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The critical groups for $K_m \vee P_n$ and $P_m \vee P_n$*

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Abstract

Let $G_1 \vee G_2$ denote the graph obtained from $G_1 + G_2$ by adding new edges from each vertex of G_1 to every vertex of G_2. In this paper, the critical groups of the graphs $K_m \vee P_n$ $(n \geq 4)$ and $P_m \vee P_n$ $(m \geq 4, n \geq 5)$ are determined.

1 Introduction

Let $G = (V, E)$ be a finite connected graph without self-loops, but with multiple edges permitted. Then the Laplacian matrix of G is the $|V| \times |V|$ matrix defined by

$$L(G)_{uv} = \begin{cases} d(u), & \text{if} \quad u = v, \\ -a_{uv}, & \text{if} \quad u \neq v, \end{cases}$$

where a_{uv} is the number of the edges joining u and v, and $d(u)$ is the degree of u.

Thinking of $L(G)$ as representing an abelian group homomorphism: $Z^{|V|} \to Z^{|V|}$, its cokernel has the form

$$Z^{|V|}/\text{im}\,(L(G)) \cong Z \oplus Z^{|V|-1}/\text{im}\left(\overline{L(G)_{uv}}\right), \tag{1.1}$$

where $\overline{L(G)_{uv}}$ is the matrix obtained from $L(G)$ by striking out row u and column v, and $\text{im}(\cdot)$ refers to the integer span of the columns of the argument. The critical group $K(G)$ is defined to be $Z^{|V|-1}/\text{im}\left(\overline{L(G)_{uv}}\right)$. It is not hard to see that this definition is independent of the choice of u and v. The critical group $K(G)$ is a finite abelian group, whose order is equal to the absolute value of $\det \overline{L(G)_{uv}}$. By the well-known Kirchhoff's Matrix-Tree Theorem [7, Theorem 13.2.1], the order $|K(G)|$

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is equal to the spanning tree number of G. For the general theory of the critical group, we refer the reader to Biggs [1, 2], and Godsil [7, Chapter 14].

Recall that an $n \times n$ integral matrix P is unimodular if $\det P = \pm 1$. So the unimodular matrices are precisely those integral matrices with integral inverses, and of course form a multiplicative group. Two integral matrices A and B of order n are equivalent (written $A \sim B$) if there are unimodular matrices P and Q such that $B = PAQ$. Equivalently, B is obtainable from A by a sequence of elementary row and column operations: (1) the interchange of two rows or columns, (2) the multiplication of any row or column by -1, (3) the addition of any integer multiple of one row (respectively, column) to another row (respectively, column). The Smith normal form (Snf) is a diagonal canonical form for our equivalence relation: every $n \times n$ integral matrix A is equivalent to a unique diagonal matrix $\mathrm{diag}(s_1(A), \ldots, s_n(A))$, where $s_i(A)$ divides $s_{i+1}(A)$ for $i = 1, 2, \ldots, n-1$. The ith diagonal entry of the Smith normal form of A is usually called the ith invariant factor of A.

It is easy to see that $A \sim B$ implies that $\mathrm{coker}(A) \cong \mathrm{coker}(B)$. Given any $n \times n$ unimodular matrices P and Q and any integral matrix A with $PAQ = \mathrm{diag}(a_1, \ldots, a_n)$, it is easy to see that $Z^{|V|}/\mathrm{im}(A) \cong (Z/a_1Z) \oplus \cdots \oplus (Z/a_nZ)$. Assume the Snf of $\overline{L(G)_{uv}}$ is $\mathrm{diag}(t_1, \ldots, t_{|V|-1})$. (In fact, every such submatrix of $L(G)$ shares the same Snf.) Then it induces an isomorphism

$$K(G) \cong (Z/t_1Z) \oplus (Z/t_2Z) \oplus \cdots \oplus \left(Z/t_{|V|-1}Z\right). \tag{1.2}$$

