9}n$. So Theorem 1.1.a is proved.

4 Proof of Theorem 1.1.b

Now G is a connected claw-free cubic graph of order $n \ge 20$. We work on the sets S and S' (obtained from S) described in Section 3. We recall that S is a set of vertex-disjoint subgraphs of G, of type in $\mathcal{H} = \{H_1, \ldots, H_{13}\}$, verifying conditions **C1, C2, C3**, and that S' is a set of vertex-disjoint subgraphs of type in $\mathcal{H}_1 = \mathcal{H} \cup \{H_{14}, H_{15}, H_{16}, H_{17}, H_{18}\}$, covering $V(G)$. We also recall that S' is such that the number p of subgraphs of S' of type H_{14} is maximum (condition **C4**).

We denote by α the maximum number of the subgraphs G_1G_2 with G_1 in S', of type H_3, and G_2 in S', of type H_1. We say that such subgraphs are of type H_{19} (see Fig. 7)

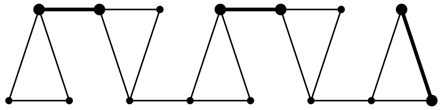


Fig. 7: Subgraph of type H_{19}

Let S''_{19} be a set of α such subgraphs. For a subgraph G' of S''_{19}, we have $\gamma_t(G') = 6 = \frac{2}{5}v(G') < \frac{10}{23}v(G')$. The new subgraphs of S''_{19}, the remaining subgraphs of type H_1 or H_3, and all the other subgraphs of S', form a set S'' of vertex-disjoint subgraphs of type in $\mathcal{H}_2 = \mathcal{H}_1 \cup \{H_{19}\}$, covering all the vertices of G. For $1 \le i \le 19$, we denote by S''_i the set of the subgraphs of S of type H_i, and s''_i is the cardinality of $S^{"}_i$. We may suppose that S'' is such that the number s''_{19} of the subgraphs of type H_{19} is maximum.

We observe that for a subgraph $H \in S''$, not of type H_3 or H_{15}, we have $\gamma_t(H) \le \frac{10}{23}v(H)$, with equality if H is of type H_{18}. If H is of type H_3 or H_{15}, we have $\gamma_t(H) = \frac{4}{9}v(H) > \frac{10}{23}v(H)$. So the critical subgraphs of S'' are those of type H_3 and H_{15}.

Suppose that there exists a subgraph $H = T_1T_2T_3$ of S''_3, such that the neighbors of the free end-vertices of H are not external. Then the free end-vertices a, b, c and d of H are of degree 2 in the subgraph G' of G, induced by the five free vertices of H. The degree in G' of the free internal vertex e of H is at most 1, and since the sum of the degrees in G' of the five vertices of G' is even, e is of degree 0 in G'. This means that e has an H-external neighbor. But now, by considering an edge between T_1 and T_3, $H' = T_1T_3T_2$ is a subgraph of type H_3, and the free internal vertex of H is a free right vertex of H' having an external neighbor. Thus, we may assume, even if it means replacing a subgraph of type H_3 by another subgraph of type H_3 with the same vertices, that for every subgraph H of S''_3, there exists at least a free end-vertex of H having an H-external neighbor. Let x be a free end-vertex of a subgraph $T_1T_2T_3$ of S''_3, having an external neighbor y. Without loss of generality, we may suppose that x is in T_3. By Lemma 2.6.a, y is either in a subgraph of S of type H_1, or the free internal vertex of a subgraph of S of type in $\mathcal{H}'$. Since y is not contained in a subgraph of S, of type H_2, y cannot be a vertex of a subgraph of S'', of type H_{14} or H_{15}. We also observe that y cannot be a vertex of a subgraph of S''_1 (for otherwise, we would have subgraphs of type H_{19} in extra). If $H = T'_1T'_2T'_3T'_4T'_5$ is a subgraph of S''_{19} with $T'_1T'_2T'_3$ in S', of type H_3, and $T'_4T'_5$ in S', of type H_1, then by Lemma 2.6.a, y cannot be a vertex of T'_1 or T'_3. We claim that y cannot be either a vertex of T'_5. Indeed, suppose the opposite. Then $(S \setminus \{T_1T_2T_3, T'_1T'_2T'_3, T'_4T'_5\}) \cup \{T_1T_2, T_3T'_5, T'_4T'_3, T'_2T'_1\}$ is a set of vertex-disjoint subgraphs of type in $\mathcal{H}$, covering m vertices, but with $f(S) - 2$ subgraphs of type in $\mathcal{H}'$, which, by minimality of $f(S)$, is not possible. So y cannot be a vertex of T'_5. Since the free internal vertex of a subgraph G' of type in $\{H_4, H_9, H_{13}\}$ has degree 3 in a subgraph $D \to G'$ of S'', y cannot be a vertex of a subgraph of S'' of type in $\{H_{16}, H_{17}, H_{18}\}$.

