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More directions in visibility graphs

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Abstract

In this paper we introduce unit bar k-visibility graphs, which are bar k-visibility graphs in which every bar has unit length. We show that almost all families of unit bar k-visibility graphs and unit bar k-visibility graphs are incomparable under set inclusion. In addition, we establish the largest complete graph that is a unit bar k-visibility graph. As well, we present a family of hyperbox visibility graphs that provide edge maximal rectangle visibility graphs in every possible standard 2-dimensional cross-section. We end with a list of open questions.

1 Background and Definitions

Classes of visibility graphs have applications in VLSI design and graph layout [6]. Let R be a set of horizontal closed line segments, or ***bars***, in the plane, at distinct heights. We say that a graph G is a ***bar visibility graph***, and R a ***bar visibility representation*** of G, if there exists a one-to-one correspondence between vertices of G and bars in R, such that there is an edge between two vertices in G if and only if there exists an unobstructed vertical line of sight between their corresponding bars. Formally, two vertices x and y in G are adjacent if and only if, for their corresponding bars X and Y in R, there exists a vertical line segment ℓ, called a ***line of sight***, whose endpoints are contained in X and Y, respectively, and which does not intersect any other bar in S. A ***bar k-visibility graph*** is a graph with a bar visibility representation in which a line of sight between bars X and Y intersects at most k additional bars [4]. A ***unit bar k-visibility graph*** is a graph which has a bar k-visibility representation in which every bar has unit length. Bar 0-visibility graphs are bar visibility graphs. Similarly, unit bar 0-visibility graphs are unit bar visibility graphs. The characterization of unit bar visibility graphs was begun in [5]. On the other hand, bar k-visibility graphs are interval graphs for large enough values of k. So bar k-visibility graphs can be thought of as existing between bar visibility graphs and interval graphs.

A ***proper*** interval graph is a graph that has an interval representation in which no interval is properly contained in another [1]. Analogously, a ***proper*** bar k-visibility graph is a graph that has a bar k-visibility representation in which no bar contains another bar when considered as intervals. We omit the proof of the following proposition since it is a straightforward modification of the technique given in [1].

Proposition 1 *A graph is a unit bar k-visibility graph if and only if it is a proper bar k-visibility graph.*

A ***rectangle visibility graph*** (RVG) is a graph G whose vertices can be represented in the plane by a set R of closed disjoint rectangles whose sides are parallel to the x- and y-axes. Two vertices u and v in G are adjacent if and only if their corresponding rectangles r_v and r_u in R have a line of sight between them, parallel to one of the axes, that intersects no other rectangle in R. The analogy in d dimensions is a d-box visibility graph in which each vertex is represented by a d-dimensional hyperbox whose sides are parallel to the standard axes in $\mathbb{R}^d$; two vertices are adjacent if and only if there is an unobstructed line of sight between them, parallel to one of the standard axes. We define d-box visibility graphs formally in Section 4.

There are two different standard definitions of visibility graphs, using lines of sight and non-degenerate rectangles of sight. In general more edges may be blocked using rectangles of sight, and thus more graphs can be represented in this way. In [4, 3] it is shown that there exists a representation of an edge-maximal visibility graph G using lines of sight if and only if there exists a representation of G using bands of sight in two dimensions; the proof generalizes to d dimensions as well. That is, there exists an edge-maximal visibility graph G using lines of sight if and only if there exists a representation of G using non-degenerate d-dimensional bands of sight. Our present interest is in edge-maximal graphs, and thus we use lines of sight.

2 Unit Bar k-Visibility Graphs

Let G be a unit bar k-visibility graph and R a set of closed horizontal unit line segments in the plane that represents G. The location of each bar A in R is uniquely determined by the x and y coordinates of its left endpoint, which we will denote by $x(A)$ and $y(A)$, respectively.

Theorem 2 *For every $k \geq 0$ the graph K_{3k+3} is a unit bar k-visibility graph, and is the largest complete unit bar k-visibility graph.*

Proof: Let G be a complete graph with n vertices, and let R be a unit bar k-visibility representation of G. We show that $n \leq 3k+3$. Suppose that A and B are the two bars in R that have the smallest and largest y-values, respectively. In other words, $y(A) < y(C) < y(B)$ for all C other than A and B in R. Without loss of generality we may assume that bars in R have distinct x-coordinates. Therefore we may also assume by symmetry that $x(A) < x(B)$.