The nonnegative integers $t_1, t_2, \ldots, t_{|V|-1}$ are also called the invariant factors of $K(G)$, and they can be computed in the following way: for $1 \le i < |V|$, $t_i = \Delta_i/\Delta_{i-1}$ where $\Delta_0 = 1$ and Δ_i is the greatest common divisor of the determinants of the $i \times i$ minors of $\overline{L(G)_{uv}}$. Since $|K(G)| = \kappa$, the spanning tree number of G, it follows that $t_1 t_2 \ldots t_{|V|-1} = \kappa$. So the invariant factors of $K(G)$ can be used to distinguish pairs of non-isomorphic graphs which have the same κ, and so there is considerable interest in their properties. If G is a simple connected graph, the invariant factor t_1 of $K(G)$ must be equal to 1; however, most of them are not easy to determine.

If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are graphs on disjoint sets of r and s vertices, respectively, their union is the graph $G_1 + G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ and their join $G_1 \vee G_2$ is the graph on $n = r + s$ vertices obtained from $G_1 + G_2$ by inserting new edges joining every vertex of G_1 to every vertex of G_2. If we use G^c to denote the complement graph of G, then $G_1 \vee G_2 = (G_1^c + G_2^c)^c$.

Compared to the number of the results on the spanning tree number κ, there are relatively few results describing the critical group structure of $K(G)$ in terms of the structure of G. There are also very few interesting infinite family of graphs for which the group structure has been completely determined (see [4, 5, 6, 8, 9, 10, 11, 12, 13] and the references therein). The aim of this paper is to describe the structure of the critical groups of two families of graphs $K_m \vee P_n$ ($n \ge 4$) and $P_m \vee P_n$ ($m \ge 4$, $n \ge 5$), where K_m is the complete graph with m vertices and P_n is the path with n vertices.

Now, we state the main results in this article as follows:

Theorem 1.1 (1) *The spanning tree number of* $K_m \vee P_n \, (n \geq 4)$ *is*

$$\frac{(m+n)^{m-1}}{2^n\sqrt{m^2+4m}}\left(\left(m+2+\sqrt{m^2+4m}\right)^n - \left(m+2-\sqrt{m^2+4m}\right)^n\right).$$

(2) *The critical group of* $K_m \vee P_n \, (n \geq 4)$ *is*

$$Z/(m+n, a_n, b_n)Z \oplus (Z/(m+n)Z)^{m-2} \oplus Z/\frac{(m+n)a_n}{(m+n, a_n, b_n)}Z,$$

where

$$\begin{cases} a_n &= \dfrac{1}{\sqrt{m^2+4m}}\left(\left(\dfrac{m+2+\sqrt{m^2+4m}}{2}\right)^n - \left(\dfrac{m+2-\sqrt{m^2+4m}}{2}\right)^n\right), \\ b_n &= e\left(\dfrac{m+2+\sqrt{m^2+4m}}{2}\right)^n - f\left(\dfrac{m+2-\sqrt{m^2+4m}}{2}\right)^n, \end{cases}$$

where

$$e = \frac{(m^2-m-(m+1)\sqrt{m^2+4m}+2mn)(m+4-\sqrt{m^2+4m})}{4m^2(m+4)},$$

$$f = \frac{(m-m^2-(m+1)\sqrt{m^2+4m}-2mn)(m+4+\sqrt{m^2+4m})}{4m^2(m+4)}.$$

Theorem 1.2 (1) *The critical group of the graph* $P_m \vee P_n \;(m \geq 4,\, n \geq 5)$ *is* $Z/tZ \oplus Z/sZ$*, where* $t = (m\beta+n, \alpha\beta-1, p'_{m-2})$*,* $s = \dfrac{(m\beta+n)p'_{m-2}}{(m\beta+n, \alpha\beta-1, p'_{m-2})}$*, and*

$$p'_{m-2} = \frac{1}{\sqrt{n^2+4n}}\left(\left(\frac{n+2+\sqrt{n^2+4n}}{2}\right)^m - \left(\frac{n+2-\sqrt{n^2+4n}}{2}\right)^m\right),$$

$$\alpha = \frac{1}{n}p'_{m-2} - \frac{m}{n},$$

$$\beta = \frac{1}{m\sqrt{m^2+4m}}\left(\left(\frac{m+2+\sqrt{m^2+4m}}{2}\right)^n - \left(\frac{m+2-\sqrt{m^2+4m}}{2}\right)^n\right) - \frac{n}{m}.$$

(2) *The spanning tree number of* $P_m \vee P_n$ *is* $(m\beta+n)p'_{m-2}$.

