AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **47** (2010), Pages 239–245

A census of one-factorizations of the complete 3-uniform hypergraph of order 9

Mahdad Khatirinejad   Patric R. J. Östergård

Department of Communications and Networking
Aalto University
P.O. Box 13000, 00076 Aalto
Finland

mahdad.khatirinejad@tkk.fi   patric.ostergard@tkk.fi

Abstract

The one-factorizations of the complete 3-uniform hypergraph with 9 vertices, K_9^3, are classified by means of an exhaustive computer search. It is shown that the number of isomorphism classes of such one-factorizations is 103 000.

1 Introduction

The complete k-uniform hypergraph on n vertices, K_n^k, has as its edges all k-subsets of the n vertices. Vertices are labelled by $\{0, \ldots, n-1\}$ throughout this paper. A *one-factor* of K_n^k is a set of n/k pairwise disjoint edges that partition the vertex set of K_n^k. A *one-factorization* of K_n^k is a partition of all edges of K_n^k into one-factors. It is clear that K_n^k has a one-factorization only if $k|n$. The converse is also true and is known as Baranyai's theorem [1].

Definition 1.1. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two sets of one-factors. We say that $\mathcal{F}_1$ and $\mathcal{F}_2$ are *isomorphic* if there is a bijection between the vertex sets of the underlying hypergraphs that maps the one-factors of $\mathcal{F}_1$ onto the one-factors of $\mathcal{F}_2$.

The number of isomorphism classes of one-factorizations of K_n^2 are 1, 1, 6, 396, 526 915 620, and 1 132 835 421 602 062 347 for $n = 4$, 6, 8, 10, 12, and 14, respectively; see [2] and [4] for the two most recent results in this sequence. Apart from this, the only nontrivial cases where the number of one-factorizations of K_n^k is known are for $n = 2k$; it is easy to see that such one-factorizations are unique (also as labelled structures).

Our goal is to continue the work on classifying one-factorizations of K_n^k up to isomorphism by considering the instance K_9^3. The one-factorizations of K_9^3 that have automorphism groups of order greater than 4 were studied already in the 1980s from a classification perspective [9]. However, as we shall later see, it turns out that there are some discrepancies between the results of [9] and the current work.

The paper is organized as follows. The idea of testing isomorphism of one-factorizations via graph isomorphism is discussed in Section 2, and the overall classification algorithm is considered in Section 3 together with the results obtained. It turns out that the number of isomorphism classes of K_9^3 is 103 000. The paper is concluded in Section 4 with a consistency check of the computer search.

2 Isomorphism testing

Since efficient tools are readily available for handling graphs and graph isomorphism testing, a conventional approach for testing isomorphism of combinatorial objects is to do this via corresponding graphs. We shall now describe how the isomorphism test between two sets with s disjoint one-factors of K_n^k with $k \geq 3$, $n \geq 3k$ may be converted into a graph isomorphism test.

To each set of disjoint one-factors $\mathcal{F} = \{F_1, \ldots, F_s\}$ with $F_j = \{e_j^1, \ldots, e_j^{n/k}\}$, $1 \leq j \leq s$, we associate the graph $G(\mathcal{F}) = (V, E)$, where

$$V = \bigcup_{j=1}^{s} \{e_j^1, \ldots, e_j^{n/k}\} \cup \{0, \ldots, n-1\}$$

and

$$\begin{aligned} E &= \left\{\{i, e_j^h\} : i \in e_j^h,\, 0 \leq i \leq n-1,\, 1 \leq h \leq n/k,\, 1 \leq j \leq s\right\} \cup \\ &\quad \left\{\{e_j^h, e_j^l\} : 1 \leq h < l \leq n/k,\, 1 \leq j \leq s\right\}. \end{aligned}$$

This graph has $n + sn/k$ vertices and $s(n + \binom{n/k}{2})$ edges.