In conclusion, we may state that if a free end-vertex of a subgraph of S''_3 has an external neighbor, this external neighbor is either the free internal vertex of a subgraph of S'', of type in $\mathcal{H}' = \{H_3, H_4, H_8, H_9, H_{13}\}$, or a free vertex of one of the triangles T_2 and T_4 of a subgraph $H = T_1T_2T_3T_4T_5$ of S''_{19}.

We denote by Ω the set of the subgraphs of S'', of type in $\{H_4, H_8, H_9, H_{13}, H_{19}\}$, with a free internal vertex adjacent to a free end-vertex of a subgraph of S''_3. We put $W_1 = \Big(\bigcup_{H \in S''_3} V(H)\Big) \cup \big(\bigcup_{H \in \Omega} V(H)\big)$, and we state:

Claim 4.1 *We have* $\gamma_t(G[W_1]) \le \dfrac{10}{23} v(G[W_1])$.

Proof. Let β be the number of the free end-vertices of the subgraphs of S''_3 having a free internal vertex of a subgraph of S'', of type in $\{H_4, H_8, H_9, H_{13}, H_{19}\}$, as external neighbor. We denote by $S''_{3,1}$ the set of the subgraphs of S''_3 whose free internal vertex has a free end-vertex of a subgraph of S''_3 as external neighbor, and $s''_{3,1}$ is its cardinality.

Let $H = T_1T_2T_3$ be a subgraph of $S''_{3,1}$. There exists a subgraph $H' = T'_1T'_2T'_3$ of S''_3, vertex-disjoint with H, such that the free vertex of T_2 is adjacent to a free end-vertex of H'. Without loss of generality, we may suppose that this free end-vertex is in T'_3. Since a vertex of T_1 cannot be adjacent to a vertex of T_3 (for otherwise we could replace the subgraphs H and H' of type H_3 by the subgraphs $T_1'T_2'$, $T_3'T_2$ and T_1T_3, of type H_1), we deduce that the four end-vertices of H have external neighbors. Let β' denote the number of the free end-vertices of the subgraphs of S_3'' having an external neighbor. We have $\beta' \geq 4s_{3,1}''+s_3''-s_{3,1}''$, that is $\beta' \geq 3s_{3,1}''+s_3''$. Since the free internal vertex of a subgraph of $S_{3,1}''$, has exactly one free end-vertex of a subgraph of S_3'' as external neighbor and since a free end-vertex of a subgraph of S_3'' has at most one external neighbor, we deduce that the number of the free end-vertices of the subgraphs of S_3'' having a free internal vertex of a subgraph of S_3'' as external neighbor, is equal to $s_{3,1}''$. We have then $\beta' = \beta + s_{3,1}''$, and from the above inequality, we deduce $\beta \geq s_3'' + 2s_{3,1}''$.