2 The critical group of $K_m \vee P_n \; (n \geq 4)$

To prove Theorem 1.1, we need the following lemmas.

Lemma 2.1 *If the graph G has n vertices, then*

$$L(K_m \vee G) \sim ((m+n)I_n - L(G^c)) \oplus (m+n)I_{m-2} \oplus I_1 \oplus 0_1. \tag{2.1}$$

Proof Note that

$$L(K_m \vee G) = \begin{pmatrix} (m+n)I_m - J_m & -J_{m\times n} \\ -J_{n\times m} & mI_n + L(G) \end{pmatrix},$$

where J_m and $J_{m\times n}$ are $m \times m$ and $m \times n$ matrices having all entries equal to 1.

Let

$$P_1 = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & 1 & 0 \\ -1 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -1 & -1 & \cdots & -1 & -1 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 1 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Then a direct calculation can show that

$$P_1 L(K_m \vee G) Q_1 = I_1 \oplus (m+n)I_{m-2} \oplus ((m+n)I_n - L(G^c)) \oplus 0_1.$$

Note that both the matrices P_1 and Q_1 are unimodular, so this lemma holds. □

In order to work out the critical group of graph $K_m \vee P_n$ $(n \geq 4)$, we only need to work on the Smith normal form of the matrix $(m+n)I_n - L(P_n^c)$.

Lemma 2.2

$$(m+n)I_n - L(P_n^c) \sim I_{n-2} \oplus \begin{pmatrix} m+n & b_n \\ 0 & a_n \end{pmatrix},$$

where

$$\begin{cases} a_n &= \dfrac{1}{\sqrt{m^2+4m}} \left(\left(\dfrac{m+2+\sqrt{m^2+4m}}{2} \right)^n - \left(\dfrac{m+2-\sqrt{m^2+4m}}{2} \right)^n \right), \\ b_n &= e \left(\dfrac{m+2+\sqrt{m^2+4m}}{2} \right)^n - f \left(\dfrac{m+2-\sqrt{m^2+4m}}{2} \right)^n, \end{cases}$$

and

$$e = \frac{(m^2 - m - (m+1)\sqrt{m^2+4m} + 2mn)(m+4-\sqrt{m^2+4m})}{4m^2(m+4)},$$

$$f = \frac{(m - m^2 - (m+1)\sqrt{m^2+4m} - 2mn)(m+4+\sqrt{m^2+4m})}{4m^2(m+4)}.$$

Proof Note that

$$(m+n)I_n - L(P_n^c) = \begin{pmatrix} m+2 & 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & m+3 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & m+3 & 0 & 1 & \cdots & 1 & 1 \\ \vdots & \ddots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 0 & m+3 & 0 & 1 \\ 1 & 1 & \cdots & 1 & 1 & 0 & m+3 & 0 \\ 1 & 1 & \cdots & 1 & 1 & 1 & 0 & m+2 \end{pmatrix}.$$

Let $P_2 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \end{pmatrix}$, $Q_2 = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}$,
and $A_2 = P_2((m+n)I_n - L(P_n^c))Q_2$. Then a direct calculation can show

$$A_2 = \begin{pmatrix} m+n & n-2 & n-2 & n-3 & n-4 & \cdots & 1 \\ 0 & m+2 & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & m+2 & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & m+2 & -1 & 0 \\ 0 & \cdots & \cdots & 0 & -1 & m+2 & -1 \\ 0 & \cdots & \cdots & \cdots & 0 & -1 & m+2 \end{pmatrix}.$$