We partition the remaining bars in R into three types. We say a bar C in R is *type I* if $x(C) < x(A)$, *type II* if $x(A) < x(C) < x(B)$, and *type III* if $x(B) < x(C)$, as shown in Figure 1.

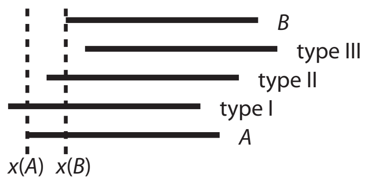

Figure 1: The three types of bars in a unit bar k-visibility representation of K_n.

Suppose there are a type I bars in R, b type II bars, and c type III bars. Then R has a total of $a+b+c+2$ bars. We assume without loss of generality that $a \geq c$. Note that every type I bar must have a vertical line of sight to every type III bar because R is a representation of a complete graph. Thus any line of sight between A and B must pass through b type II bars, and either a type I bars or c type III bars. If $b+c > k$, then there are no lines of sight between A and B, which contradicts the fact that G is a complete graph. Therefore $b+c \leq k$.

Let D be the type I bar with smallest x-coordinate. Then more than k type I bars below D block visibility between D and A, and more than k type I bars above D block visibility between D and B. Therefore $a \leq 2k+1$ and hence R has at most $a+(b+c)+2 \leq (2k+1)+k+2 = 3k+3$ bars. Therefore K_{3k+4} is not a unit bar k-visibility graph.

Conversely, Figure 2 shows a unit bar k-visibility representation of K_{3k+3}.

□

3 Incomparability of Bar Visibility Graph Families

Let $\mathcal{B}_k$ be the set of all bar k-visibility graphs, and let $\mathcal{U}_k$ be the set of all unit bar k-visibility graphs. Hartke, Vandenbussche, and Wenger proved that $\mathcal{B}_i \not\subseteq \mathcal{B}_k$ if $i \neq k$ [7]. Since every unit bar k-visibility representation is also a bar k-visibility representation, it follows that $\mathcal{U}_k \subseteq \mathcal{B}_k$ for all $k \geq 0$. In this section we prove additionally that $\mathcal{U}_i \not\subseteq \mathcal{U}_k$ if $i \neq k$, $\mathcal{U}_i \not\subseteq \mathcal{B}_k$ if $i \neq k$, and $\mathcal{B}_i \not\subseteq \mathcal{U}_k$ for all $i, k \geq 0$.

Theorem 3 *The sets $\mathcal{U}_i$ and $\mathcal{U}_k$ are incomparable under set inclusion for $i \neq k$.*

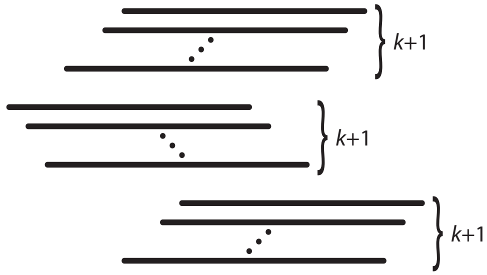

Figure 2: A unit bar k-visibility representation of K_{3k+3}.

Proof: By Theorem 2, K_{3k+3} is a unit bar k-visibility graph and not a unit bar i-visibility graph for $k > i$, and hence $\mathcal{U}_k \not\subseteq \mathcal{U}_i$ for $k > i$.

Conversely, the proof of Theorem 4 below shows that the graph $K_k \vee C_5$ is a unit bar i-visibility graph and not a unit bar k-visibility graph for $k > i$, and hence $\mathcal{U}_i \not\subseteq \mathcal{U}_k$ for $k > i$. □

In the following, $G_1 \vee G_2$ is the standard ***join*** of G_1 and G_2,obtained from the disjoint union of G_1 and G_2 by adding the edges $\{uv \,|\, u \in V(G_1), v \in V(G_2)\}$.