Example 2.1. *Let* $F_1 = \{\{0,1,2\}, \{3,4,5\}, \{6,7,8\}\}$ *and* $F_2 = \{\{0,1,3\}, \{4,6,8\}, \{2,5,7\}\}$ *be two disjoint one-factors of* K_9^3*. The graph* $G(\mathcal{F})$ *associated to* $\mathcal{F} = \{F_1, F_2\}$ *is depicted in Figure 1.*

Proposition 2.2. *Two sets of one-factors* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *are isomorphic if and only if the graphs* $G(\mathcal{F}_1)$ *and* $G(\mathcal{F}_2)$ *are isomorphic.*

Proof. Since the vertices $0, \ldots, n-1$ do not occur in cliques of size greater than 2, any isomorphism ψ between $G(\mathcal{F}_1)$ and $G(\mathcal{F}_2)$ must map the vertices $0, \ldots, n-1$ in $\mathcal{F}_1$ to the vertices $0, \ldots, n-1$ in $\mathcal{F}_2$. Moreover, as (n/k)-cliques in $G(\mathcal{F}_1)$ are mapped to (n/k)-cliques in $G(\mathcal{F}_2)$, it follows that one-factors from $\mathcal{F}_1$ are mapped to one-factors from $\mathcal{F}_2$, so ψ induces an isomorphism from $\mathcal{F}_1$ to $\mathcal{F}_2$. The converse is straightforward. □

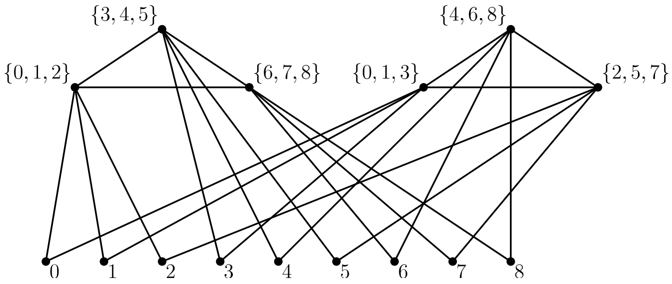

Figure 1: Graph from two one-factors

3 Classification

Any parts of the approach to be discussed can be generalized directly to arbitrary parameters. However, for the sake of clarity, the specific case of K_9^3 is considered throughout the presentation. The hypergraph K_9^3 has $\binom{9}{3}\binom{6}{3}/3! = 280$ one-factors, and a one-factorization of K_9^3 has $\binom{9}{3}/3 = 28$ one-factors.

To classify all one-factorizations of K_9^3, since each edge has to be covered by exactly one one-factor, we may assume that the edges $\{0,1,i\}$, $2 \leq i \leq 8$, belong to the first seven one-factors. To this end, we define a *seed* to be a set of seven one-factors $\{F_1,\ldots,F_7\}$ so that there exist $0 \leq a < b \leq 8$ such that

$$\{\{a,b,i\} : 0 \leq i \leq 8, i \neq a,b\} \subset \bigcup_{j=1}^{7} F_j.$$

Note that since F_j and F_k are disjoint when $j \neq k$, each $\{a,b,i\}$ must belong to a separate F_j. Every one-factorization contains exactly $\binom{9}{2} = 36$ seeds.

We start by classifying the seeds up to isomorphism. This is done by a backtrack search, adding one-factors that contain an edge of the form $\{0,1,i\}$ one at a time and carrying out isomorph rejection. To this end, the *nauty* library [7] by McKay is used to handle graphs obtained via the construction in Section 2. Table 1 shows the number of nonisomorphic sets of one-factors when building up a seed in this manner. The number of one-factors in a partial seed is denoted by m. Consequently, there are 208 non-isomorphic seeds.

Table 1: Number of partial seeds

m	1	2	3	4	5	6	7
# of partial seeds	1	2	11	45	156	277	208

By extending each classified seed in all possible ways, we can visit every isomorphism class of one-factorizations. The problem of finding all one-factorizations that contain a given seed is an instance of the *exact cover problem*: Given a finite set U and a collection $\mathcal{C}$ of subsets of U, find all partitions of U consisting of sets in $\mathcal{C}$. To extend seeds, we must cover the remaining uncovered edges of K_9^3 exactly once, in all possible ways using the one-factors of K_9^3. To solve the exact cover instances, the *libexact* library [5] by Kaski and Pottonen was used. The instances of finding one-factorizations from the given seeds lead to a total of 8 185 376 solutions.

There are several methods available for isomorph rejection among the final solutions; see [3, Chapter 4] for an extensive survey. Since the number of nonisomorphic one-factorizations turns out not to be too large, one could here utilize the straightforward method of *recorded objects*, maintaining a catalogue of nonisomorphic objects and comparing any new candidates with those in the catalogue. However, in this work the method of *canonical augmentation*, due to McKay [8], was utilized, which is not limited by the amount of memory available.