For $i = 0, \dots, n-3$, let $M_{i+1} = \begin{pmatrix} I_i & 0_{i\times 1} & & 0_{i\times(n-i-1)} & \\ 0_{1\times i} & 1 & m+2 & -1 & 0_{1\times(n-i-3)} \\ 0_{(n-i-1)\times(i+1)} & & & I_{n-i-1} & \end{pmatrix}$,
and $M_{n-1} = \begin{pmatrix} I_{n-2} & 0_{(n-2)\times 1} & 0_{(n-2)\times 1} \\ 0_{1\times(n-2)} & 1 & m+2 \\ 0_{1\times(n-1)} & & 1 \end{pmatrix}$. Let $M = M_1 \dots M_{n-1}$; then

$$A_2M = \begin{pmatrix} m+n & b_2 & b_3 & b_4 & \cdots & b_{n-1} & b_n \\ 0 & a_2 & a_3 & a_4 & \cdots & a_{n-1} & a_n \\ 0_{(n-2)\times 1} & & & -I_{n-2} & & & 0_{(n-2)\times 1} \end{pmatrix},$$

where $0_{i\times j}$ is an $i \times j$ zero matrix, and the numbers a_l, b_l satisfy the following recurrence relations and initial values:

$$\begin{cases} a_l = (m+2)a_{l-1} - a_{l-2}, & l \geq 3, \\ a_1 = 1, \quad a_2 = m+2; & \\ b_l = (m+2)b_{l-1} - b_{l-2} + (n-l+1), & l \geq 3, \\ b_1 = 0, \quad b_2 = n-2. & \end{cases} \tag{2.2}$$

Let $P_3 = \begin{pmatrix} I_2 & \begin{matrix} b_2 & b_3 & \cdots & b_{n-2} & b_{n-1} \\ a_2 & a_3 & \cdots & a_{n-2} & a_{n-1} \end{matrix} \\ 0_{(n-2)\times 2} & I_{n-2} \end{pmatrix}$. Then

$$P_3 A_2 M = \begin{pmatrix} m+n & \multirow{2}{*}{$0_{2\times(n-2)}$} & b_n \\ 0 & & a_n \\ 0_{(n-2)\times 1} & -I_{n-2} & 0_{(n-2)\times 1} \end{pmatrix} \sim I_{n-2} \oplus \begin{pmatrix} m+n & b_n \\ 0 & a_n \end{pmatrix}. \tag{2.3}$$

Solving recursion (2.2) by using standard methods yields

$$a_l = \frac{1}{\sqrt{m^2+4m}} \left(\left(\frac{m+2+\sqrt{m^2+4m}}{2} \right)^l - \left(\frac{m+2-\sqrt{m^2+4m}}{2} \right)^l \right),$$

and

$$b_l = e \left(\frac{m+2+\sqrt{m^2+4m}}{2} \right)^l - f \left(\frac{m+2-\sqrt{m^2+4m}}{2} \right)^l - \frac{n-l}{m},$$

where

$$e = \frac{(m^2 - m - (m+1)\sqrt{m^2+4m} + 2mn)(m+4-\sqrt{m^2+4m})}{4m^2(m+4)},$$

$$f = \frac{(m - m^2 - (m+1)\sqrt{m^2+4m} - 2mn)(m+4+\sqrt{m^2+4m})}{4m^2(m+4)}.$$

In fact, $a_n = U_{n-1}(m)$, where $U_n(x)$ is the Chebyshev polynomial of the second kind. For the details of $U_n(x)$, we can see Section 2 in [3] or Section 3 in [14] and the references therein. □

Proof of Theorem 1.1 Note that every line sum of the Laplacian matrix of a graph is 0, so we have

$$L(G) \sim \mathrm{Snf}(\overline{L(G)_{uv}}) \oplus 0_1, \quad \text{for every } u, v \in V(K_m \vee P_n). \tag{2.4}$$

It follows from (2.1) and (2.3) that

$$L(K_m \vee P_n) \sim I_{n-1} \oplus \begin{pmatrix} m+n & b_n \\ 0 & a_n \end{pmatrix} \oplus (m+n) I_{m-2} \oplus 0_1. \tag{2.5}$$