Theorem 4 *The sets $\mathcal{B}_i$ and $\mathcal{U}_k$ are incomparable under set inclusion for $i \neq k$.*

Proof: In [7], Hartke, Vandenbussche, and Wenger showed that the graph $K_k \vee C_5$ is not a bar k-visibility graph, but is a bar i-visibility graph for $i < k$. So $K_k \vee C_5$ is also not a unit bar k-visibility graph. On the other hand, the bar k-visibility representation of $K_k \vee C_5$ given in [7] is easily modified to obtain a unit bar k-visibility representation of $K_k \vee C_5$. Hence $\mathcal{U}_i, \mathcal{B}_i \not\subseteq \mathcal{U}_k, \mathcal{B}_k$ for $i < k$.

The graph K_4 is a bar 0-visibility graph and not a unit bar 0-visibility graph [5], so $\mathcal{B}_0 \not\subseteq \mathcal{U}_0$. Finally, the claw $K_{1,3}$ is a bar k-visibility graph for all $k \geq 0$, since Figure 3 shows a bar k-visibility representation of $K_{1,3}$ that is valid for any $k \geq 0$. On the other hand, one can easily check all arrangements of four unit-length bars to verify that none are a bar k-visibility representation of $K_{1,3}$ for any $k \geq 1$, so $K_{1,3}$ is not a unit bar k-visibility graph for any $k \geq 1$. So $\mathcal{B}_k \not\subseteq \mathcal{U}_i$ for any $k, i \geq 1$. □

Figure 3: A non-unit bar k-visibility representation of $K_{1,3}$.

In summary, $\mathcal{U}_k \subseteq \mathcal{B}_k$, but all other pairs of families of unit or general bar k-visibility graphs, even for different values of k, are incomparable under set inclusion.

4 A Family of d-Box Visibility Graphs

A ***d-box*** in $\mathbb{R}^d$ is a set of the form $\{(x_1, x_2, \ldots, x_d) \,|\, a_1 \le x_1 \le b_1, a_2 \le x_2 \le b_2, \ldots, a_k \le x_d \le b_d\}$ for some constants a_i, b_i, $1 \le i \le k$. Given a set of d-boxes R in $\mathbb{R}^d$ and two d-boxes X and Y in R, a ***line of sight*** between X and Y is a line segment parallel to one of the coordinate axes in $\mathbb{R}^d$, with one endpoint contained in X, one endpoint contained in Y, and no other points contained in any box in R. A graph G is a ***d-box visibility graph*** if there exists a set of d-boxes R in $\mathbb{R}^d$ and a one-to-one correspondence between vertices in G and d-boxes in R, such that there is an edge between two vertices in G if and only if there exists a line of sight between their corresponding boxes. A 2-box visibility graph is an RVG [8].

In this section we construct a family of d-box visibility representations inspired by the particularly elegant representation of K_8 as an RVG given in Figure 4. In our construction, each cross-section by an axis-aligned 2-dimensional plane that passes through the origin induces a representation with layers of four rectangles, each consecutive pair of which are equivalent to the eight rectangles in Figure 4, in the sense that their x- and y-coordinates occur in the same increasing order.

Our intention is to maximize the number of edges in certain 2-dimensional cross-sections of our particular representations of d-box visibility graphs and thus we assume without loss of generality that the boxes are pairwise disjoint. In other treatments of this subject, objects that represent vertices are allowed to intersect along a boundary for the purpose of representing a specific graph or specific family of graphs.

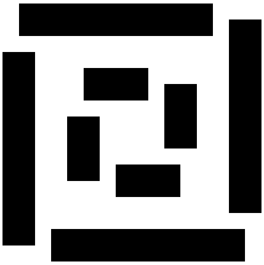

Figure 4: An elegant representation of K_8 as a rectangle visibility graph.

We show below that our construction yields a d-box visibility graph with n vertices and at least $d(3n - 4d - 2)$ edges. For ease in computation we restrict n to $n \equiv 0 \pmod{2d}$, so in particular we choose n to be even. A d-box visibility graph with $d(3n - 4d - 2)$ edges is not edge-maximal when $d = 3$ for large enough n. In particular, by a construction given by Bose et al [2] of a representation of K_{22} by a set of closed disjoint parallel rectangles in $\mathbb{R}^3$, one can construct a 3-box visibility graph with $n = 22r$ vertices and $231r$ edges by using r isolated copies of K_{22}. When $d = 3$ and $n = 22r$, the number of edges of the graph in our construction is $3(3(22r) - 12 - 2) = 198r - 42$.