For $i \in \{4, 8, 9, 13, 19\}$, let β_i be the number of the free end-vertices of the subgraphs of S_3'' having a free internal vertex of a subgraph of S_i'', as external neighbor. Let β_{19}' be the number of the subgraphs of S_{19}'' having one free internal vertex adjacent to a free end-vertex of a subgraph of S_3'' and let β_{19}'' be the number of the subgraphs of S_{19}'' having two free internal vertices adjacent to two free end-vertices of subgraphs of S_3''. For $i \in \{4, 8, 9, 13\}$, the number of the subgraphs of S_i'' whose free internal vertex is adjacent to a free end-vertex of a subgraph of S_3'' is β_i. We have then $\beta = \beta_4 + \beta_8 + \beta_9 + \beta_{13} + \beta_{19}' + 2\beta_{19}''$. The orders of the graph-chains H_3, H_4, H_8, H_9, H_{13} and H_{19} are, respectively, 9, 10, 14, 15, 19 and 15. We then get $v(G[W_1]) = 9s_3'' + 10\beta_4 + 14\beta_8 + 15\beta_9 + 19\beta_{13} + 15\beta_{19}' + 15\beta_{19}''$

Since the subgraphs of type H_3, H_4, H_8, H_9, H_{13} and H_{19} have total dominating sets with respectively 4, 4, 6, 6, 8 and 6 vertices, we deduce:

$$\gamma_t(G[W_1]) \leq 4s_3'' + 4\beta_4 + 6\beta_8 + 6\beta_9 + 8\beta_{13} + 6\beta_{19}' + 6\beta_{19}'' \tag{4.1}$$

The inequality:

$$\begin{aligned} &4s_3'' + 4\beta_4 + 6\beta_8 + 6\beta_9 + 8\beta_{13} + 6\beta_{19}' + 6\beta_{19}'' \leq \\ &\quad \tfrac{10}{23}(9s_3'' + 10\beta_4 + 14\beta_8 + 15\beta_9 + 19\beta_{13} + 15\beta_{19}' + 15\beta_{19}'') \end{aligned} \tag{4.2}$$

is equivalent to the inequality:

$$2s_3'' \leq 12\beta_{19}' + 12\beta_{19}'' + 8\beta_4 + 2\beta_8 + 12\beta_9 + 6\beta_{13}. \tag{4.3}$$

We have $\beta \geq s_3'' + 2s_{3,1}''$; hence

$$2s_3'' \leq 2\beta \tag{4.4}$$

Since $\beta = \beta_4 + \beta_8 + \beta_9 + \beta_{13} + \beta_{19}' + 2\beta_{19}''$, it is easy to see that

$$2\beta \leq 12\beta_{19}' + 12\beta_{19}'' + 8\beta_4 + 2\beta_8 + 12\beta_9 + 6\beta_{13}. \tag{4.5}$$

From (4.4) and (4.5), we deduce (4.3) and (4.2). From (4.1) and (4.2), we deduce $\gamma_t(G[W_1]) \leq \frac{10}{23}v(G[W_1])$, and the claim is proved. ■

If $S_{15}'' = \emptyset$, Theorem 1.1.b is proved. So we suppose that $S_{15}'' \neq \emptyset$ and then let $H = D_{1,1}T_{1,1}D_{1,2}T_{1,2}D_{1,3}$ be a subgraph of S_{15}'' ($D_{1,2}$ is the diamond not in $W(S)$). Now H has two free end-vertices (one in $D_{1,1}$ and one in $D_{1,3}$). We claim that these two free end-vertices are not adjacent. Indeed, suppose the opposite. Then $(S \setminus \{D_{1,1}T_{1,1}, D_{1,3}T_{1,2}\}) \cup \{D_{1,2}T_{1,1}, T_{1,2}D_{1,3}D_{1,1}\}$ would be a set of vertex-disjoint subgraphs of type in $\mathcal{H}$, covering $m+4$ vertices, which is not possible. So the two free end-vertices of H are not adjacent, and since G is connected of order $n \geq 20$, at least one of them has an H-external neighbor. Without loss of generality, we may suppose that the free vertex x of $D_{1,3}$ has an H-external neighbor y (which is in $W(S)$, by Lemma 2.3.b). We state:

Claim 4.2 *y is either the free internal vertex of a subgraph H' of S'' of type in $\{H_4, H_9, H_{13}\}$, or a free right vertex of a subgraph H' of S'' of type H_2.*