The framework of canonical augmentation can be used for isomorph-free exhaustive generation by applying the following two tests to a completed one-factorization $\mathcal{F}$:

(i) If $\mathcal{F}$ is obtained by extending a seed $\mathcal{S}$, check whether $\mathcal{F}$ is the (lexicographic) minimum of its $\mathrm{Aut}(\mathcal{S})$-orbit.

(ii) Identify a canonical $\mathrm{Aut}(\mathcal{F})$-orbit of seeds contained by $\mathcal{F}$, and then check whether the seed from which $\mathcal{F}$ was extended is in the canonical orbit.

If $\mathcal{F}$ satisfies both tests, we accept $\mathcal{F}$, otherwise we reject $\mathcal{F}$. In order to speed up the test (ii), we introduce the following invariant of a seed of a one-factorization.

For any two sets f and g, we define

$$\chi(f,g) = \begin{cases} 1 & \text{if } f \cap g \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Let $F = \{f_1, f_2, f_3\}$ and $G = \{g_1, g_2, g_3\}$ be two one-factors of K_9^3. We define

$$\alpha(F,G) = \sum_{i=1}^{3} \sum_{j=1}^{3} \chi(f_i, g_j).$$

It is easy to check that $\alpha(F,G) \in \{6,7,9\}$ for any two one-factors F and G.

Suppose $\{F_1, \ldots, F_7\}$ is a seed. We further define

$$d(F_j) = \sum_{i \neq j} \alpha(F_i, F_j),$$

and to each seed $\mathcal{S} = \{F_1, \ldots, F_7\}$ we associate the multiset

$$d(\mathcal{S}) = \{d(F_1), \ldots, d(F_7)\}.$$

The canonical Aut($\mathcal{F}$)-orbit of seeds in (ii) is required to have the lexicographically smallest invariant $d(\mathcal{S})$. Finally, *nauty* is used to complete the test if the invariant test is passed. Note that Aut($\mathcal{F}$)-orbits of seeds correspond to Aut($\mathcal{F}$)-orbits of 2-subsets of vertices.

We find that among all one-factorizations that extend the classified seeds, there are exactly 103 000 isomorphism classes of one-factorizations. In Table 2, we display the possible automorphism group orders and the associated number of isomorphism classes of one-factorizations. The current implementation carries out the entire classification in 3 hours of CPU time, with a 3-GHz processor.

Table 2: Orders of the automorphism groups

$\lvert\mathrm{Aut}(\mathcal{F})\rvert$	#	$\lvert\mathrm{Aut}(\mathcal{F})\rvert$	#
1	99 453	16	2
2	3 151	18	3
3	151	24	5
4	111	36	1
6	84	42	1
7	2	54	2
8	10	56	1
9	1	336	1
12	17	432	1
14	2	1 512	1

The results of the current work can now be compared with the classification of one-factorizations of K_9^3 that have automorphism groups of order greater than 4 that was carried out in [9]. There are discrepancies between the results of these two studies; 130 one-factorizations with such automorphism groups are claimed in [9], whereas the number obtained here is 134. The consistency check carried out as discussed later in Section 4 gives confidence in the results presented here, but several specific errors in the old study were also pinpointed—both for the set of one-factorizations and for the claimed automorphism group orders. For example, the automorphism group of object #6 in [9, Table 1] is stated to have order 36 but it has order 18.

One-factorizations of K_9^3 can also be viewed as resolutions of 2-$(9, 3, 7)$ balanced incomplete block designs, so the number of inequivalent such resolutions is at least 103 000, improving on the bound given in [6]. But it seems apparent that the objects classified here form only a tiny fraction of such objects. Anyway, the resolutions of the unique *simple* 2-$(9, 3, 7)$ design are now known.

4 A consistency check

Any computer search is prone to many types of errors. To gain confidence in the correctness of a classification, there are several methods available [3, Chapter 10]. In the current work, we have used data from the computer search to arrive at a double counting argument. We shall here discuss this argument for the final classification as well as for the classification of seeds.

