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Optimal pebbling in products of graphs

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Abstract

We prove a generalization of Graham's Conjecture for optimal pebbling with arbitrary sets of target distributions. We provide bounds on optimal pebbling numbers of products of complete graphs and explicitly find optimal t-pebbling numbers for specific such products. We obtain bounds on optimal pebbling numbers of powers of the cycle C_5. Finally, we present explicit distributions which provide asymptotic bounds on optimal pebbling numbers of hypercubes.

1 Introduction

For a graph $G = (V, E)$, a function $D : V \to \mathbb{N}$ is called a *distribution on the vertices of* G, or a *distribution on* G. We usually imagine that $D(v)$ pebbles are placed on v for each vertex $v \in V$. Let $|D|$ denote the *size* of D, i. e. $|D| = \sum_{v \in V} D(v)$. For two distributions D and D' on G, we say that D *contains* D' if $D'(v) \leq D(v)$ for all $v \in V$. We allow pebbling moves on the graph, and define the *pebbling number*, the *optimal pebbling number*, the *t-pebbling number*, and the *optimal t-pebbling number* of a graph as follows:

Definitions: A *pebbling move* in G takes two pebbles from a vertex $v \in V$, which contains at least two pebbles, and places a pebble on a neighbor of v. For two distributions D_1 and D_2, we say that D_2 is *reachable* from D_1 if there is some sequence of pebbling moves beginning with D_1 and resulting in a distribution which contains D_2. We say the distribution D is *solvable*, (respectively, *t-fold solvable*), if every distribution with one pebble (respectively, t pebbles) on a single vertex is reachable from D.

The traditional *pebbling number* and *t-pebbling number* of a graph G, denoted $\pi(G)$ and $\pi_t(G)$ respectively, were defined by Chung [1]. The *optimal pebbling number*, denoted $\pi^*(G)$, was defined by Pachter, Snevily, and Voxman [9]. We generalize to define the *optimal t-pebbling number* of G, denoted $\pi_t^*(G)$. We give those definitions now.

Definitions (Chung [1] and Pachter et al. [9]): The *t-pebbling number* of G is the smallest number $\pi_t(G)$ such that every distribution D with $|D| \geq \pi_t(G)$ is t-fold solvable. The *optimal t-pebbling number* of G, denoted $\pi_t^*(G)$, is the smallest number such that some distribution with $\pi_t(G)$ pebbles is t-fold solvable. In both cases we omit the t when $t = 1$. Thus, the *pebbling number* of G is $\pi(G) = \pi_1(G)$ and the *optimal pebbling number* of G is $\pi^*(G) = \pi_1^*(G)$.

The pebbling number was generalized in [5] to allow for an arbitrary set of target distributions. We define this generalization and extend it to define the optimal pebbling number of a set of distributions on G.

Definitions ([5]): Let $\mathcal{S}$ be a set of distributions on a graph G. We say a distribution D is *$\mathcal{S}$-solvable* if every distribution in $\mathcal{S}$ is reachable from D. The *pebbling number of $\mathcal{S}$ in G*, denoted $\pi(G, \mathcal{S})$, is the smallest number such that every distribution D with $|D| \geq \pi(G, \mathcal{S})$ is $\mathcal{S}$-solvable. The *optimal pebbling number of $\mathcal{S}$ in G*, denoted $\pi^*(G, \mathcal{S})$, is the smallest number such that some distribution D with $|D| = \pi^*(G, \mathcal{S})$ is $\mathcal{S}$-solvable.

If $\mathcal{S}_t(G)$ consists of all distributions with t pebbles on a single vertex, we have $\pi(G, \mathcal{S}_1) = \pi(G)$, $\pi^*(G, \mathcal{S}_1(G)) = \pi^*(G)$, $\pi(G, \mathcal{S}_t(G)) = \pi_t(G)$, and $\pi^*(G, \mathcal{S}_t(G)) = \pi_t^*(G)$.

2 Graham's Conjecture and Generalizations in Optimal Pebbling

Graham's Conjecture asserts a bound on the pebbling number of the Cartesian product of two graphs.

Definition: If $G = (V, E)$ and $G' = (V', E')$ are two graphs, their Cartesian product is the graph $G \square G'$ whose vertex set is the product

$$V_{G \square G'} = V \times V' = \{(x, x') : x \in V, x' \in V'\},$$

and whose edges are given by

$$E_{G \square G'} = \{((x, x'), (y, x')) : (x, y) \in E, x' \in V'\} \cup \{((x, x'), (x, y')) : (x', y') \in E', x \in V\}.$$

We also write G^d for the graph $G\Box G\Box\cdots\Box G$ with d copies of G in the product. Throughout this paper we follow the convention that $G=(V,E)$ and $G'=(V',E')$.

Chung [1] attributed Conjecture 2.1 to Graham.

Conjecture 2.1 (Graham's Conjecture) *For any graphs G and G', we have $\pi(G\Box G')\leq\pi(G)\pi(G')$.*

Conjecture 2.1 was generalized in [5] to accommodate the more general definitions of pebbling numbers with arbitrary sets of target distributions. The following definition of products of distributions first appeared in [3] and the definition of products of sets of distributions appeared in [5].

Definition ([3, 5]): If D and D' are distributions on G and G' respectively, then we define $D\cdot D'$ as the distribution on $G\Box G'$ such that

$$(D\cdot D')((x,x'))=D(x)D'(x')$$

for every vertex $(x,x')\in V(G\Box G')$. Similarly, if $\mathcal{S}$ and $\mathcal{S}'$ are sets of distributions on G and G' respectively, then $\mathcal{S}\cdot\mathcal{S}'$ is the set of distributions on $G\Box G'$ given by

$$\mathcal{S}\cdot\mathcal{S}'=\{D\cdot D' : D\in\mathcal{S} \text{ and } D'\in\mathcal{S}'\}$$

Also, for any integer s we define the distribution sD by $(sD)(x)=sD(x)$ for all $x\in V$.

Conjecture 2.2 ([5]) *For all graphs G and G', and all sets of distributions $\mathcal{S}$ and $\mathcal{S}'$ on G and G' respectively, we have $\pi(G\Box G',\mathcal{S}\cdot\mathcal{S}')\leq\pi(G,\mathcal{S})\pi(G',\mathcal{S}')$.*

In this section we prove the analog of Conjecture 2.2 for optimal pebbling.

Theorem 2.3 *Let D be an $\mathcal{S}$-solvable distribution on G and let D' be an $\mathcal{S}'$-solvable distribution on G'. Then $D\cdot D'$ is an $(\mathcal{S}\cdot\mathcal{S}')$-solvable distribution on $G\Box G'$. In particular, we have $\pi^*(G\Box G',\mathcal{S}\cdot\mathcal{S}')\leq\pi^*(G,\mathcal{S})\pi^*(G',\mathcal{S}')$.*

To show this, we first establish a few lemmas.

Lemma 2.4 *If D_1 and D_2 are distributions on the graph G such that D_2 is reachable from D_1, then for any integer s, the distribution sD_2 is reachable from sD_1.*

Proof: The distribution sD_1 may be regarded as s distinct copies of D_1. We can reach D_2 from each copy of D_1, so sD_2 is reachable from sD_1. □

Lemma 2.5 *Let G and G' be graphs. If D_1 and D_2 are distributions on G such that D_2 is reachable from D_1, then for any distribution D' on G', $D_2\cdot D'$ is reachable from $D_1\cdot D'$.*

Proof: For each $(x_i, y_j) \in V(G\Box G')$, the number of pebbles on (x_i, y_j) in the distribution $D_1 \cdot D'$ is given by $(D_1 \cdot D')((x_i, y_j)) = D_1(x_i)D'(y_j)$. Fix $y_j \in V'$. We write $G\Box\{y_j\}$ for the subgraph of $G\Box G'$ induced by the vertices whose second coordinate is y_j. Then $G\Box\{y_j\} \cong G$, and if we restrict $D_1 \cdot D'$ to $G\Box\{y_j\}$, we obtain the distribution $D'(y_j)D_1$. Since y_j is fixed, $D'(y_j)$ is a constant, so by Lemma 2.4, the distribution $D'(y_j)D_2$ is reachable in $G\Box\{y_j\}$. Repeating this for each $y_j \in V'$, we end up with a distribution in which each (x_i, y_j) has at least $D_2(x_i)D'(y_j) = (D_2 \cdot D')((x_i, y_j))$ pebbles, so $D_2 \cdot D'$ is reachable from $D_1 \cdot D'$. □

We are now ready to prove Theorem 2.3.