We construct the representation with n boxes

$$R = \{B_0, \ldots, B_{\frac{n}{2}-1}\} \cup \{\overline{B}_0, \ldots, \overline{B}_{\frac{n}{2}-1}\}.$$

For each box $X \in R$, we give the d-tuple $C(X)$ of coordinates of its center and the d-tuple $L(X)$ of the lengths of its sides. Let $0 \le i \le n/2-1$. By the division algorithm, there exist unique integers q_i and r_i such that $i = q_i d + r_i$ with $0 \le r_i < d-1$. Box B_i will be in position r_i in layer q_i of the representation, and box $\overline{B}_i$ will be in the same layer in the opposite position, as in Figure 6.

We place the centers of boxes $B_0, \ldots, B_{n/2-1}$ as follows. For $0 \le i \le n/2-1$ let $C(B_i)$ and $C(\overline{B}_i)$ denote the (d-dimensional) centers of boxes B_i and $\overline{B}_i$, respectively, and let $X|_m$ denote the mth coordinate of vector X. Because of the notation used to parameterize a box by way of the division algorithm, it will be convenient to call the first coordinate of a vector the 0th coordinate. Then define $C(B_i) = \mathbf{a}_i + \mathbf{p}_i + \mathbf{s}_i$ and $C(\overline{B}_i) = -\mathbf{a}_i - \mathbf{p}_i + \mathbf{s}_i$, where for $i = 0, \ldots, n/2-1$ and $m = 0, \ldots, d-1$, $\mathbf{a}_i$, $\mathbf{p}_i$, and $\mathbf{s}_i$ are the d-dimensional vectors with

$$\mathbf{a}_i|_m = \begin{cases} 2(d-1)(4q_i+3) & \text{if } m = r_i \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

$$\mathbf{p}_i|_m = \begin{cases} 2r_i - 2m - 2d & \text{if } m < r_i \\ 0 & \text{if } m = r_i \\ 2m - 2r_i & \text{if } m > r_i, \end{cases} \tag{2}$$

and

$$\mathbf{s}_i|_m = q_i(4d-5) \quad \text{for } m = 0, 1, \ldots, d-1. \tag{3}$$

Similarly, let $L(B_i)$ and $L(\overline{B}_i)$ denote the side lengths of boxes B_i and $\overline{B}_i$, respectively, and define $L(B_i)$ and $L(\overline{B}_i)$ to be the d-dimensional vectors with

$$L(B_i)|_m = L(\overline{B}_i)|_m = \begin{cases} 4(d-1) & \text{if } m = r_i \\ 8(d-1)(2q_i+1) & \text{otherwise.} \end{cases} \tag{4}$$

We say that two sets of rectangles R_1 and R_2 are ***equivalent*** if there is a one-to-one correspondence between rectangles in R_1 and rectangles in R_2 such that the x- and y-coordinates of the corners of the rectangles occur in the same order. Note that if two graphs have equivalent rectangle visibility representations, they must be isomorphic graphs.

In what follows, let x and y be any two axes in d-space with $0 \le x, y \le d-1$, and call the intersections of the boxes B_i and $\overline{B}_i$ with the xy-plane S_i and $\overline{S}_i$, respectively. Let R' be the set of rectangles obtained by intersecting the xy-plane with R, and let G' be the rectangle visibility graph with rectangle visibility representation R'. Let $R'(q, q+1)$ be the set of rectangles of R' with $q_i = q$ or $q_i = q+1$, for some integer q between 0 and $n/2d - 1$.

Lemma 5 *The set of rectangles $R'(q, q+1)$ is equivalent to the representation in Figure 4.*

Proof: First we claim that B_i or $\overline{B_i}$ intersects the xy-plane if and only if $r_i = x$ or $r_i = y$. To see this, observe that B_i intersects the xy-plane if and only if $|C(B_i)_m| < L(B_i)_m/2$ for every coordinate $m \neq x, y$. But $|C(B_i)_{r_i}| = 2(d-1)(4q_i+3) + q_i(4d-5) > 2(d-1) = L(B_i)_{r_i}/2$, and thus B_i intersects the xy-plane only if $r_i = x$ or $r_i = y$. Since $|C(B_i)_m| < 4(d-1)(2q_i+1) = L(B_i)_m/2$ for $m \neq r_i$, the claim is true for B_i. A similar argument establishes the claim for $\overline{B_i}$.