Therefore by (2.4) and (2.5), we have

$$\mathrm{Snf}(\overline{L(G)_{uv}}) \sim I_{n-1} \oplus \begin{pmatrix} m+n & b_n \\ 0 & a_n \end{pmatrix} \oplus (m+n) I_{m-2}. \tag{2.6}$$

It is easy to see that the invariant factors of the matrix $\begin{pmatrix} m+n & b_n \\ 0 & a_n \end{pmatrix} \oplus (m+n)I_{m-2}$ are: $(m+n, a_n, b_n)$, $m+n$ (with multiplicity $m-2$), $\frac{(m+n)a_n}{(m+n, a_n, b_n)}$, where $(m+n, a_n, b_n)$ stands for the greatest common divisor of $m+n$, a_n, and b_n. So this theorem holds. □

Remark 2.3 It is known that the Laplacian eigenvalues of P_n are: $0, 2+2\cos\left(\frac{\pi j}{n}\right)$ $(1 \le j \le n-1)$; and the Laplacian eigenvalues of K_m are: $0, m$ (with multiplicity $m-1$). Then it follows from Theorem 2.1 in [10] that the Laplacian eigenvalues of $K_m \vee P_n$ are: 0, $m+n$ (with multiplicity m), $m+2+2\cos\left(\frac{j\pi}{n}\right)$, where $1 \le j \le n-1$. Then by the well-known Kirchhoff Matrix-Tree Theorem we know that the spanning tree number of $K_m \vee P_n$ is $\kappa(K_m \vee P_n) = (m+n)^{m-1} \prod_{j=1}^{n-1} \left(m+2+2\cos\left(\frac{\pi j}{n}\right)\right)$. Recalling the first part of Theorem 1.1, we have

$$\begin{aligned} &\prod_{j=1}^{n-1} \left(m+2+2\cos\left(\tfrac{\pi j}{n}\right)\right) \\ &= \frac{1}{2^n\sqrt{m^2+4m}} \left(\left(m+2+\sqrt{m^2+4m}\right)^n - \left(m+2-\sqrt{m^2+4m}\right)^n\right). \end{aligned} \tag{2.7}$$

Example 2.4

If $m=1$, then $G = K_1 \vee P_n$, the fan graph. From (1) of Theorem 1.1 we have the spanning tree number of $K_1 \vee P_n$ is $\frac{1}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n\right]$, which is the result of Theorem 2 in [3].

Example 2.5

If $m=3, n=4$, then $a_4 = 115, b_4 = 59$. If $m=4, n=4$, then $a_4 = 204, b_4 = 83$. If $m=4, n=5$, then $a_5 = 1189, b_5 = 730$. So it follows from Theorem 1.1 that we have the following:

$$\mathrm{Snf}(K_3 \vee P_4) = I_4 \oplus \mathrm{diag}(7, 805, 0); \tag{2.8}$$
$$\mathrm{Snf}(K_4 \vee P_4) = I_4 \oplus \mathrm{diag}(8, 8, 1632, 0); \tag{2.9}$$
$$\mathrm{Snf}(K_4 \vee P_5) = I_5 \oplus \mathrm{diag}(9, 9, 10701, 0). \tag{2.10}$$

Note that one can use MAPLE to check the results of (2.8), (2.9) and (2.10).

3 The critical group of $P_m \vee P_n$ $(m \ge 4,\ n \ge 5)$

In this section we will work on the critical group of $P_m \vee P_n (m \ge 4, n \ge 5)$. Let L' be the submatrix of $L(P_m \vee P_n)$ resulting from the deletion of the last row and the $(m+1)$-th column. Thus $L' = \begin{pmatrix} nI_m + L(P_m) & -J_{m\times(n-1)} \\ -J_{(n-1)\times m} & U \end{pmatrix}$, where U is the submatrix obtained from $mI_n + L(P_n)$ by deleting its first column and last row. Now we discuss the Smith normal form of the matrix L'.