Proof. $\mathfrak{S} = (S \setminus \{D_{1,3}T_{1,2}\}) \cup \{D_{1,2}T_{1,2}\}$ is a set of vertex-disjoint subgraphs of $\mathcal{H}$ covering m vertices, verifying conditions C1, C2, C3, C4, and $D_{1,3}$ is not in $W(\mathfrak{S})$. Clearly, y which is in $W(S)$ is also in $W(\mathfrak{S})$. By Lemma 2.3.b (applied to $\mathfrak{S}$), y is either the free internal vertex of a subgraph H' of $\mathfrak{S}$ of type in $\{H_4, H_9, H_{13}\}$ (and H' is also in S), or a free right vertex of a subgraph H' of $\mathfrak{S}$ of type H_2 (and H' is also in S). Since the free internal vertex of a subgraph G' of type in $\{H_4, H_9, H_{13}\}$ has degree 3 in a subgraph $D \to G'$ of S'', we know that y cannot be a vertex of a subgraph of S'', of type in $\{H_{16}, H_{17}, H_{18}\}$. Since the free right vertices of a subgraph G' of type H_2 have degree 3 in a subgraph $G'(D, D')$ of S'', it follows that y cannot be a vertex of a subgraph of S'', of type H_{14}.

We claim that y cannot be a free vertex of a subgraph $H' = D'_{1,1}T'_{1,1}D'_{1,2}T'_{1,2}D'_{1,3}$ of S'', of type H_{15}. Indeed, suppose the opposite. Then y is necessarily the free vertex of one of the proper triangles of H', say $T'_{1,2}$. Consider the p vertex-disjoint subgraphs $G_i(D_i, D'_i)$, $1 \leq i \leq p$, of type H_{14}, with D_i, D'_i in $V(G) \setminus W(S)$ and G_i in S, of type H_2. Clearly, for $1 \leq i \leq p$, the diamonds D_i, D'_i are in $V(G) \setminus W(\mathfrak{S})$ and G_i is in $\mathfrak{S}$, of type H_2. Denote by H'' the subgraph $D'_{1,3}T'_{1,2}$ of S of type H_2. We observe that $H''(D_{1,3}, D'_{1,2})$ is a subgraph of type H_{14} with $D_{1,3}$, $D'_{1,2}$ in $V(G) \setminus W(\mathfrak{S})$ and H'' in $\mathfrak{S}$, of type H_2, vertex-disjoint with the subgraphs $G_i(D_i, D'_i)$, $1 \leq i \leq p$. By maximality of p this is impossible, and consequently, y cannot be a free vertex of a subgraph of S'' of type H_{15}. In conclusion, y is either the free internal vertex of a subgraph H' of S'' of type in $\{H_4, H_9, H_{13}\}$, or a free right vertex of a subgraph H' of S'' of type H_2, and so the result is proved. ■

When y is the free internal vertex of a subgraph H' of type in $\{H_4, H_9, H_{13}\}$, we get a subgraph $H \to H'$. If H' is of type H_4, we call it a subgraph of type H_{20}, if H' is of type H_9, we call it a subgraph of type H_{21}, and if H' is of type H_{13}, we call it a subgraph of type H_{22} (see Fig. 8). When y is a free right vertex of a subgraph H' of S'', of type H_2, we get a chain-graph $H\bar{H}'$, and we call it a subgraph of type H_{23} (Fig. 8). For a graph G' of type in $\{H_{20}, H_{21}, H_{22}, H_{23}\}$, we have $\gamma_t(G') < \frac{10}{23}v(G')$.

We state also:

Claim 4.3 *Let $H = D_4T_4$ be a subgraph of S'' of type H_2. The two free right vertices of H cannot be both adjacent to two end-vertices of two distinct subgraphs of S'' of type H_{15}.*