Proof of Theorem 2.3: Let D and D' be $\mathcal{S}$- and $\mathcal{S}'$-solvable distributions on G and G' respectively. To show that $D \cdot D'$ is $(\mathcal{S} \cdot \mathcal{S}')$-solvable on $G\Box G'$, let Δ be a distribution in $\mathcal{S} \cdot \mathcal{S}'$. Then we can write $\Delta = D_i \cdot D'_j$ for some $D_i \in \mathcal{S}$ and $D'_j \in \mathcal{S}'$. Also, D_i is reachable from D and D'_j is reachable from D'. Thus, by Lemma 2.5, $D_i \cdot D'_j$ is reachable from $D \cdot D'_j$, which is reachable from $D \cdot D'$.

If we choose D and D' so that $|D| = \pi^*(G)$ and $|D'| = \pi^*(G')$, we have

$$|D \cdot D'| = \sum_{x_i \in V} \sum_{y_j \in V'} D(x_i)D'(y_j) = \sum_{x_i \in V} D(x_i) \sum_{y_j \in V'} D'(y_j) = |D||D'|.$$

Thus, $D \cdot D'$ is an $(\mathcal{S} \cdot \mathcal{S}')$-solvable distribution on $G\Box G'$ with

$$|D||D'| = \pi^*(G, \mathcal{S})\pi^*(G', \mathcal{S});$$

therefore $\pi^*(G\Box G', \mathcal{S} \cdot \mathcal{S}') \le \pi^*(G, \mathcal{S})\pi^*(G', \mathcal{S}')$, as desired. □

Theorems 2.6 and 2.7 follow immediately from Theorem 2.3. Fu and Shiue [2] announced Theorem 2.7, the optimal pebbling analog to Graham's Conjecture. Shiue proved it in [10].

Theorem 2.6 *For all graphs G and G' and all positive integers s and t, we have $\pi^*_{st}(G\Box G') \le \pi^*_s(G)\pi^*_t(G')$.*

Theorem 2.7 (Fu and Shiue [2, 10]) *For all graphs G and G', we have $\pi^*(G\Box G') \le \pi^*(G)\pi^*(G')$.*

3 Products of Complete Graphs

Our work in Section 2 puts an upper bound on $\pi^*_t(G\Box G')$. In this section, we improve those bounds when G and G' are complete graphs. Our main result is Theorem 3.1.

Theorem 3.1 *For any graph G and any positive integer t, we have $\left\lceil \left(\frac{n}{n+1}\right) \pi^*_{2t}(G) \right\rceil \le \pi^*_t(G\Box K_n) \le \pi^*_{2t}(G)$.*

Lemma 3.2 helps us find the optimal t-pebbling number of a complete graph (Theorem 3.3). We denote $V(K_n)$ by $\{v_1, v_2, \ldots, v_n\}$.

Definition: Given any distribution of pebbles on the vertices of the graph G, we say the vertex v is *odd* or *even*, depending on whether it has an odd or an even number of pebbles on it.

Lemma 3.2 *Let t and n be positive integers, and suppose we have a t-fold solvable distribution with $\pi_t^*(K_n)$ pebbles on the vertices of K_n. Then:*

1. *If the vertex v_i is odd, then every other vertex has at least as many pebbles as v_i.*
2. *There are at most two odd vertices.*
3. *If there are two odd vertices in K_n, then moving a pebble from one of these vertices to the other creates another t-fold solvable distribution.*

In particular, some t-fold solvable distribution of $\pi_t^*(K_n)$ pebbles on K_n has at most one odd vertex.

Proof: Removing a pebble from an odd vertex v_i does not affect the number of pebbles that may be moved to any other vertex; thus, every other vertex may still receive t pebbles. Since there would now be fewer than $\pi_t^*(K_n)$ pebbles, v_i could no longer receive t pebbles. If another vertex v_j started with fewer pebbles than v_i, we could use the pebbles now on v_i and v_j to put at least as many pebbles on v_i as on v_j, and any pebbles that could be moved to v_j from other vertices could also be moved to v_i. Thus, we could put at least as many pebbles on v_i as on v_j, contradicting our assertion that t pebbles can be moved to v_j, but not to v_i. Therefore, every other vertex has at least as many pebbles as the odd vertex v_i.

If there are two or more odd vertices in K_n, we remove a pebble from each of these vertices and add two pebbles to any vertex, say v_1. Now every vertex can receive at least as many pebbles as it could from the original distribution: if the target originally was odd, the first move would be from v_1 to the target. We therefore have a t-fold solvable distribution in which every vertex is even. Furthermore, if we originally had three or more odd vertices, this distribution would have fewer pebbles, contradicting the hypothesis that the original distribution had $\pi_t^*(K_n)$ pebbles. □

Theorem 3.3 *For any positive integers n and t, let $q = t \operatorname{div} (n+1)$ and let $r = t \bmod (n+1)$. Thus, $t = (n+1)q + r$. Then $\pi_t^*(K_n)$ is given by*

$$\pi_t^*(K_n) = \begin{cases} 2t - 2q = 2nq + 2r & \text{if } r < n \\ 2t - 2q - 1 = 2nq + 2n - 1 & \text{if } r = n \end{cases}$$

In particular, $\pi_t^*(K_n) = 2t$ if and only if $t < n$.

Proof: First note that if we put $2q+2r$ pebbles on one vertex and we put $2q$ pebbles on every other vertex, then we can move an additional $(n-1)q + r$ pebbles onto any vertex that starts with $2q$ pebbles, and we can move $(n-1)q$ additional pebbles onto the vertex that starts with $2q+2r$ pebbles. In either case, we can move at least $t=(n+1)q+r$ pebbles to any target, including the pebbles that start there. Thus, $\pi_t^*(K_n)\le 2nq+2r$.

We now consider whether a t-fold solvable distribution in K_n could have fewer than $2nq+2r$ pebbles. Let v_i be the vertex with the fewest pebbles, and suppose it has p_i pebbles. Adding $t-p_i$ pebbles to v_i costs at least $2(t-p_i)$ pebbles. Therefore, including the pebbles that started on v_i, the original distribution has at least $2t-p_i$ pebbles. If this is less than $2nq+2r=2t-2q$, then $p_i>2q$.

If $p_i\ge 2q+2$, every vertex has at least $2q+2$ pebbles, and so the distribution uses at least $(2q+2)n=2nq+2n\ge 2nq+2r$ pebbles. Therefore, we assume $p_i=2q+1$. Now by Lemma 3.2, we may assume every other vertex has at least $2q+2$ pebbles, so we have already accounted for $(2q+2)n-1=2nq+2n-1$ pebbles. The only way this can be smaller than $2nq+2r$ is if $r=n$. Now we simply observe that if $r=n$ and $2q+1$ pebbles are on v_i and $2q+2$ pebbles are on every other vertex, then a total of $(2q+1)+(n-1)(q+1)=(n+1)q+n=t$ pebbles can be moved to v_i, and similarly, $(2q+2)+(n-2)(q+1)+q=(n+1)q+n=t$ pebbles can be moved to any other vertex. Finally, $\pi_t^*(K_n)=2t$ if and only if $q=0$ and $r<n$, i. e. if and only if $t<n$. □

The optimal t-pebbling number is not generally monotone, in the following sense. If it is large for a particular graph, it can be reduced significantly by the addition of a single vertex adjacent to all others. However, for complete graphs the parameter is nondecreasing.

Proposition 3.4 *For every graph G and every positive integer n, we have $\pi_t^*(G\Box K_n)\le\pi_t^*(G\Box K_{n+1})$.*

Proof: Given any distribution $D: G\Box K_{n+1}\to\mathbb{N}$, let $g(D): G\Box K_n\to\mathbb{N}$ be the distribution on $G\Box K_n$ defined by

$$(g(D))(w,v_i)=\begin{cases} D(w,v_i) & \text{if } i<n\\ D(w,v_n)+D(w,v_{n+1}) & \text{if } i=n.\end{cases}$$

Then any pebbling move from D to D' in $G\Box K_{n+1}$ can be shadowed by moves from $g(D)$ to a distribution that contains $g(D')$ in $G\Box K_n$: moves from (w,v_n) to (w,v_{n+1}) or *vice versa* may be ignored, other moves from D to D' either from, to, or within $G\Box\{v_{n+1}\}$ can be made from $g(D)$ to $g(D')$ using $G\Box\{v_n\}$ instead, and moves from D to D' that do not use $G\Box\{v_{n+1}\}$ can be made unchanged from $g(D)$ to $g(D')$. Therefore, if D is a t-fold solvable distribution on $G\Box K_{n+1}$ then $g(D)$ is a t-fold solvable distribution on $G\Box K_n$. Since $|g(D)|=|D|$, we have $\pi_t^*(G\Box K_n)\le\pi_t^*(G\Box K_{n+1})$. □

Corollary 3.5 follows from Proposition 3.4 by induction on n, starting with $n=m$ as a basis.