Let $M = \begin{pmatrix} T_m & 0_{m\times(n-1)} \\ 0_{(n-1)\times m} & I_{n-1} \end{pmatrix}$ and $N = \begin{pmatrix} T_m^{-1} & 0_{m\times(n-1)} \\ 0_{(n-1)\times m} & T_{n-1}^{-1} \end{pmatrix}$, where $T_m = (t_{ij})$ is an unimodular matrix of order m with its entries $t_{ij} = \begin{cases} 1, & \text{if} \quad i=j, \\ 1, & \text{if} \quad j=1, \\ 0, & \text{otherwise.} \end{cases}$

Moreover, we let $P = (p_{ij})$ be an unimodular matrix of order $m+n-1$ with its entries

$$p_{ij} = \begin{cases} -1, & \text{if} \quad i=1,\ j=m+1, \\ 1, & \text{if} \quad i=m+1,\ j=1, \\ 1, & \text{if} \quad i=j\neq 1,\, m+1, \\ 0, & \text{otherwise,} \end{cases}$$

$$B_{11} = \begin{pmatrix} m & 1 & 1 & \cdots & \cdots & \cdots & 1 \\ 0 & n+3 & -1 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & n+2 & -1 & \cdots & \cdots & 0 \\ 0 & 1 & -1 & n+2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 1 & 0 & \cdots & -1 & n+2 & -1 \\ 0 & 1 & 0 & \cdots & \cdots & -1 & n+1 \end{pmatrix}_{m\times m},$$

$$B_{12} = \begin{pmatrix} 1 & 0_{1\times(n-2)} \\ 0_{(m-1)\times(n-1)} & \end{pmatrix}_{m\times(n-1)}, \quad B_{21} = \begin{pmatrix} n & -1 & 0_{1\times(m-2)} \\ & 0_{(n-2)\times m} & \end{pmatrix}_{(n-1)\times m},$$

and

$$B_{22} = \begin{pmatrix} -1 & -1 & -1 & \cdots & \cdots & -1 & -1 \\ m+3 & -1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & m+2 & -1 & \cdots & \cdots & 0 & 0 \\ 1 & -1 & m+2 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 1 & 0 & \cdots & -1 & m+2 & -1 & 0 \\ 1 & 0 & \cdots & \cdots & -1 & m+2 & -1 \end{pmatrix}_{(n-1)\times(n-1)}.$$

Then a direct calculation shows that

$$PNL'M = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

The sequences p_i and c_i will be used in the following Lemma 3.1, where

$$\begin{cases} p_k = (n+1)p_{k-1} + 1, & k \geq 1, \\ p_0 = n+3. \end{cases} \tag{3.1}$$

and

$$\begin{cases} c_k = (m+1)c_{k-1} + 1, & k \geq 1, \\ c_0 = m+3. \end{cases} \tag{3.2}$$

Lemma 3.1 *We have the following equivalence of matrices:*

$$L' \sim I_{m+n-3} \oplus \begin{pmatrix} p'_{m-2} & 0 \\ \alpha\beta - 1 & m\beta + n \end{pmatrix},$$

where

$$p'_{m-2} = \frac{1}{\sqrt{n^2+4n}} \left(\left(\frac{n+2+\sqrt{n^2+4n}}{2} \right)^m - \left(\frac{n+2-\sqrt{n^2+4n}}{2} \right)^m \right),$$

$$\alpha = \frac{1}{n} p'_{m-2} - \frac{m}{n},$$

$$\beta = \frac{1}{m\sqrt{m^2+4m}} \left(\left(\frac{m+2+\sqrt{m^2+4m}}{2} \right)^n - \left(\frac{m+2-\sqrt{m^2+4m}}{2} \right)^n \right) - \frac{n}{m}.$$