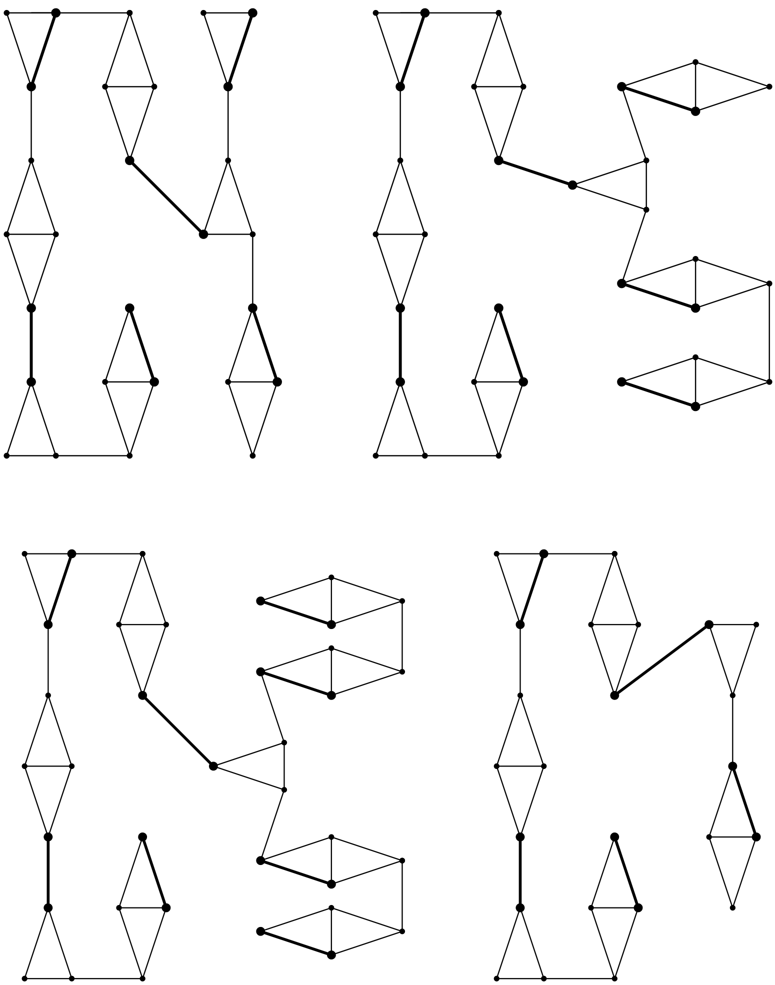


Fig. 8: Graphs H_{20}, H_{21}, H_{22} and H_{23}

Proof. Suppose the opposite. Then there exist two vertex-disjoint subgraphs $D_1T_1D_2T_2D_3$ and $D_1'T_1'D_2'T_2'D_3'$ of S'', of type H_{15}, such that the free vertex of D_3 is adjacent to a free vertex of T_4 and the free vertex of D_3' is adjacent to the other free vertex of T_4.

Then $\mathfrak{S} = (S \setminus \{D_3T_2, D_3'T_2'\}) \cup \{D_2T_2, D_2'T_2'\}$ is a set of vertex-disjoint subgraphs of type in $\mathcal{H}$, covering m vertices and verifying conditions C1, C2 and C3. But then the number of the vertex-disjoint subgraphs $G'(D, D')$, with D and D' in $V(G) \setminus W(\mathfrak{S})$ and G' in $\mathfrak{S}$, of type H_2, is at least $p + 1$ (because $H(D_3, D_3')$ is such a subgraph), which, by maximality of p, is not possible. So the result is proved. $\blacksquare$

Let u be the maximum number of vertex-disjoint subgraphs of type in $\{H_{20}, H_{21}, H_{22}, H_{23}\}$, using subgraphs of S'' of type in $\{H_{15}, H_4, H_9, H_{13}, H_2\}$.