Corollary 3.5 *For every graph G and all positive integers m and n with $m\le n$, we have $\pi_t^*(G\Box K_m)\le\pi_t^*(G\Box K_n)$.* □

Definitions: Given a distribution $D : V(G\Box G') \to \mathbb{N}$ on $G\Box G'$ and a subset $S \subseteq V'$, we define the distribution $f_S(D) : V \to \mathbb{N}$ on G by

$$(f_S(D))(u) = \sum_{v\in V'\setminus S} D(u,v) + 2\sum_{v\in S} D(u,v).$$

for every $u \in V$. In other words, we count every pebble on a vertex whose coordinate in G' is in S twice and every other pebble once. If the vertices of G' are $\{v_1, v_2, \ldots, v_n\}$, we define $f_i(D)$ by

$$(f_i(D))(u) = (f_{\{v_i\}}(D))(u) = \sum_{j\neq i} D(u,v_j) + 2D(u,v_i).$$

Lemmas 3.6 and 3.7 are key to proving Theorem 3.8, which is the upper bound in Theorem 3.1.

Lemma 3.6 *Let S be any nonempty subset $S \subseteq V'$, and suppose there is a sequence of pebbling moves in $G\Box G'$ from D_0 to D_k. Then there is a sequence of pebbling moves in G from $f_S(D_0)$ to a distribution that contains $f_S(D_k)$. In particular, if $|f_S(D_0)| < \pi^*_{2t}(G)$, then D_0 cannot be t-fold solvable in $G\Box G'$.*

Proof: Let $D_0, D_1, \ldots, D_k$ be the sequence of distributions in $G\Box G'$ after each pebbling move. We show by induction that we can shadow each pebbling move in $G\Box G'$ with moves in G. Toward that end, suppose that there is a sequence of pebbling moves in G from $f_S(D_0)$ to a distribution that contains $f_S(D_i)$. The basis $i = 0$ is trivial.

Suppose going from D_i to D_{i+1} requires a move from (u, v_1) to (u, v_2). Then the pebbles involved in the move add either four or two pebbles to u in $f_S(D_i)$, depending on whether $v_1 \in S$, and they add either two pebbles or one pebble to u in $f_S(D_{i+1})$, depending on whether $v_2 \in S$ or not. In either case, $f_S(D_i)$ contains $f_S(D_{i+1})$, and we can simply ignore the extra pebbles.

Otherwise, going from D_i to D_{i+1} requires a move from (u_1, v) to (u_2, v). If $v \in S$ the pebbles involved in this move add four pebbles to u_1 and two pebbles to u_2 in $f_S(D_i)$ and $f_S(D_{i+1})$, and if $v \notin S$, they add two pebbles to u_1 and one pebble to u_2 in $f_S(D_i)$ and $f_S(D_{i+1})$, respectively. The latter case simply requires a pebbling move from u_1 to u_2 in G to get from $f_S(D_i)$ to $f_S(D_{i+1})$; the former case requires two such moves.

In any of these cases, we can go from $f_S(D_0)$ to a distribution that contains $f_S(D_i)$ to one that contains $f_S(D_{i+1})$. Continuing this process, we reach a distribution that contains $f_S(D_k)$.

Now if $|f_S(D_0)| < \pi^*_{2t}(G)$, there is some vertex $x \in V$ such that $2t$ pebbles cannot be moved onto x by any sequence of pebbling moves starting from $f_S(D_0)$. Therefore, we cannot reach any distribution D_k in $G\Box G'$ for which $(f_S(D_k))(x) \geq 2t$. In particular, for any $s \in S$, we cannot move t pebbles onto the vertex (x, s). □

Lemma 3.7 tells us that if some copy of G in $G\Box K_n$ starts with a single pebble, then that pebble does not help us reach vertices in any other copy of G.

Lemma 3.7 *Let $D : V(G\Box K_n) \to \mathbb{N}$ be a distribution of pebbles on $G\Box K_n$, and suppose there is at most one pebble on some $G\Box\{v_i\}$. Let D' be the distribution on $G\Box K_n$ obtained by removing that pebble, or let $D' = D$ if there is no such pebble. Let $S = V(K_n)\setminus\{v_i\}$, and let D'' be any distribution of pebbles on $G\Box S$ that is reachable from D. Then D'' is reachable from D'.*

Proof: If there are no pebbles on $G\Box\{v_i\}$ and $D = D'$, there is nothing to prove, so we assume there is a pebble on $G\Box\{v_i\}$ in D. Paint this pebble gold, and assume it survives every pebbling move in the sequence from D to D'' in which it participates.

If the gold pebble never leaves $G\Box\{v_i\}$, we can make the same moves in D' as in D and ignore the moves involving the gold pebble. Otherwise, let v_j be the vertex in K_n involved in the first move of the gold pebble from (x, v_i) to (x, v_j). We examine the moves by the gold pebble before it leaves $G\Box\{v_i\}$. Note that every such move consumes a nongold pebble that was moved onto $G\Box\{v_i\}$ from a different copy of G. Our approach is to move those pebbles to $G\Box\{v_j\}$ instead.

Thus, from D', we ignore all moves from D involving the gold pebble before it first leaves $G\Box\{v_i\}$. We replace all other moves to, from, or within $G\Box\{v_i\}$ with moves to, from, or within $G\Box\{v_j\}$, ignoring moves between $G\Box\{v_i\}$ and $G\Box\{v_j\}$. Now the pebble that would have been removed from (x, v_i) when the gold pebble moved to (x, v_j) reaches (x, v_j) in place of the gold pebble. This pebble can replace of the gold pebble on all subsequent moves. The result of these changes is that all pebbles that ended up on $G\Box S$ starting from D end up on the same vertices starting from D', except that the gold pebble is replaced by a different pebble. □

Notation: Suppose we have a distribution of pebbles on $G\Box K_n$. For each i with $1 \le i \le n$, we let p_i be the number of pebbles on $G\Box\{v_i\}$, and we assume without loss of generality that $p_1 \le p_2 \le \cdots \le p_n$.

Theorem 3.8 gives the upper bound from Theorem 3.1.

Theorem 3.8 *For any graph G and any positive integer n, we have $\pi_t^*(G\Box K_n) \le \pi_{2t}^*(G)$. Furthermore, equality holds when $2n \ge \pi_{2t}^*(G) + 1$.*

Proof: We first note that if D is a $2t$-fold solvable distribution on G, then placing $D(x)$ pebbles on the vertex (x, v_1) for every $x \in V$ creates a distribution from which t pebbles can be moved to the vertex (x_i, v_j) since we can first move $2t$ pebbles to (x_i, v_1). Therefore, $\pi_t^*(G\Box K_n) \le \pi_{2t}^*(G)$.

Now suppose $2n \ge \pi_{2t}^*(G) + 1$, and let D be a distribution on $G\Box K_n$ with $\pi_{2t}^*(G) - 1$ pebbles or fewer. Then either $p_1 = 0$ or $p_1 = p_2 = 1$; otherwise, we would have $1 \le p_1$ and $2 \le p_2 \le p_3 \le \cdots \le p_n$. But then $|D| \ge 2n - 1 \ge \pi_{2t}^*(G)$, contrary to our assumption that D has at most $\pi_{2t}^*(G) - 1$ pebbles.

If $p_1 = 0$, then $f_1(D)$ has at most $\pi_{2t}^*(G) - 1$ pebbles, so $2t$ pebbles cannot be moved onto some $x \in V$ starting from $f_1(D)$. Therefore, by Lemma 3.6, t pebbles cannot be moved onto (x, v_1) starting from D. On the other hand, if $p_1 = p_2 = 1$, let D' be the distribution on $G\Box K_n$ with the lone pebble on $G\Box\{v_2\}$ removed. Then $|f_1(D')| \le \pi_{2t}^*(G) - 1$, since the pebble on $G\Box\{v_1\}$ that is counted twice is offset by the pebble that is removed from $G\square\{v_2\}$. As before, Lemma 3.6 shows that t pebbles cannot be moved to some (x, v_1) in $V(G\square K_n)$ starting from D'. But now applying Lemma 3.7 with $i = 2$ shows that t pebbles cannot be moved to (x, v_1) from D in this case either. Therefore, $\pi_t^*(G\square K_n) = \pi_{2t}^*(G)$. □

Applying Theorem 3.8 inductively gives Corollary 3.9.

Corollary 3.9 *For any graph G, any positive integer t, and any sequence of integers $n_1, n_2, \ldots n_d$, we have*

$$\pi_t^*(G\square K_{n_1}\square K_{n_2}\square\cdots\square K_{n_d}) \le \pi_{2^d t}^*(G).$$

Furthermore, equality holds if $2n_i \ge \pi_{2^d t}^*(G) + 1$ for each n_i.