Proof Let

$$M_1 = \begin{pmatrix} m & 1 & 1 & 1 & \cdots & \cdots & 1 \\ 0 & n+3 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & n+2 & -1 & \cdots & \cdots & 0 \\ 0 & 1 & -1 & n+2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 1 & 0 & \cdots & -1 & n+2 & -1 \\ 0 & 1 & 0 & \cdots & \cdots & -1 & n+1 \end{pmatrix}_{m \times m}$$

and

$$M_2 = \begin{pmatrix} -1 & -1 & -1 & \cdots & \cdots & -1 & -1 \\ m+3 & -1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & m+2 & -1 & \cdots & \cdots & 0 & 0 \\ 1 & -1 & m+2 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 1 & 0 & \cdots & -1 & m+2 & -1 & 0 \\ 1 & 0 & \cdots & \cdots & -1 & m+2 & -1 \end{pmatrix}_{(n-1)\times(n-1)};$$

then we can rewrite L' as $L' = \begin{pmatrix} M_1 & B_{12} \\ B_{21} & M_2 \end{pmatrix}$.

Let $B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$ where

$$B_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & p_0 & 1 & 0 & \cdots & 0 \\ 0 & p_1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & p_{m-3} & 0 & 0 & \cdots & 1 \end{pmatrix}_{m\times m}, \quad B_2 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ c_0 & 1 & 0 & \cdots & 0 \\ c_1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n-3} & 0 & 0 & \cdots & 1 \end{pmatrix}_{(n-1)\times(n-1)}.$$

Then it is easy to check that

$$L'B = \begin{pmatrix} M_1' & B_{12} \\ B_{21} & M_2' \end{pmatrix},$$

where

$$M_1' = \begin{pmatrix} m & \alpha & 1 & 1 & \cdots & \cdots & 1 \\ 0 & 0 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & n+2 & -1 & \cdots & \cdots & 0 \\ 0 & 0 & -1 & n+2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & n+2 & -1 \\ 0 & p'_{m-2} & 0 & \cdots & \cdots & -1 & n+1 \end{pmatrix}_{m\times m},$$

and $p'_{m-2} = (n+1)p_{m-3} - p_{m-4} + 1$, $\alpha = 1 + \sum_{k=0}^{m-3} p_k$.

From (3.1), we get

$$p_k = (x+y)\left(\frac{n+2+\sqrt{n^2+4n}}{2}\right)^k + (x-y)\left(\frac{n+2-\sqrt{n^2+4n}}{2}\right)^k - \frac{1}{n},$$

where $x = \frac{n^2+3n+1}{2n}$, $y = \frac{n^2+5n+5}{2\sqrt{n^2+4n}}$. And now we can easily get the expression of p'_{m-2} and α by a direct calculation.

$$M_2' = \begin{pmatrix} -\beta & -1 & -1 & -1 & \cdots & -1 & -1 \\ 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & m+2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & m+2 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & m+2 & -1 & 0 \\ 0 & 0 & \cdots & \cdots & -1 & m+2 & -1 \end{pmatrix}_{(n-1)\times(n-1)},$$

and $\beta = 1 + \sum_{k=0}^{n-3} c_k$.

From (3.2), we obtain

$$c_k = (u+v)\left(\frac{m+2+\sqrt{m^2+4m}}{2}\right)^k + (u-v)\left(\frac{m+2-\sqrt{m^2+4m}}{2}\right)^k - \frac{1}{m},$$

where $u = \frac{m^2+3m+1}{2m}$, $v = \frac{m^2+5m+5}{2\sqrt{m^2+4m}}$. Thus we get β by a direct calculation.

Now we deal with the matrix $L'B$. In the following, we will use r_i to denote the ith row of matrix $L'B$.

For $i = 2, \ldots, m-2$, we first add $(n+2)r_i$ to r_{i+1} and add $-r_i$ to r_{i+2}; then add r_i to r_1, and add $(n+1)r_{m-1}$ to r_m. After that we have

$$M_1' \sim M_1'' = \begin{pmatrix} m & \alpha & 0_{1\times(m-2)} \\ 0_{(m-2)\times 2} & & -I_{(m-2)} \\ 0 & p'_{m-2} & 0_{1\times(m-2)} \end{pmatrix}_{m\times m}.$$

For $i = m+2, \ldots, m+n-2$, we first add $(m+2)r_i$ to r_{i+1} and add $-r_i$ to r_{i+2}; then add $-r_i$ to r_{m+1}, and add $-r_{m+n-1}$ to r_{m+1}. After that we have

$$M_2' \sim M_2'' = \begin{pmatrix} -\beta & 0_{1\times(n-2)} \\ 0_{(n-2)\times 1} & -I_{(n-2)} \end{pmatrix}_{(n-1)\times(n-1)}.$$

Note that the matrices B_{12} and B_{21} are not influenced in the operations on $L'B$.