There are eight rectangles in $R'(q, q+1)$, and since each rectangle has two distinct corner x-coordinates and two distinct corner y-coordinates, these rectangles have a total of 16 potentially distinct x-coordinates and 16 potentially distinct y-coordinates. Specifically, if rectangle S_i has $r_i = x$ then S_i has center $(2(d-1)(4q_i+3) + q_i(4d-5), 2(y-x) + q_i(4d-5))$ and side lengths $4(d-1)$ and $8(d-1)(2q_i+1)$; if rectangle $\overline{S}_i$ has $r_i = x$ then $\overline{S}_i$ has center $-(2(d-1)(4q_i+3) + q_i(4d-5), -2(y-x)+q_i(4d-5))$ and side lengths $4(d-1)$ and $8(d-1)(2q_i+1)$; if rectangle S_i has $r_i = y$ then S_i has center $(2(y-x-d)+q_i(4d-5), 2(d-1)(4q_i+3)+q_i(4d-5))$ and side lengths $8(d-1)(2q_i+1)$ and $4(d-1)$; if rectangle $\overline{S}_i$ has $r_i = y$ then $\overline{S}_i$ has center $(-2(y-x-d) + q_i(4d-5), -2(d-1)(4q_i+3) + q_i(4d-5))$ and side lengths $8(d-1)(2q_i+1)$ and $4(d-1)$.

Figure 5 shows the order in which these corners should occur. In this figure, rectangle S_i with $q_i = q$ and $r_i = x$ is denoted by (q, x), and rectangle $\overline{S}_i$ with $q_i = q$ and $r_i = x$ is denoted by $(q, -x)$.

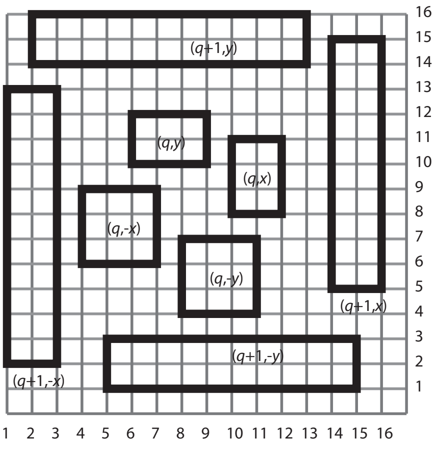

Figure 5: The ordering of the x- and y-coordinates of two layers of R'.

We verify that these x- and y-coordinates occur in the desired order in the following table. The coordinates in the table are listed in the same order that they appear in Figure 5. Note that $0 \leq x < y \leq d-1$, so $1 \leq y - x \leq d-1$. Each expression can be seen to be less than the one below it.

□

	x-coordinates	y-coordinates
1.	$-4dq_i - 12d + 3q_i + 11$	$-4q_id - 12d + 3q_i + 11$
2.	$-4dq_i - 10d + 2(y-x) + 3q_i + 7$	$-4q_id - 8d - 2(y-x) + 3q_i + 7$
3.	$-4dq_i - 8d + 3q_i + 7$	$-4q_id - 8d + 3q_i + 7$
4.	$-4dq_i - 8d + 3q_i + 8$	$-4q_id - 8d + 3q_i + 8$
5.	$-4dq_i - 6d - 2(y-x) + 3q_i + 7$	$-4q_id - 8d + 2(y-x) + 3q_i + 7$
6.	$-4dq_i - 6d + 2(y-x) + 3q_i + 4$	$-4q_id - 4d - 2(y-x) + 3q_i + 4$
7.	$-4dq_i - 4d + 3q_i + 4$	$-4q_id - 4d + 3q_i + 4$
8.	$-4dq_i - 2d - 2(y-x) + 3q_i + 4$	$-4q_id - 4d + 2(y-x) + 3q_i + 4$
9.	$12dq_i + 2(y-x) + 2d - 13q_i - 4$	$12q_id + 4d - 2(y-x) - 13q_i - 4$
10.	$12dq_i + 4d - 13q_i - 4$	$12q_id + 4d - 13q_i - 4$
11.	$12dq_i + 6d - 2(y-x) - 13q_i - 4$	$12q_id + 4d + 2(y-x) - 13q_i - 4$
12.	$12dq_i + 8d - 13q_i - 8$	$12q_id + 8d - 13q_i - 8$
13.	$12dq_i + 2(y-x) + 14d - 13q_i - 17$	$12q_id + 16d - 2(y-x) - 13q_i - 17$
14.	$12dq_i + 16d - 13q_i - 17$	$12q_id + 16d - 13q_i - 17$
15.	$12dq_i + 18d - 2(y-x) - 13q_i - 17$	$12q_id + 16d + 2(y-x) - 13q_i - 17$
16.	$12dq_i + 20d - 13q_i - 21$	$12q_id + 20d - 13q_i - 21$