Proof: We fix d and t, and prove by induction on k that

$$\pi_{2^{d-k}t}^*(G\square K_{n_1}\square K_{n_2}\square\cdots\square K_{n_k}) \le \pi_{2^d t}^*(G), \tag{1}$$

and that equality holds when each n_i satisfies $2n_i \ge \pi_{2^d t}^*(G) + 1$. The basis $k = 0$ is trivial, so we assume that (1) holds for some k with $0 \le k < d$. Applying Theorem 3.8 and then applying (1) gives

$$\begin{aligned}\pi_{2^{d-k-1}t}^*(G\square K_{n_1}\square K_{n_2}\square\cdots\square K_{n_k}\square K_{n_{k+1}}) \le \\ \pi_{2^{d-k}t}^*(G\square K_{n_1}\square K_{n_2}\square\cdots\square K_{n_k}) \le \pi_{2^d t}^*(G).\end{aligned}$$

as desired. Furthermore, equality still holds if n_{k+1} satisfies $2n_{k+1} \ge \pi_{2^d t}^*(G) + 1$. □

Corollary 3.10 *For all positive integers t, and any product of d complete graphs, we have*

$$\pi_t^*(K_{n_1}\square K_{n_2}\square\cdots\square K_{n_d}) = 2^d t$$

if and only if each $n_i \ge 2^{d-1}t + 1$.

Proof: Applying Corollary 3.9 with G equal to the trivial graph gives $\pi_t^*(K_{n_1}\square K_{n_2}\square \cdots\square K_{n_d}) \le \pi_{2^d t}^*(G) = 2^d t$. Furthermore, equality holds when each n_i satisfies $2n_i \ge 2^d t + 1$, or equivalently, $n_i \ge 2^{d-1}t + 1$. On the other hand, if $n_i \le 2^{d-1}t$ for some i, we assume without loss of generality that $n_1 \le 2^{d-1}t$. Now applying Corollary 3.9 with $G = K_{n_1}$ gives $\pi_t^*(K_{n_1}\square K_{n_2}\square\cdots\square K_{n_d}) \le \pi_{2^{d-1}t}^*(K_{n_1})$, and by Proposition 3.3, $\pi_{2^{d-1}t}^*(K_{n_1}) \le 2^d t - 1$ when $n_1 \le 2^{d-1}t$. □

We can now prove Theorem 3.1.

Proof of Theorem 3.1: The upper bound is given by Theorem 3.8. To establish the lower bound, suppose we have a t-fold solvable distribution D of $P = \pi_t^*(G\square K_n)$ pebbles on $G\square K_n$. Since $p_1 \le p_2 \le \cdots \le p_n$, we have $p_1 \le \frac{P}{n}$. Now by Lemma 3.6, $\pi_{2t}^*(G) \le |f_1(D)| = P + p_1 \le P + \frac{P}{n} = \left(\frac{n+1}{n}\right) P$. Since P must be an integer, we have $P \ge \left\lceil \left(\frac{n}{n+1}\right) \pi_{2t}^*(G)\right\rceil$. □

For the smallest of products, we are able to get exact results for all t. These exhibit a nice pattern that we will say more about subsequently. First we present an obvious proposition.

Proposition 3.11 *For any graph G and any positive integers s and t, we have $\pi^*_{s+t}(G) \leq \pi^*_s(G)+\pi^*_t(G)$. Similarly, for regular pebbling, we have $\pi_{s+t}(G) \leq \pi_s(G)+\pi_t(G)$.*

Proof: We can place $\pi^*_s(G)$ red pebbles and $\pi^*_t(G)$ blue pebbles on G in such a way that s red pebbles and t blue pebbles can be moved to any target vertex.

For regular pebbling, we note that from any placement of $\pi_s(G)+\pi_t(G)$ pebbles, if we arbitrarily paint $\pi_s(G)$ pebbles red and $\pi_t(G)$ pebbles blue, then s red pebbles and t blue pebbles can be moved to any target vertex. □

Proposition 3.12 *To find the optimal t-pebbling number of $K_2 \square K_2$, let $q = t$ div 9 and $r = t$ mod 9. Then*

$$\pi^*_t(K_2 \square K_2) = \begin{cases} 3 & \text{if } t = 1 \\ 16q + 2r & \text{if } r \in \{0,1,2,3,4,5\} \text{ and } t \neq 1 \\ 16q + 2r - 1 & \text{if } r \in \{6,7,8\}. \end{cases}$$

In each case except $t = 1$, the lower bound from Theorem 3.1 is tight.

Proof: If $t = 1$, we note that two pebbles are not enough to reach every vertex: if we put them on different vertices, the unoccupied vertices cannot be reached, and if we put them on the same vertex, the antipodal vertex is unreachable. On the other hand, three pebbles are sufficient, since we can put two pebbles on (v_0, v_0) and one on (v_1, v_1).

For $2 \leq t \leq 10$, we consider Table 1: The second row of this table gives the

t	2	3	4	5	6	7	8	9	10
$\lceil \frac{2}{3}\pi^*_{2t}(K_2) \rceil$	4	6	8	10	11	13	15	16	18
Optimal Distribution	2, 0	2, 1	2, 2	3, 2	4, 2	4, 3	4, 4	4, 4	5, 4
$p_{00}, p_{01}/p_{10}, p_{11}$	0, 2	1, 2	2, 2	2, 3	2, 3	3, 3	3, 4	4, 4	4, 5

Table 1: Computing $\pi^*_t(K_2 \square K_2)$ for $2 \leq t \leq 10$

lower bound for $\pi^*_t(K_2 \square K_2)$ from Theorem 3.1, and the last row gives a solvable distribution with the given number of pebbles. Therefore, the bound is tight.

Finally, for $t \geq 11$, we assume by induction on t that the lower bound is tight for $t' = t - 9$, and we show that $\pi^*_t(K_2 \square K_2) = \pi^*_{t'}(K_2 \square K_2) + 16$. Comparing the computation of the lower bound for $\pi^*_t(K_2 \square K_2)$ to that of $\pi^*_{t'}(K_2 \square K_2)$, we have $2t = 2t' + 18$, so $2t$ div $3 = 2t'$ div $3 + 6$, and $2t$ mod $3 = 2t'$ mod 3. Thus, $\pi^*_{2t}(K_2) = \pi^*_{2t'}(K_2) + 24$, and the lower bound from Theorem 3.1 gives $\pi^*_t(K_2 \square K_2) \geq \pi^*_{t'}(K_2 \square K_2) + 16$. On the other hand, Proposition 3.11 tells us that $\pi^*_t(K_2 \square K_2) \leq \pi^*_{t'}(K_2 \square K_2) + \pi^*_9(K_2 \square K_2) = \pi^*_{t'}(K_2 \square K_2) + 16$. Therefore, $\pi^*_t(K_2 \square K_2) = \pi^*_{t'}(K_2 \square K_2) + 16$, as required. □

We can compute $\pi^*_t(K_2 \square K_3)$ similarly.

Proposition 3.13 *The optimal t-pebbling number of $K_2 \Box K_3$ is*

$$\pi_t^*(K_2 \Box K_3) = \max\left(\left\lceil \frac{2}{3}\pi_{2t}^*(K_3)\right\rceil, \left\lceil \frac{3}{4}\pi_{2t}^*(K_2)\right\rceil\right).$$

In particular, if $q = t \operatorname{div} 6$ and $r = t \bmod 6$, then

$$\pi_t^*(K_2 \Box K_3) = \begin{cases} 12q & \text{if } r = 0 \\ 12q + 2r + 1 & \text{otherwise.} \end{cases}$$

Proof: For $1 \le t \le 6$, we use Table 2. For larger t, we note from Proposition 3.11

$\mathbf{t}$	1	2	3	4	5	6
$\lceil \frac{3}{4}\pi_{2t}^*(\mathbf{K_2})\rceil$	3	5	6	9	11	12
$\lceil \frac{2}{3}\pi_{2t}^*(\mathbf{K_3})\rceil$	3	4	7	8	11	12
Optimal Distribution	$2,0,0$	$2,0,1$	$2,0,2$	$2,2,2$	$2,2,2$	$2,2,2$
$p_{00}, p_{01}, p_{02}/p_{10}, p_{11}, p_{12}$	$1,0,0$	$0,2,0$	$1,2,0$	$1,1,1$	$2,2,1$	$2,2,2$

Table 2: Computing $\pi_t^*(K_2 \Box K_3)$ for $1 \le t \le 6$

that $\pi_{t'+6}^*(K_2 \Box K_3) \le \pi_{t'}^*(K_2 \Box K_3) + \pi_6^*(K_2 \Box K_3) = \pi_{t'}^*(K_2 \Box K_3) + 12$, which agrees with the asserted lower bound. □

Corollary 3.10 shows that for small values of t, the upper bound in Theorem 3.1 is tight for products of complete graphs. It was obtained by applying Theorem 3.8 inductively, with G being the trivial graph. If we apply the lower bound in Theorem 3.1 inductively with G being the trivial graph, we get a lower bound on the optimal t-pebbling number of a product of complete graphs. Theorem 3.15 shows that this lower bound is asymptotically tight as t gets large. We begin with Lemma 3.14.