So

$$\begin{aligned} L' \sim L'B &\sim \begin{pmatrix} M_1'' & B_{12} \\ B_{21} & M_2'' \end{pmatrix} \sim I_{m+n-4} \oplus \begin{pmatrix} m & \alpha & 1 \\ 0 & p'_{m-2} & 0 \\ n & -1 & -\beta \end{pmatrix} \\ &\sim I_{m+n-4} \oplus \begin{pmatrix} m & \alpha & 1 \\ 0 & p'_{m-2} & 0 \\ m\beta+n & \alpha\beta-1 & 0 \end{pmatrix} \\ &\sim I_{m+n-3} \oplus \begin{pmatrix} p'_{m-2} & 0 \\ \alpha\beta-1 & m\beta+n \end{pmatrix}. \end{aligned}$$

□

Proof of Theorem 1.2 From Lemma 3.1, we immediately have this theorem. □

Remark 3.2 It is known that the Laplacian eigenvalues of P_m are: $0, 2+2\cos\left(\frac{\pi j}{m}\right)$, $(1 \le j \le m-1)$. Then it follows from Theorem 2.1 in [10] that the Laplacian eigenvalues of $P_m \vee P_n$ are: 0, $m+n$, $n+2+2\cos\left(\frac{i\pi}{m}\right)(1 \le i \le m-1)$, $m+2+2\cos\left(\frac{j\pi}{n}\right)(1 \le j \le n-1)$. Then by the well-known Kirchhoff Matrix-Tree Theorem we know that the spanning tree number of $P_m \vee P_n$ is $\kappa(P_m \vee P_n) =$

$\prod_{i=1}^{m-1}\left(n+2+2\cos\left(\frac{i\pi}{m}\right)\right)\prod_{j=1}^{n-1}\left(m+2+2\cos\left(\frac{j\pi}{n}\right)\right)$. Recalling Theorem 1.2, we have

$$\begin{aligned}&\prod_{i=1}^{m-1}\left(n+2+2\cos\left(\tfrac{i\pi}{m}\right)\right)\prod_{j=1}^{n-1}\left(m+2+2\cos\left(\tfrac{j\pi}{n}\right)\right)\\ &=\frac{\left(\left(n+2+\sqrt{n^2+4n}\right)^m-\left(n+2-\sqrt{n^2+4n}\right)^m\right)\left(\left(m+2+\sqrt{m^2+4m}\right)^n-\left(m+2-\sqrt{m^2+4m}\right)^n\right)}{2^{m+n}\sqrt{(n^2+4n)(m^2+4m)}}.\end{aligned}\tag{3.3}$$

Example 3.3

If $m=4$ and $n=5$, then $p_3'=329$, $\alpha=65$, $\beta=296$. If $m=4$ and $n=6$, then $p_4'=496$, $\alpha=82$, $\beta=1731$. If $m=5$ and $n=5$, then $p_3'=2255$, $\alpha=450$, $\beta=450$. So it follows from Theorem 3.2 that we have the following:

$$\mathrm{Snf}(P_4\vee P_5) = I_7\oplus \mathrm{diag}(391181,0); \tag{3.4}$$
$$\mathrm{Snf}(P_4\vee P_6) = I_8\oplus \mathrm{diag}(3437280,0); \tag{3.5}$$
$$\mathrm{Snf}(P_5\vee P_5) = I_7\oplus \mathrm{diag}(451,11275,0). \tag{3.6}$$

Here we also note that one can use MAPLE to check the results of (3.4), (3.5) and (3.6).

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