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Existence of doubly near resolvable $(v, 4, 3)$-BIBDs

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Abstract

The existence of doubly near resolvable $(v, 2, 1)$-BIBDs was established by Mullin and Wallis in 1975. For doubly near resolvable $(v, 3, 2)$-BIBDs, the existence problem was investigated by Lamken in 1994, and completed by Abel, Lamken and Wang in 2007. In this paper, we look at doubly near resolvable $(v, 4, 3)$-BIBDs; we establish that these exist whenever $v \equiv 1 \pmod 4$ except for $v = 9$ and possibly for $v \in \{17, 213\}$. Further, when $v \notin \{5, 9, 17, 185, 205, 213\}$, our designs are also $(1, 3; 4)$-frames of type 1^v.

1 Introduction

A (v, k, λ)-BIBD is a pair $(X, \mathcal{B})$ where X is a set of v points, and $\mathcal{B}$ is a collection of size k subsets of X (called blocks) such that any two points appear together in exactly λ blocks.

A $(v, k, k-1)$-BIBD is called *near resolvable*, or a NR$(v, k, k-1)$-BIBD if its blocks can be partitioned into holey resolution classes, $R_1, R_2, \ldots, R_v$ each containing precisely $v - 1$ points of the design. The classes $R_1, R_2, \ldots, R_v$ are said to form a holey resolution.

Two holey resolutions $R = (R_1, R_2, \ldots, R_v)$ and $R^* = (R_1^*, R_2^*, \ldots, R_v^*)$ are said to be orthogonal if R and R^* have at most one common block. (In this definition, we note that the blocks of the design are considered to be labeled, so that if two different blocks consist of the same points, they are treated as being distinct.) A NR$(v, k, k-1)$-BIBD containing two orthogonal near resolutions is called a *doubly near resolvable BIBD*, denoted as a DNR$(v, k, k-1)$-BIBD.

The blocks of a DNR$(v, k, k-1)$-BIBD can be put in the cells of a $v \times v$ array so that the blocks in any row or any column form one of its holey resolution classes. This is done by placing the block from $R_i \cap R_j^*$ in the cell indexed as (i, j) for all i, j.

If a DNR$(v, k, k-1)$-BIBD has the additional property that its holey resolution classes R_i, R_i^* can be ordered so that (1) The point missing in the blocks of R_i is also the one missing in the blocks of R_i^* and (2) R_i, R_i^* have no common (labeled) blocks for any i, then the DNR$(v, k, k-1)$-BIBD is called a $(1, k-1; k)$-frame of type 1^v.

In this paper, we shall investigate the problem of existence of DNR$(v, 4, 3)$-BIBDs and $(1, 3; 4)$-frames of type 1^v. Up to now, this problem is a long way from being solved, although Wu et al. [15] have established existence when v is a prime power $\equiv 5 \pmod 8$. A necessary condition for these designs to exist is $v \equiv 1 \pmod 4$. The blocks of all 11 $(9, 4, 3)$-BIBDs can be found in [12]; it is easy to check that none of these designs is doubly near resolvable, since none of them has every block disjoint from two other blocks. In this paper we establish the following result.

Theorem 1.1. *1. A DNR$(v, 4, 3)$-BIBD exists for all $v \equiv 1 \pmod 4$ except for $v = 9$ and possibly for $v \in \{17, 213\}$.*

2. A $(1, 3; 4)$-frame of type 1^v exists for all $v \equiv 1 \pmod 4$ except for $v \in \{5, 9\}$ and possibly for $v \in \{17, 185, 205, 213\}$.

2 Frames and other auxiliary designs

Several of our constructions for doubly near resolvable BIBDs make use of frames. To describe these constructions, we need some extra definitions:

A (K, λ) *group divisible design* (GDD) is a triple $(X, \mathcal{G}, \mathcal{B})$ where X is a set of points, $\mathcal{G}$ is a partition of X into subsets (called groups) and $\mathcal{B}$ is a collection of subsets of X with sizes in K (called blocks) such that (1) no two points in the same group lie in any block, and (2) any two points in different groups appear together in exactly λ blocks. If $\mathcal{G}$ contains t_1 groups of size g_1 plus t_2 groups of size g_2, t_3 groups of size g_3, ..., t_n groups of size g_n, then the GDD is said to have type $g_1^{t_1} g_2^{t_2} \dots g_n^{t_n}$. Notationally, we permit $g_i = g_j$ for $i \neq j$. If $K = \{k\}$, we more commonly write k instead of $\{k\}$.

A (k, λ)-frame is a (k, λ)-GDD $(X, \mathcal{G}, \mathcal{B})$ in which the collection of blocks $\mathcal{B}$ can be partitioned into *holey resolution classes* R_i each of which is a partition of $X - G$ for some group $G \in \mathcal{G}$. The "type" of the frame is the same as for the corresponding GDD. Also the total number of holey parallel classes missing any group of size g_i is $\lambda(|X| - g_i)/(k-1)$. The holey resolution classes R_i are said to form a holey resolution, R.