Lemma 3.14 *Let $n_1, n_2, \ldots, n_d$ be a sequence of positive integers and let $T_j = \prod_{i=1}^{j}(n_i + 1)$. Then for any integer k, putting $2^d k$ pebbles on each vertex of $G = K_{n_1} \Box K_{n_2} \Box \cdots \Box K_{n_d}$ creates a kT_d-fold solvable distribution. Thus, $\pi_{kT_d}^*(G) = 2^d k \prod_{i=1}^{d} n_i$.*

Proof: If $d = 0$ the products are all empty, so $T_0 = 1$ and G is the trivial graph. Clearly, putting k pebbles on the lone vertex gives an optimal k-fold solvable distribution, as required. For larger d, we first show the specified distribution is kT_d-fold solvable. Toward that end, let $(x_1, x_2, \ldots, x_d)$ be the target vertex in $K_{n_1} \Box K_{n_2} \Box \cdots \Box K_{n_d}$. If we have $2^d k$ pebbles on each vertex, then for each $v \in V(K_{n_d})$, we have $2^{d-1}(2k)$ pebbles on each vertex of $K_{n_1} \Box K_{n_2} \Box \cdots \Box K_{n_{d-1}} \Box \{v\} \cong$

$K_{n_1} \square K_{n_2} \square \cdots \square K_{n_{d-1}}$. Therefore, by induction on d, we assume that we can put $2kT_{d-1}$ pebbles on $(x_1, x_2, \ldots, x_{d-1}, v)$. But now we have $2kT_{d-1}$ pebbles on $(x_1, x_2, \ldots, x_{d-1}, x_d)$ and we can move an additional kT_{d-1} pebbles from $(x_1, x_2, \ldots, x_{d-1}, v)$ to $(x_1, x_2, \ldots, x_{d-1}, x_d)$ for every vertex $v \neq x_d$. Thus, we can move a total of $(n_d + 1)kT_{d-1} = kT_d$ pebbles onto $(x_1, x_2, \ldots, x_{d-1}, x_d)$, as required, and so $\pi^*_{kT_d}(G) \leq 2^d k \prod_{i=1}^{d} n_i$.

Conversely, we know from Theorem 3.1 that

$$\pi^*_{kT_d}(K_{n_1} \square K_{n_2} \square \cdots \square K_{n_d}) \geq \left\lceil \left(\frac{n_d}{n_d + 1} \right) \pi^*_{2kT_d}(K_{n_1} \square K_{n_2} \square \cdots \square K_{n_{d-1}}) \right\rceil .$$

Now $2kT_d = 2k(n_d + 1)T_{d-1}$, and we may assume by induction on d that

$$\pi^*_{2k(n_d+1)T_{d-1}}(K_{n_1} \square K_{n_2} \square \cdots \square K_{n_{d-1}}) = 2^{d-1}(2k(n_d + 1)) \prod_{i=1}^{d-1} n_i = 2^d k(n_d + 1) \prod_{i=1}^{d-1} n_i.$$

Multiplying this number by $\frac{n_d}{n_d+1}$ gives an integer, which equals its own ceiling. Therefore,

$$\pi^*_{kT_d}(K_{n_1} \square K_{n_2} \square \cdots \square K_{n_d}) \geq 2^d k n_d \prod_{i=1}^{d-1} n_i = 2^d k \prod_{i=1}^{d} n_i,$$

which agrees with our upper bound. Therefore,

$$\pi^*_{kT_d}(K_{n_1} \square K_{n_2} \square \cdots \square K_{n_d}) = 2^d k \prod_{i=1}^{d} n_i.$$

□

Theorem 3.15 *Let $n_1, n_2, \ldots, n_d$ be a sequence of positive integers. Then for some constant C,*

$$\pi^*_t(K_{n_1} \square K_{n_2} \square \cdots \square K_{n_d}) \leq 2^d t \prod_{i=1}^{d} \frac{n_i}{n_i + 1} + C.$$

Proof: Let $G = K_{n_1} \square K_{n_2} \square \cdots \square K_{n_d}$, let $T = \prod_{i=1}^{d}(n_i + 1)$, and let C be given by

$$C = \max_{t<T} \left(\pi^*_t(G) - 2^d t \prod_{i=1}^{d} \frac{n_i}{n_i + 1} \right).$$

For any $t > T$, we can let $q = t \text{ div } T$ and $r = t \bmod T$. Then applying Proposition 3.11, we have

$$\pi^*_t(G) = \pi^*_{qT+r}(G) \leq \pi^*_{qT}(G) + \pi^*_r(G) \leq q\pi^*_T(G) + \pi_r(G).$$

From Lemma 3.14, we know that $\pi^*_{qT}(G) = 2^d q\prod_{i=1}^{d} n_i = 2^d qT\prod_{i=1}^{d}\frac{n_i}{n_i+1}$, and from the definition of C, we have

$$\begin{aligned}\pi^*_t(G) &\leq 2^d qT\prod_{i=1}^{d}\frac{n_i}{n_i+1} + 2^d r\prod_{i=1}^{d}\frac{n_i}{n_i+1} + C \\ &= 2^d(qT+r)\prod_{i=1}^{d}\frac{n_i}{n_i+1} + C \\ &= 2^d t\prod_{i=1}^{d}\frac{n_i}{n_i+1} + C,\end{aligned}$$

as desired. □

4 Optimal Fractional Pebbling

Fractional distributions and *fractional pebbling moves* were defined in [4]. These are continuous analogs of pebbling concepts. Moews [8] previously called them *continuous distributions*, and *continuous pebbling moves*, and he defined the *continuous optimal pebbling number* of a graph. We give these definitions now.

Definitions [4]: A *fractional distribution* on G is a function $D : V \to \mathbb{R}^+ \cup \{0\}$. Again, a distribution represents a placement of pebbles on the vertices of G, though we now allow a nonintegral number of pebbles. A *fractional pebbling move* consists of removing $2k$ pebbles from one vertex and adding k pebbles to an adjacent vertex. As in an integer-valued distribution, the *size* of D is given by $|D| = \sum_{v\in V} D(v)$, and D is *fractionally solvable*, (or simply *solvable* if there is no ambiguity), in case for every vertex v, it is possible to reach v with one pebble through some sequence of fractional pebbling moves, starting from D.

Moews [8] defined the *continuous optimal pebbling number* of a graph and denoted it by $\mathsf{ofc}(G)$. The *optimal fractional pebbling number* of the graph G, which we denote $\hat{\pi}^*(G)$, was defined in [4]. We give these definitions now.

Definitions [4, 8]: The *continuous optimal pebbling number* of a graph G, is the smallest number $\mathsf{ofc}(G)$ such that some fractional distribution D with $|D| =\mathsf{ofc}(G)$ is solvable using fractional pebbling moves. The *optimal fractional pebbling number* $\hat{\pi}^*(G)$ is given by

$$\hat{\pi}^*(G) = \liminf_{t\to\infty}\frac{\pi^*_t(G)}{t}.$$

Theorem 4.1 was shown in [4].

Theorem 4.1 ([4]) *Every graph G satisfies $\hat{\pi}^*(G) =\mathsf{ofc}(G)$. Furthermore, $\hat{\pi}^*(G)$ is rational for any graph G and every graph has an optimal fractional distribution in which the number of pebbles on each vertex is rational.* □

Moews [8] proved Theorem 4.2, and used it to give a nonconstructive proof of Theorem 4.3, which relates the optimal pebbling number of G^d to the continuous optimal pebbling number of G.

Theorem 4.2 (Moews [8]) *For all graphs G and G', we have $\hat{\pi}^*(G\Box G') = \hat{\pi}^*(G)\hat{\pi}^*(G')$.*

Theorem 4.3 (Moews [8]) *For all graphs G, we have $\pi^*(G^d) \le c\,(\hat{\pi}^*(G))^d \cdot d^k$ for some constants c and k.*

Theorem 4.5 generalizes some of our results from Section 3. We begin with Lemma 4.4.

Lemma 4.4 *For every graph G we have $\pi_t^*(G) \ge \hat{\pi}^*(G)t$ for all t.*

Proof: Suppose by contradiction that there is some t such that $\pi_t^*(G) < \hat{\pi}^*(G)t$. Let D be a t-fold solvable distribution on G with $|D| = \pi_t^*(G)$. Then the fractional distribution $\hat{D}$ given by $\hat{D}(v) = \frac{D(v)}{t}$ for all v is fractionally solvable. So, $\mathsf{ofc}(G) \le \frac{\pi_t^*(G)}{t} < \hat{\pi}^*(G)$, contradicting Theorem 4.1. □

It is shown in [6] that the lim inf in the definition of $\hat{\pi}^*(G)$ can be replaced by a limit. This implies that for large enough t, $\pi_t^*(G) \le \hat{\pi}^*(G)t + \varepsilon t$ for every $\varepsilon > 0$. Here we tighten the lower order term.