During the main search, we record (i) the order of the automorphism group $\mathrm{Aut}(\mathcal{S}_i)$ for each seed $\mathcal{S}_i$, (ii) the total number M_i of one-factorizations found by the exact cover algorithm as extensions of $\mathcal{S}_i$, and (iii) the order of the automorphism group $\mathrm{Aut}(\mathcal{F}_j)$ for each isomorphism class $\mathcal{F}_j$ of one-factorizations. Using these numbers and the orbit-stabilizer theorem, we are able to count the total number of one-factorizations of K_9^3 in two different ways:

$$\frac{1}{\binom{9}{2}} \sum_{i=1}^{208} \frac{9! \cdot M_i}{|\mathrm{Aut}(\mathcal{S}_i)|} = \sum_{i=1}^{103\,000} \frac{9!}{|\mathrm{Aut}(\mathcal{F}_i)|}. \tag{1}$$

Note that the scaling on the left-hand side of (1) is necessary because the left-hand side sum counts every one-factorization once for each of the $\binom{9}{2}$ seeds that occur in a one-factorization. Using the classification data, both the left-hand side and the right-hand side of (1) evaluate to 36 696 023 040.

Let us next consider the details for validating the classification of seeds. All one-factorizations of the seed must contain the pair $\{0, 1\}$, so only automorphisms on the set $\{2, 3, \ldots, 8\}$ are considered below. If from N partial seeds $\mathcal{F}_1, \ldots, \mathcal{F}_N$ with $m-1$ (≥ 0) one-factors we get N' partial seeds $\mathcal{F}'_1, \ldots, \mathcal{F}'_{N'}$ with m one-factors, and M_i is the number of candidate one-factors when extending $\mathcal{F}_i$, then by the orbit-stabilizer theorem the total number of partial seeds of size m is

$$\frac{1}{m} \sum_{i=1}^{N} \frac{7! \cdot M_i}{|\mathrm{Aut}(\mathcal{F}_i)|} = \sum_{i=1}^{N'} \frac{7!}{|\mathrm{Aut}(\mathcal{F}'_i)|}. \tag{2}$$

For $m = 1, \ldots, 7$, using the seed classification data, both sides of (2) evaluate to 70, 1 890, 25 410, 182 910, 701 820, 1 323 420, and 942 900, respectively. Thus all steps in the classification of seeds passed (2).

Acknowledgement

The authors thank the Computing Centre of Helsinki University of Technology TKK for providing computing resources and Harri Haanpää, Petteri Kaski, and Olli Pottonen for useful discussions. The first author thanks the Natural Sciences and Engineering Research Council of Canada (NSERC), and the second author thanks the Academy of Finland (Grants No. 110196 and 130142) for their support.

References

[1] Zs. Baranyai, On the factorization of the complete uniform hypergraph, in *Infinite and Finite Sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. I*, (A. Hajnal, R. Rado, and V. T. Sós, Eds.), Colloq. Math. Soc. Jánōs Bolyai **10**, North-Holland, Amsterdam, 1975, pp. 91–108.

[2] J. H. Dinitz, D. K. Garnick, and B. D. McKay, There are 526,915,620 nonisomorphic one-factorizations of K_{12}, *J. Combin. Des.* **2** (1994), 273–285.

[3] P. Kaski and P. R. J. Östergård, *Classification Algorithms for Codes and Designs*, Springer, Berlin, 2006.

[4] P. Kaski and P. R. J. Östergård, There are 1,132,835,421,602,062,347 nonisomorphic one-factorizations of K_{14}, *J. Combin. Des.* **17** (2009), 147–159.

[5] P. Kaski and O. Pottonen, `libexact` user's guide, version 1.0, HIIT Technical Reports 2008-1, Helsinki Institute for Information Technology HIIT, 2008.

[6] R. Mathon and A. Rosa, 2-(v, k, λ) designs of small order, in *Handbook of Combinatorial Designs* (C. J. Colbourn and J. H. Dinitz, Eds.), 2nd ed., Chapman & Hall/CRC Press, Boca Raton, 2007, pp. 25–58.

[7] B. D. McKay, *nauty* user's guide (version 1.5), Technical Report TR-CS-90-02, Computer Science Department, Australian National University, Canberra, 1990.

[8] B. D. McKay, Isomorph-free exhaustive generation, *J. Algorithms* **26** (1998), 306–324.

[9] R. Mathon and A. Rosa, A census of 1-factorizations of K_9^3: solutions with group of order > 4, *Ars Combin.* **16** (1983), 129–147.

(Received 22 Oct 2009)