Theorem 6 *The set R is a valid d-box visibility representation, and the graph G with d-box visibility representation R has at least $d(3n-4d-2)$ edges.*

Proof: First we show that any pair of distinct boxes X and Y in R are disjoint. Suppose X has $r_i = x$ and Y has $r_i = y$. Then the xy-plane intersects both X and Y by the proof of Lemma 5. Call the intersections of X and Y with the xy-plane S_1 and S_2. By Lemma 5, S_1 and S_2 are disjoint. Then since X and Y are axis-aligned, X and Y are also disjoint.

Next we count the vertices and edges in G'. Since R has $2d$ rectangles in each layer and R' has 4 rectangles in each layer, R' has $2n/d$ total rectangles and G' has $2n/d$ vertices. Note that since we chose $n \equiv 0 \pmod{2d}$, this is an integer. To count the edges in G', by Lemma 5, any rectangle S_i or $\overline{S}_i$ sees every rectangle in its own layer, the previous layer, and the next layer. So each rectangle sees 11 others, for a total of $11/2(2n/d)$ edges in G'. In addition, there are two extra visibilities in R' between each layer l and layer $l+2$, for a total of n/d additional edges. But we overcount on the first two and last two layers: the first and last layers have 9 fewer edges and the second and second-to-last layers have 1 fewer edges. So G' has a total of $11/2(2n/d) + n/d - 20 = 12n/d - 20$ edges.

Finally we show that G has at least $d(3n-4d-2)$ total edges. Each subgraph G' has $12n/d - 20$ edges, and there are $\binom{d}{2} = d^2/2 - d/2$ such subgraphs, one for each pair of axes. On the other hand, edges between boxes with the same value of r_i are counted once for each G' they appear in. Consider the subgraph G_{r_i} of any of the rectangle visibility graphs G' formed from rectangles with a fixed value of r_i. By a similar argument as the one which counted the edges of G', G_{r_i} has $5/2(n/d) + 1/2(n/d) - 6 = 3n/d - 6$ edges. There are d such subgraphs, one for each value of r_i. Each edge in G_{r_i} is counted $d-1$ times over all the graphs G', one for each other axis the r_i-axis is paired with, and should be counted only once, so is overcounted $d-2$ times. So the total number of edges in G is $(12n/d-20)(d^2/2-d/2)-(3n/d-6)d(d-2)=d(3n-3d-4)$. □

In [8] the authors show that an RVG on n vertices has at most $6n-20$ edges; we note that our construction for $d=2$ gives an alternative layout of an edge-maximal rectangle visibility graph.

4.1 Two Examples of d-Box Visibility Graphs

An example of a 2-box visibility graph is shown in rightmost picture in Figure 6; the first two pictures demonstrate the use of $\mathbf{a}_i$ and $\mathbf{p}_i$. The rightmost picture in Figure 6 is also a representation of an edge-maximal rectangle (or 2-box) visibility graph with 12 vertices; for ease in viewing, the boxes are labeled by i and $\bar{i}$ rather than B_i and $\overline{B_i}$ respectively. The black dot in the center of each frame is located at the origin (0,0) as a point of reference. The table following Figure 6 gives the evolution of the coordinates of the midpoints as well as the dimensions of each box.