Theorem 4.5 *For every graph G there is a constant C such that $\pi_t^*(G) \le \hat{\pi}^*(G)t + C$.*

Proof: Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let a and b be integers satisfying $\hat{\pi}^*(G) = \frac{a}{b}$. From Theorem 4.1, there is some fractionally solvable fractional distribution $\hat{D}$ on G with $|\hat{D}| = \frac{a}{b}$ such that $\hat{D}(v_i) = \frac{a_i}{b_i}$ for some integers a_i and b_i. Under $\hat{D}$, if $\mathrm{dist}(v_i, v_j) = \delta$, then $\left(\frac{a_i}{b_i}\right) 2^{-\delta}$ pebbles could be sent from v_i to v_j by making fractional pebbling moves toward v_j. Since $\hat{D}$ is fractionally solvable, every vertex v satisfies

$$\sum_i \left(\frac{a_i}{b_i}\right) 2^{-\mathrm{dist}(v_i,v)} \ge 1.$$

Let $l = \mathrm{lcm}(b_1, b_2, \ldots, b_n)$ and let $k = 2^d l$, where $d = \mathrm{diam}(G)$. Given an integer t, the division algorithm produces integers q and r such that $t = kq + r = 2^d lq + r$ and $0 \le r \le k - 1$. Consider the distribution D on G given by $D(v_i) = kq\hat{D}(v_i) + r$ for all i. Under D, we have $D(v_i) = \left(\frac{a_i}{b_i}\right) 2^d lq + r$ for all i. Since l is a multiple of b_i, $D'(v_i) = D(v_i) - r$ is a multiple of 2^d. So, under the distribution D', it is possible to send $\left(\frac{a_i}{b_i}\right) 2^{d-\delta} lq$ pebbles from v_i to v_j. So, starting from D', the number of pebbles that can be sent to a root v is at least

$$\sum_i \left(\frac{a_i}{b_i}\right) 2^{d-\mathrm{dist}(v_i,v)} lq = 2^d lq \sum_i \left(\frac{a_i}{b_i}\right) 2^{-\mathrm{dist}(v_i,v)} \ge 2^d lq = t - r.$$

Thus, D' is $(t-r)$-fold solvable on G, meaning D is t-fold solvable on G. Since n and k are constants, letting $C = n(k-1)$ gives

$$\pi_t^*(G) \;\leq\; kq\left(\frac{a}{b}\right) + nr \;\;\leq\;\; \left(\frac{a}{b}\right)t + n(k-1) = \hat{\pi}^*(G)t + C,$$

as desired. □

We note that $\hat{\pi}^*(K_n) = \frac{2n}{n+1}$, $\hat{\pi}^*(K_2 \Box K_2) = \frac{16}{9}$, and $\hat{\pi}^*(K_2 \Box K_3) = 2 = \frac{12}{6}$. Thus, these specific cases of Theorem 4.5 are witnessed by Theorem 3.3 and Propositions 3.12 and 3.13, respectively.

5 Products of C_5

If we apply Theorem 4.3 to C_5, we find $\pi^*(C_5^d) \leq c2^d d^k$ for some constant k, since $\hat{\pi}^*(C_5) = 2$. However, Moews's proof of Theorem 4.3 was nonconstructive. It does not give distributions for small values of d, and it gives no information for small values of d. We give distributions that show that $\pi^*(C_5^d) \leq 4\sqrt{5}^{\,d-1}$. We let the vertices of C_5 be $\{v_0, v_1, v_2, v_3, v_4\}$. We begin by finding t-fold solvable distributions A_t on $C_5 \Box C_5$ for $t = 1$, $t = 2$, and $t = 4$.

Notation: We denote by A_1 the distribution with four pebbles on (v_0, v_0) and two pebbles each on (v_2, v_2) and (v_3, v_3), by A_2 the distribution with four pebbles each on (v_0, v_0), (v_2, v_2), and (v_3, v_3), and by A_4 the distribution with 4 pebbles on each $(v_i, v_{2i \bmod 5})$, $0 \leq i \leq 4$. We write B for the $\frac{1}{4}A_4$, i. e. the distribution with one pebble on each $(v_i, v_{2i \bmod 5})$. B is shown in Figure 1 (filled-in vertices are occupied, dark edges give the neighborhoods of the occupied vertices, and the edges wrap around in the obvious ways).

Proposition 5.1 *For each $t \in \{1, 2, 4\}$, the distribution A_t is t-fold solvable on $C_5 \Box C_5$.*

Proof: First note that $\{v_2, v_3\} \Box \{v_2, v_3\} \cong K_2 \Box K_2$. We call these vertices the *corners* of the graph, imagining (v_0, v_0) to be the center. If we have two pebbles each on (v_2, v_2) and (v_3, v_3), we have the 2-fold solvable distribution in Proposition 3.12, so two pebbles can be moved any of the corners, and one pebble can be moved to any vertex adjacent to these corners. The rest of the vertices are within two steps from (v_0, v_0), so they can be reached from the four pebbles from there. This takes care of the $t = 1$ case.

For $t = 2$, we instead have four pebbles each on (v_2, v_2) and (v_3, v_3), so we can consider these to be two groups which each have two pebbles on both (v_2, v_2) and (v_3, v_3). Therefore, we can put four pebbles on any corner. Then the vertices whose distance from (v_0, v_0) is zero or one can receive two pebbles from that vertex. The vertices whose distance from (v_0, v_0) is three or four can receive two pebbles from the nearest corner, and those whose distance from (v_0, v_0) is two can receive one pebble from (v_0, v_0) and one from the nearest corner.

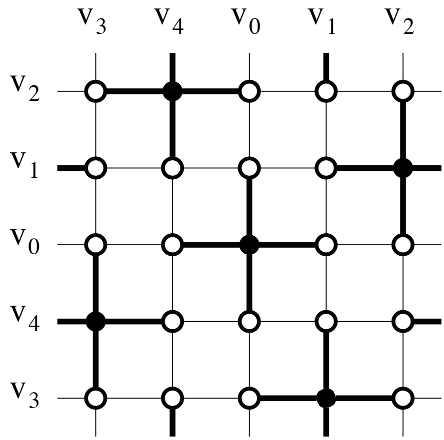

Figure 1: The distribution B on $C_5 \square C_5$

When $t = 4$, the symmetry of A_4 allows us to consider only one target, say (v_0, v_1). This vertex can receive two pebbles from (v_0, v_0), and one each from (v_3, v_1) and (v_1, v_2). □

Theorem 5.2 *Let G be any graph and let D be a t-fold solvable distribution on G in which the number of pebbles on every vertex is a multiple of four. Then the distribution $B \cdot D$ is a t-fold solvable distribution in $(C_5 \square C_5) \square G$ in which the number of pebbles on every vertex is a multiple of four. Note that the number of pebbles in $B \cdot D$ is $5|D|$. In particular, by induction on m, we have $\pi_t^*(C_5^{2m} \square G) \leq 5^m |D|$.*

Proof: Let the target vertex in $C_5 \square C_5 \square G$ be (v_i, v_j, y). Since $D(x)$ is a multiple of four, we write $\frac{1}{4}D$ for the distribution with $\frac{1}{4}D(x)$ pebbles on $x \in V$, and we write $\mathcal{S}_{1/4}$ for the set $\mathcal{S}_{1/4} = \left\{\frac{1}{4}D\right\}$. Now $B \cdot D = (4B) \cdot \left(\frac{1}{4}D\right)$. By Proposition 5.1, $4B$ is $\mathcal{S}_4(C_5 \square C_5)$-solvable, so by Theorem 2.3, $B \cdot D$ is $\left(\mathcal{S}_4(C_5 \square C_5) \cdot \mathcal{S}_{1/4}\right)$-solvable in $C_5 \square C_5 \square G$. That is, from $B \cdot D$ we can reach the distribution in which $4\left(\frac{1}{4}D(y_k)\right) = D(y_k)$ pebbles are on (v_i, v_j, y_k) for every $y_k \in V$. But now the distribution on the vertices in $(v_i, v_j) \square G \cong G$ is D. Since D is t-fold solvable, we can put t pebbles on (v_i, v_j, y_k). Clearly, $(B \cdot D)((v_i, v_j, y_k)) = B((v_i, v_j))D(y_k)$ is a multiple of four, since $D(y_k)$ is a multiple of four. □

Corollary 5.3 *For all integers $m \geq 0$, we have $\pi^*(C_5^{2m+1}) \leq 4 \cdot 5^m$.*

Proof: We apply Theorem 5.2 to the distribution with four pebbles on a single vertex of $G = C_5$. □

A natural question at this point is what bounds we can get for $\pi^*(C_5^{2m})$. We create a solvable distribution F on C_5^4, and we use F to start an induction with Theorem 5.2 for even products similar to the argument for Corollary 5.3.