Let $G_1 \to G'_1, \ldots, G_k \to G'_k, G_{k+1}\bar{G}'_{k+1}, \ldots, G_u\bar{G}'_u$ be a family of u vertex-disjoint subgraphs with G_i in S''_{15} for $1 \le i \le u$, $G'_i \in S''$ of type in $\{H_4, H_9, H_{13}\}$ for $1 \le i \le k$, and G'_i in S''_2 for $k+1 \le i \le u$. We claim that this family use all the subgraphs of S''_{15}. Indeed, suppose that there exists a subgraph H of S''_{15} which is not used. It was seen (Claim 4.2) that there exists a subgraph H' of S'', of type in $\{H_2, H_4, H_9, H_{13}\}$, forming with H a subgraph G'' of type in $\{H_{20}, H_{21}, H_{22}, H_{23}\}$. It is easy to see that if H' is of type in $\{H_4, H_9, H_{13}\}$, it is distinct from the subgraphs G'_i, $1 \le i \le k$ (an internal vertex has at most one external neighbor) and that if H' is of type H_2, it is distinct from the subgraphs G'_i, $k+1 \le i \le u$ (by Claim 4.3). But then $G_1 \to G'_1, \ldots, G_k \to G'_k, G_{k+1}\bar{G}'{}_{k+1}, \ldots, G_u\bar{G}'{}_u$ and G'' are $u+1$ vertex-disjoint subgraphs of type in $\{H_{20}, H_{21}, H_{22}, H_{23}\}$, using subgraphs of S'' of type in $\{H_{15}, H_4, H_9, H_{13}, H_2\}$, which, by maximality of u, is not possible. So, all the subgraphs of S''_{15} are used. Since the vertices of $G[W_1]$ are either vertices of subgraphs of type in $\{H_3, H_{19}\}$, or vertices of subgraphs of type in $\{H_4, H_8, H_9, H_{13}\}$ with the free internal vertex adjacent to a free end-vertex of a subgraph of type H_3, the u cited subgraphs are vertex-disjoint with the subgraph $G[W_1]$. The subgraph $G[W_1]$, the subgraphs $G_1 \to G'_1, \ldots, G_k \to G'_k, G_{k+1}\bar{G}'{}_{k+1}, \ldots, G_u\bar{G}'{}_u$ and the remaining subgraphs of S'' form a set $\mathfrak{S}''$ of vertex-disjoint subgraphs whose covering G and for $H \in \mathfrak{S}''$, we have $\gamma_t(H) \le \frac{10}{23}v(H)$. It is easy to see that this implies $\gamma_t(G) \le \frac{10}{23}n$, and consequently Theorem 1.1.b is proved.

5 To reach or not to reach

Consider a connected claw-free cubic graph G of order $n \ge 10$. It was seen that $\gamma_t(G) \le \dfrac{4}{9}n$ holds, and that if $n \ge 20$, we have $\gamma_t(G) \le \frac{10}{23}n < \frac{4}{9}n$. Consequently, if we want $\gamma_t(G) = \frac{4}{9}n$, necessarily $n = 18$ and necessarily also, G is a vertex-disjoint union of subgraphs of type H_3 or H_{15}. If there are no subgraphs of type H_{15}, there are no diamonds, and then we have $\gamma_t(G) \le \frac{2}{5}n < \frac{4}{9}n$ (see [4]), and therefore the bound is not reached. Consequently, there are subgraphs of type H_{15}, and as a subgraph of this type has 18 vertices, G has a subgraph of type H_{15} as spanning subgraph, and the two free end-vertices of this spanning subgraph are not adjacent. This yields two possible graphs (Fig. 9), and it is easy to verify that the bound is effectively reached by these two graphs. In conclusion, the first upper bound is reached only for the two graphs of Fig. 9.

Now we will prove that the second bound is not reached; in other words, we will prove that for any connected claw-free cubic graph G of order $n \ge 20$, we have $\gamma_t(G) < \frac{10}{23}n$.

Suppose that there exists a connected claw-free cubic graph G of order $n \ge 20$, with $\gamma_t(G) = \frac{10}{23}v(G)$. Within the notation already used, we deduce:

- S'' has no subgraphs of type in $\{H_1, H_2, H_5, H_6, H_7, H_{10}, H_{11}, H_{12}, H_{14}, H_{15},$