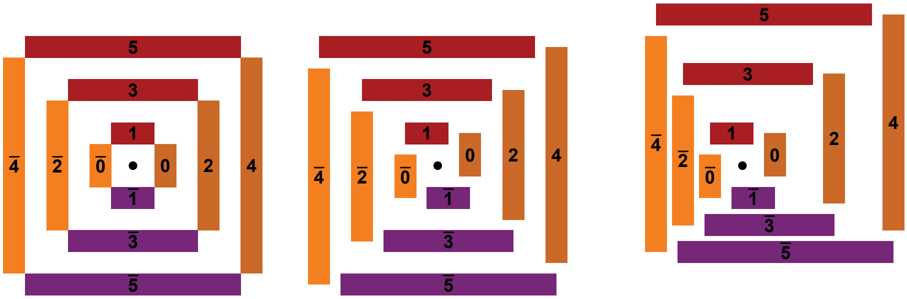

Figure 6: The construction in two dimensions. The centers of the boxes in the first two figures are coordinatized by $\pm\mathbf{a}_i$ and $\pm(\mathbf{a}_i+\mathbf{p}_i)$. The centers of the boxes on the right are $C(B_i)=\mathbf{a}_i+\mathbf{p}_i+\mathbf{s}_i$ and $C(\overline{B}_i)=-\mathbf{a}_i-\mathbf{p}_i+\mathbf{s}_i$.

A representation R for $d=3$ and $n=18$ is shown on the right in Figure 7. In the first three pictures in Figure 7, several boxes of R are removed in order to expose boxes that would otherwise be hidden from view.

Box	Midpt.: $\mathbf{a}_i$	Midpt.: $\mathbf{a}_i + \mathbf{p}_i$	Midpt.: $\mathbf{a}_i + \mathbf{p}_i + \mathbf{s}_i$	Dimensions
0	(6,0)	(6,2)	(6,2)	4×8
1	(0,6)	(-2,6)	(-2,6)	8×4
2	(14,0)	(14,2)	(17,5)	4×24
3	(0,14)	(-2,14)	(1,17)	24×4
4	(22,0)	(22,2)	(28,8)	4×40
5	(0,22)	(-2,22)	(4,28)	40×4
Box	Midpt.: $-\mathbf{a}_i$	Midpt.: $-\mathbf{a}_i - \mathbf{p}_i$	Midpt.: $-\mathbf{a}_i - \mathbf{p}_i + \mathbf{s}_i$	Dimensions
$\overline{\mathbf{0}}$	(-6,0)	(-6,-2)	(-6,-2)	4×8
$\overline{\mathbf{1}}$	(0,-6)	(2,-6)	(2,-6)	8×4
$\overline{\mathbf{2}}$	(-14,0)	(-14,-2)	(-11,1)	4×24
$\overline{\mathbf{3}}$	(0,-14)	(2,-14)	(5,-11)	24×4
$\overline{\mathbf{4}}$	(-22,0)	(-22,-2)	(-16,4)	4×40
$\overline{\mathbf{5}}$	(0,-22)	(2,-22)	(8,-16)	40×4

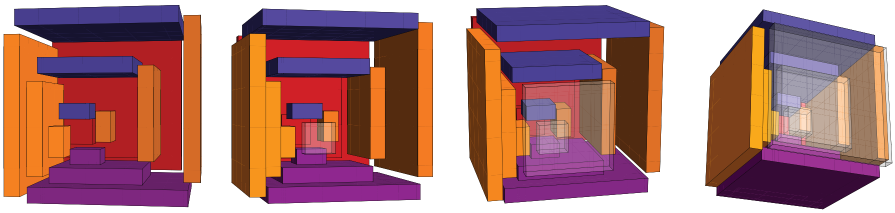

Figure 7: An example of a 3-box visibility graph with 18 vertices and 120 edges.

5 Open Questions

There are several interesting questions that follow naturally from this work. What is the largest number of edges in a d-box visibility graph? What is the largest complete d-box visibility graph? For $d = 3$ Bose et al [2] show that the answer lies between 22 and 56. Further, we can amalgamate both bar k-visibility and d-box visibility graphs to d-box k-visibility graphs, and ask for the largest complete graph and the largest number of edges of such graphs. It would also be interesting to investigate other standard graph parameters, such as the chromatic number of unit bar k-visibility graphs, d-box visibility graphs, and d-box k-visibility graphs. For example, the thickness of the 2-box visibility graph in Figure 6 is 2, and thus the chromatic number is at most 12.

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