Notation: Let F be the distribution of 44 pebbles on C_5^4 given by

$$F(v_i, v_j, v_k, v_m) = \begin{cases} A_4(v_k, v_m) & \text{if } i = j = 0 \\ A_2(v_k, v_m) & \text{if } i = j = 2 \text{ or } i = j = 3 \\ 0 & \text{otherwise.} \end{cases}$$

Note that if we denote the empty distribution by A_0, then we may more simply write

$$F(v_i, v_j, v_k, v_m) = A_r(v_k, v_m),$$

where $r = A_1(v_i, v_j)$.

Proposition 5.4 *Every occupied vertex in F has four pebbles, and F is solvable in C_5^4.*

Proof: Every occupied vertex of both A_4 and A_2 has four pebbles in $C_5 \square C_5$, so this holds for F in C_5^4 as well. To show F is solvable, let the target vertex in C_5^4 be (v_i, v_j, v_k, v_m). By construction, the distribution of pebbles on $(v_0, v_0) \square C_5 \square C_5 \cong C_5 \square C_5$ is A_4. Therefore, by Proposition 5.1, four pebbles can be moved to (v_0, v_0, v_k, v_m) using only the pebbles on $(v_0, v_0) \square C_5 \square C_5$. Similarly, and simultaneously, by Proposition 5.1, two pebbles can be moved to both (v_2, v_2, v_k, v_m) and (v_3, v_3, v_k, v_m). At this point, the distribution of pebbles on $C_5 \square C_5 \square (v_k, v_m) \cong C_5 \square C_5$ is A_1, so one pebble may be moved to (v_i, v_j, v_k, v_m), again by Proposition 5.1. □

The use of Theorem 5.2 on the distribution A_1 would give a better coefficient of $\frac{8}{5}$ in Theorem 5.5; however, A_1 does not qualify since some vertices get only 2 pebbles.

Theorem 5.5 *We have $\pi^*(C_5^{2m}) \le \frac{44}{25}(5^m)$. For all $d \ge 1$ we have $\pi^*(C_5^d) \le 4\sqrt{5}^{d-1}$.*

Proof: If $d = 2$, Proposition 5.1 shows that $\pi^*(C_5 \square C_5) \le 8 < \frac{44}{5}$. If $d = 4$, Proposition 5.4 shows that $\pi^*(C_5^4) \le 44$. For $d = 2m$ with $m > 2$, Theorem 5.2 implies that $\pi^*(C_5^{2m}) = \pi^*(C_5^{2(m-2)} \square C_5^4) \le 5^{m-2}|F| = \frac{44}{25}(5^m)$. Since $\frac{44}{25} < \frac{4}{\sqrt{5}}$, the second part follows for even d.

If $d = 2m+1$, Corollary 5.3 gives us $\pi^*(C_5^d) \le 4 \cdot 5^m = 4 \cdot 5^{\frac{d-1}{2}} = 4\sqrt{5}^{d-1}$. □

We can generalize the construction of F and the proof of Proposition 5.4 to obtain Theorem 5.6.

Theorem 5.6 *Let $\mathcal{S}$ be a set of distributions on G, suppose D is an $\mathcal{S}$-solvable distribution, and suppose $\{D'_r\}_{r \ge 1}$ is a family of distributions on G' such that each D'_r is r-fold solvable. Let $\Delta : V(G \square G') \to \mathbb{N}$ be the distribution on $G \square G'$ defined by*

$$\Delta((v, w)) = D'_{D(v)}(w).$$

Then Δ is $(\mathcal{S} \cdot \mathcal{S}_1(G'))$-solvable in $G \square G'$. That is, for any distribution $\overline{D} \in \mathcal{S}$ and any vertex $w \in V'$, a copy of $\overline{D}$ can be moved to the vertices of $G \square \{w\}$.

Proof: Let w be the chosen vertex in G'. Then for any $v \in V$, restricting Δ to the vertices $\{v\}\Box G'$ gives the distribution $D'_{D(v)}$ in $\{v\}\Box G'$. Since $D'_{D(v)}$ is $D(v)$-fold solvable in G', we can move $D(v)$ pebbles to (v, w) for each $v \in V$. After these moves, the distribution of pebbles on $G\Box\{w\}$ is D. Since D is $\mathcal{S}$-solvable, we can put a copy of any $\overline{D} \in \mathcal{S}$ on $G\Box\{w\}$, as desired. □

We note that the proofs of Theorems 2.6 and 2.7 essentially involved letting each $D'_r = rD'$, where D' is the distribution of pebbles on G' defined in the proof of Theorem 2.3. Corollary 5.7 is a stronger result.

Corollary 5.7 *Let $\mathcal{S}$ be a set of distributions on G, and suppose D is an $\mathcal{S}$-solvable distribution. Then for any graph G', we have*

$$\pi^*(G\Box G', \mathcal{S} \cdot \mathcal{S}_1(G')) \leq \sum_{v \in V} \pi^*_{D(v)}(G').$$

Proof: We simply apply Theorem 5.6 and use a family of distributions $\{D'_t\}$ in which each D'_t is optimal, i. e. $|D'_t| = \pi^*_t(G')$. □

6 Hypercubes

In this section, we give optimal pebbling distributions on the d-dimensional hypercube $Q^d \cong K_2^d$. We consider the vertices of Q^d to be all bitstrings of length d, or equivalently, all vectors in the d-dimensional vector space $\mathbb{F}_2^d$ over the two-element field $\mathbb{F}_2$. There is an edge between two vertices when the Hamming distance between the corresponding bitstrings is 1. Given two bitstrings $\mathbf{v_1} \in V(Q^{d_1})$ and $\mathbf{v_2} \in V(Q^{d_2})$, we write $\mathbf{v_1} \cdot \mathbf{v_2}$ for the bitstring in $V(Q^{d_1+d_2})$ obtained by concatenating the bits in $\mathbf{v_1}$ and $\mathbf{v_2}$. We also write $\mathbf{0^k}$ and $\mathbf{1^k}$ for the bitstrings $00\ldots0$ and $11\ldots1$, respectively, and we call the number of 1's in a bitstring its *weight*.

Since the continuous optimal pebbling number of K_2 is $\hat{\pi}^*(K_2) = \frac{4}{3}$, Theorem 4.3 implies Theorem 6.1, which Moews also proved directly.

Theorem 6.1 (Moews [8]) *The optimal pebbling number of Q^d satisfies $\pi^*(Q^d) \leq (c\frac{4}{3}^d d^k)$ for some constants c and k.*

Theorem 6.1 gives the best known bound for hypercubes, but it does not give explicit distributions, which is our aim. The d^{th} root of Moews's result tends to about 1.33, and Proposition 6.2 gives an example, the d^{th} root of whose size is roughly 1.41. Our new construction in Theorem 6.7 improves that number below 1.38. Proposition 6.2 gives solvable distributions on Q^d for all d. These were first given in Pachter, Snevily, and Voxman [9].

Proposition 6.2 (Pachter et al. [9]) *If $d = 2k$, the distribution on Q^d obtained by putting 2^k pebbles on $\mathbf{0^d}$ and 2^{k-1} pebbles on $\mathbf{1^d}$ is solvable. If $d = 2k + 1$, the* *distribution on* Q^d *given by putting* 2^k *pebbles on both* $\mathbf{0^d}$ *and* $\mathbf{1^d}$ *is solvable. Thus, the optimal pebbling number of a hypercube satisfies*

$$\pi^*(Q^{2k}) \leq 3 \cdot 2^{k-1}$$
$$\pi^*(Q^{2k+1}) \leq 2^{k+1}.$$

In particular, $\pi^*(Q^d) \leq 1.5\sqrt{2}^d$.

Proof: In both cases whether d is even or odd, every vertex whose weight is at most k can receive at least one pebble from the pebbles on $\mathbf{0^k}$ in the given distribution, and every vertex with larger weight can receive a pebble from those on $\mathbf{1^k}$. □

We give a construction for extending the distributions in Proposition 6.2 to distributions on larger cubes with better asymptotic bounds than those in the Proposition. This construction is based on an argument similar to the proof of Theorem 5.6 using distributions on K_2 obtained from Theorem 3.3. First recall the distributions on K_2 from Theorem 3.3; we will use these in Theorem 6.4.