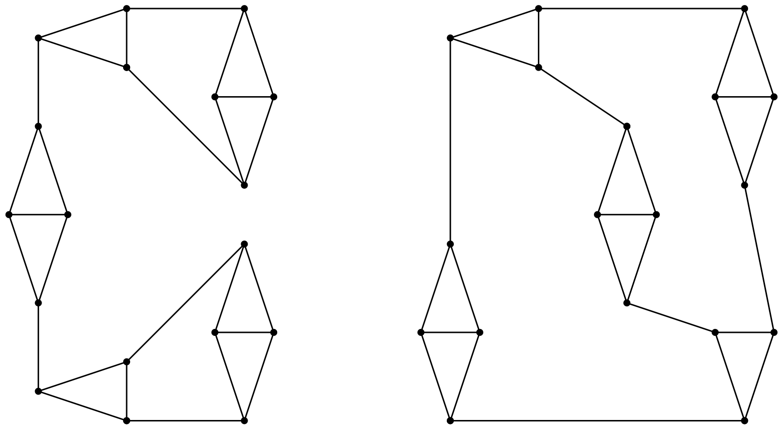


Fig. 9: The graphs G with $\gamma_t(G) = \frac{4}{9}v(G)$

$H_{16}, H_{17}\}$. Consequently, the subgraphs of S'' are of type in $\{H_3, H_4, H_8, H_9, H_{13}, H_{18}, H_{19}\}$.

- Necessarily we have $\gamma_t(G[W_1]) = \frac{10}{23}v(G[W_1])$, and then all the large inequalities (4.1)–(4.5) are equalities. It follows that $\beta'_{19} = \beta''_{19} = \beta_4 = \beta_9 = \beta_{13} = 0$, and then S'' has no subgraphs of type in $\{H_4, H_9, H_{13}, H_{19}\}$ (because for such a subgraph H we have $\gamma_t(H) < \frac{10}{23}v(H)$). So the subgraphs of S'' are of type in $\{H_3, H_8, H_{18}\}$.

- We have $s''_{3,1} = 0$, and then $s''_3 = \beta = \beta_8$. This means that each subgraph of type H_3 has exactly one end-vertex having an external neighbor, and this external neighbor is the free-internal of a subgraph of type H_8. This gives the graph in Fig. 10 that we call graph of type H_{24}.

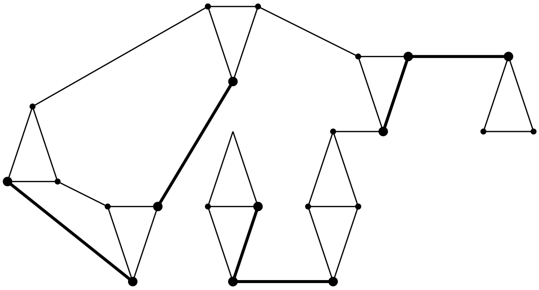


Fig. 10: Graph of type H_{24}

We conclude that G is vertex-disjoint union of subgraphs of type H_{18}, H_{24} and H_8. But as $\gamma_t(H_8) < \frac{3}{7}v(H_8)$ and $\gamma_t(H) = \frac{10}{23}v(H)$ for a subgraph H of type in H_{18} or H_{24}, G is the vertex-disjoint union of subgraphs of type H_{18} or H_{24}.

Since G is connected of even order, there exists a subgraph G' of G whose vertex set is is the union of the vertex sets of two vertex-disjoint subgraphs H and H' of type H_{18} or H_{24} and whose edges are those of H and H' and a single edge between a free vertex of H and a free vertex of H'. A simple verification (but a little long), shows that in any case we have $\gamma_t(G') < \frac{10}{23}n$, which implies $\gamma_t(G) < \frac{10}{23}n$.

We finish with:

Open problem. For an integer $k \geq 2$, we denote by $\mathcal{G}_k$ the set of the connected claw-free cubic graphs G with $v(G) \geq 2k$, and we put $f_t(k) = \sup\left(\frac{\gamma_t(G)}{v(G)},\ G \in \mathcal{G}_k\right)$.
Q1. For given $k \geq 2$, are there graphs $G \in \mathcal{G}_k$ such that $\gamma_t(G) = f_t(k) \times v(G)$?
Q2. It is easy to see that the sequence $f_t(k)$, $k \geq 2$, converges. What is the value of $\lim_{k\to+\infty} f_t(k)$?

In this paper we have proved that $f_t(5) = \frac{4}{9}$ and $f_t(10) \geq \frac{10}{23}$. We have $f_t(2) = f_t(3) = f_t(4) = \frac{1}{2}$. For $k \geq 2$, we have $\frac{1}{3} \leq f_t(k)$ and $f_t(k+1) \leq f_t(k)$. This justifies the convergence.

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