Definition: We let $\mathcal{D}$ be the family $\mathcal{D} = \{D_r\}_{r\geq 0}$ of distributions on K_2 given by

$$D_{3k}(x_0) = 2k \qquad D_{3k+1}(x_0) = 2k+2 \qquad D_{3k+2}(x_0) = 2k+2$$
$$D_{3k}(x_1) = 2k \qquad D_{3k+1}(x_1) = 2k \qquad D_{3k+2}(x_1) = 2k+1$$

Proposition 6.3 *D_r is r-fold solvable, and in each case, we have* $|D_r| = \left\lceil \frac{4r}{3} \right\rceil \leq \frac{4}{3}r + \frac{2}{3}$.

Proof: For each $r \neq 0$, D_r is the r-fold solvable distribution from the proof of Theorem 3.3. Counting pebbles, we have $|D_{3k}| = 4k$, $|D_{3k+1}| = 4k+2$, and $|D_{3k+2}| = 4k+3$. In each case, $|D_r| = \left\lceil \frac{4}{3}r \right\rceil$. □

In the spirit of Theorem 5.6, we want to extend a solvable distribution D on a graph G to a distribution D' on $G \square K_2$. We hope that $|D'| \approx \frac{4}{3}|D|$. Unfortunately, the extra $\frac{2}{3}$ in Proposition 6.3 can cause problems. For example, if D has a single pebble on a large number of vertices, those pebbles each give rise to two pebbles in D'. We can get an extra $\frac{2}{3}$ for each occupied vertex in D. We define the *support* of D to keep track of this information.

Definition: The *support* of a distribution D on the graph G, denoted $\sigma(D)$, is the set of occupied vertices in D; i. e. $\sigma(D) = \{v \in V(G) : D(v) > 0\}$.

Theorem 6.4 *Let D be a t-fold solvable distribution on Q^d. For each $\mathbf{v} \in V(Q^d)$, define $D'(\mathbf{v}\cdot 0)$ and $D'(\mathbf{v}\cdot 1)$ by*

$$D'(\mathbf{v}\cdot 0) = D_{D(\mathbf{v})}(x_0)$$
$$D'(\mathbf{v}\cdot 1) = D_{D(\mathbf{v})}(x_1)$$

Then D' is t-fold solvable on Q^{d+1}. Furthermore, the number of pebbles in D' is at most $\frac{4}{3}|D| + \frac{2}{3}|\sigma(D)|$, *and* $|\sigma(D')| \leq 2|\sigma(D)|$.

Proof: Let the target in Q^{d+1} be $\mathbf{v} \cdot b$, where $\mathbf{v} \in V(Q^d)$ and $b \in \{0,1\}$. For each $\mathbf{v_i} \in V(Q^d)$, the distribution of pebbles on $\mathbf{v_i} \Box K_2 \cong K_2$ is $D_{D(\mathbf{v_i})}$. Since this distribution is $D(\mathbf{v_i})$-fold solvable in K_2, we can put $D(\mathbf{v_i})$ pebbles on $\mathbf{v_i} \cdot b$. If we do this for each $\mathbf{v_i} \in V(Q^d)$, the distribution of pebbles on $Q^d \Box \{b\} \cong Q^d$ is D. Since D is t-fold solvable on Q^d, we can put t pebbles on $\mathbf{v} \cdot b$. The total number of pebbles in D' is

$$|D'| = \sum_{\mathbf{v} \in \sigma(D')} D'(\mathbf{v}) = \sum_{\mathbf{v} \in \sigma(D)} |D_{D(\mathbf{v})}| \leq \sum_{\mathbf{v} \in \sigma(D)} \left(\frac{4}{3} D(\mathbf{v}) + \frac{2}{3} \right) = \frac{4}{3}|D| + \frac{2}{3}|\sigma(D)|.$$

Finally, $\sigma(D') \subseteq \sigma(D) \Box \{0,1\}$. □

Theorem 6.5 describes what happens when we apply Theorem 6.4 repeatedly.

Theorem 6.5 *Let D be a solvable distribution on a graph G with $s = |\sigma(D)|$, and let D_m be the result of applying Theorem 6.4 m times to D. Then D_m is a solvable distribution on $G \Box K_2^m \cong G \Box Q^m$ such that $|\sigma(D_m)| \leq 2^m s$, and $|D_m| \leq \left(\frac{4}{3}\right)^m |D| + 2^m s - \left(\frac{4}{3}\right)^m s$.*

Proof: There is nothing to show if $m = 0$, so we suppose by induction that for some $i \geq 0$, D_i is a solvable distribution on $G \Box Q^i$ with $|D_i| \leq \left(\frac{4}{3}\right)^i |D| + 2^i s - \left(\frac{4}{3}\right)^i s$, and $|\sigma(D_i)| \leq 2^i |\sigma(D)|$. Then applying Theorem 6.4 to D_i, we find that D_{i+1} is a solvable distribution on $G \Box Q^{i+1}$ with $|\sigma(D_{i+1})| \leq 2|\sigma(D_i)| = 2^{i+1}s$. Furthermore, we have

$$|D_{i+1}| \leq \frac{4}{3}|D_i| + \frac{2}{3}|\sigma(D_i)| \leq \frac{4}{3}\left[\left(\frac{4}{3}\right)^i |D| + 2^i s - \left(\frac{4}{3}\right)^i s \right] + \frac{2}{3}(2^i s).$$

Multiplying through by the $\frac{4}{3}$ and noting that $\frac{4}{3}(2^i s) + \frac{2}{3}(2^i s) = 2^{i+1}s$, we have

$$|D_{i+1}| \leq \left(\frac{4}{3}\right)^{i+1} |D| + \frac{4}{3}(2^i s) - \left(\frac{4}{3}\right)^{i+1} s + \frac{2}{3}(2^i s) = \left(\frac{4}{3}\right)^{i+1} |D| + 2^{i+1}s - \left(\frac{4}{3}\right)^{i+1} s,$$

completing the induction. □

Corollary 6.6 *Let D_0 be the distribution with 2^k pebbles on both $\mathbf{0^{2k+1}}$ and $\mathbf{1^{2k+1}}$ in Q^{2k+1}, and let D_m be the resulting distribution in Q^{2k+m+1} obtained by applying Theorem 6.4 m times. Then*

$$|D_m| \leq \left(\frac{4}{3}\right)^m (2^{k+1}) + 2^{m+1} - 2\left(\frac{4}{3}\right)^m.$$

Proof: We apply Theorem 6.5 to D_0, noting that $|D_0| = 2^{k+1}$ and $s = |\sigma(D_0)| = 2$. □

For large m, the term $2\left(\frac{4}{3}\right)^m$ in Corollary 6.6 is small compared to 2^{m+1}. By controlling the relationship between k and m, we can ensure that the first two terms are roughly equal. Using logarithms to solve the equation $\left(\frac{4}{3}\right)^m (2^{k+1}) \approx 2^{m+1}$, or $2^k \approx \left(\frac{3}{2}\right)^m = 1.5^m$ together with the observation that $d = 2k + m + 1$. We obtain the constants in Theorem 6.7.

Theorem 6.7 *Given an integer d, let $k = \left\lceil \frac{\log_2 1.5}{\log_2 4.5}(d-1) \right\rceil \approx 0.2696(d-1)$, and let $m = d - 1 - 2k \approx 0.4608(d-1)$. Then the distribution D_m on $Q^{2k+m+1} = Q^d$ from Corollary 6.6 with these values of k and m satisfies $|D_m| \leq c \cdot 2^m \approx c \cdot 1.3763^d$ for some constant c.*

Proof: We define K and M by $K = \frac{\log_2 1.5}{\log_2 4.5}(d-1)$ and $M = \frac{1}{\log_2 4.5}(d-1)$. We note that

$$2K + M = \frac{2\log_2 1.5 + 1}{\log_2 4.5}(d-1) = \frac{\log_2 (1.5^2 \cdot 2)}{\log_2 4.5}(d-1) = d - 1.$$

Since $k = \lceil K \rceil$, we have $K \leq k < K+1$, and since $2k + m = 2K + M$, this implies $M - 2 < m \leq M$. Furthermore, $K = M \log_2 1.5$; therefore, $2^K = 1.5^M$, or equivalently, $\left(\frac{4}{3}\right)^M 2^K = 2^M$. Thus, applying Corollary 6.6 gives

$$|D_m| \leq \left(\frac{4}{3}\right)^m 2^{k+1} + 2^{m+1} \leq \left(\frac{4}{3}\right)^M 2^{K+2} + 2^{M+1} = 6 \cdot 2^M \leq 6 \cdot 2^{m+2},$$

as desired. □

Remark

In recent unpublished work, H.-L. Fu, K.-C. Huang, and C.-L. Shiue independently established results similar to Corollary 6.6 and Theorem 6.7, using the probabilistic method.

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