A $(k, k-1)$-frame of any type is said to be a $(1, k-1; k)$-frame of that type if it possesses two holey resolutions $R = \{R_1, R_2, \dots, R_v\}$ and $R^* = \{R_1^*, R_2^*, \dots, R_v^*\}$ such that

(1) R_i, R_j^* never have more than one block in common for any i, j;

(2) If R_i R_j^* miss the same group G, then R_i, R_j^* have no blocks in common.

As was the case for DNR$(v, k, k-1)$-BIBDs, the blocks here are considered to be labeled so that different blocks containing exactly the same points are considered distinct. If we label the points of such a $(1, k-1; k)$-frame as $1, 2, \dots, v$, and the indices of the holey resolution classes R_i, R_i^* are such that both R_i, R_i^* miss the group containing point i, then as with a DNR$(v, k, k-1)$-BIBD, by putting the block from $R_i \cap R_j^*$ in the (i, j) cell of a $v \times v$ square, we have a square in which

(1) row i (respectively, column j) contains the blocks from R_i (respectively, R_j^*);

(2) The sub-squares indexed as $G \times G$ for any group G are empty.

In [8] several $(1, 2; 3)$-frames of types 3^m and 9^m were constructed, and used later in [10] to construct several DNR$(v, 3, 2)$-BIBDs and $(1, 2; 3)$-frames of types 1^v. Later the existence problem for DNR$(v, 3, 2)$-BIBDs and $(1, 2; 3)$-frames of type 1^v was completed (except for one case, $v = 13$) in [10] and [2]. We summarize these results in the following theorem:

Theorem 2.1. *Let $v \equiv 1 \pmod 3$. Then a DNR$(v, 3, 2)$-BIBD exists except for $v = 7$. Also, a $(1, 2; 3)$-frame of type 1^v exists except for $v = 4, 7$ and possibly for $v = 13$.*

Existence of a $(1, 1; 2)$-frame of type 1^v is equivalent to that of a Room square of side v, or a doubly resolvable $(v + 1, 2, 1)$-BIBD. Such layouts are known to exist for all odd $v > 6$; see [11]. A DNR$(v, 2, 1)$-BIBD exists for all odd $v > 2$, except $v = 5$.

A *transversal design*, TD(k, m) is a $(k, 1)$-GDD of group type m^k. A resolvable TD(k, m) (denoted by RTD(k, m)) is equivalent to a TD$(k + 1, m)$. It is well known that a TD(k, m) is equivalent to $k - 2$ mutually orthogonal Latin squares (MOLS) of order m. In this paper, we mainly employ the following known results on TDs.

Lemma 2.2. [1]

1. *A TD$(6, m)$ exists for all $m \geq 4$ except for $m = 6$ and possibly for $m \in \{10, 14, 18, 22\}$.*
2. *If $v = p_1^{e_1} p_2^{e_2} \ldots, p_n^{e_n}$, where each e_i is a positive integer and each p_i is prime, then there exists a TD(k, v) for $k = Min\{1 + p_i^{e_i} | i = 1, 2, \ldots, n\}$.*

The next three recursive construction theorems are similar to those used in [7, 8, 10]. The first two are variations of Wilson's fundamental construction for GDDs [14] and are adapted to give constructions for $(1, k-1; k)$-frames. The third one provides a method of filling in the groups of such frames to obtain $(1, k - 1; k)$-frames of type 1^u.

Construction 2.3. *(Weighting) Let $(X, \mathcal{G}, \mathcal{B})$ be a GDD with index unity, and let $w : X \to Z^+ \cup \{0\}$ be a weight function on X. Suppose that for each block $B \in \mathcal{B}$, there exists a $(1, k-1; k)$-frame of type $\{w(x) : x \in B\}$. Then there is a $(1, k-1; k)$-frame of type $\{\sum_{x \in G} w(x) : G \in \mathcal{G}\}$.*

Construction 2.4. *If a $(1, k-1; k)$-frame of type $g_1^{t_1} g_2^{t_2} \ldots g_n^{t_n}$ and a TD$(k+2,\ m)$ both exist, then a $(1, k-1; k)$-frame of type $(mg_1)^{t_1} (mg_2)^{t_2} \ldots (mg_n)^{t_n}$ also exists.*

Construction 2.5. *Suppose a $(1, k-1; k)$-frame of type $g_1^{t_1} g_2^{t_2} \ldots g_n^{t_n}$ exists. Suppose also, $b \in \{0, 1\}$ and there exists a $(1, k-1; k)$-frame of type $1^{g_i + b}$ for $i = 1, 2, \cdots, n$. Then there exists a $(1, k-1; k)$-frame of type 1^u for $u = b + \sum_{i=1}^{n} g_i t_i$.*

3 Direct constructions

Most of the designs constructed in this section will be obtained by constructions which make use of starters and adders. Several authors have used starter-adder constructions for designs which are related to doubly near resolvable and doubly resolvable designs — see for instance [2, 7, 8, 9, 10, 11, 16].

Let $G = \{0, g_1, g_2, \ldots, g_{mv-1}\}$ be an additive abelian group of order $mv = kt + 1$, and let G_2 be a size m subgroup of G. A $(1, k-1; k)$ starter of type m^v over G is a collection $S = S_1, S_2, \ldots, S_t$ of $t = m(v-1)/k$ blocks of size k, such that: (1) $S_1 \cup S_2 \cup \ldots, S_t = G \backslash G_2$ and (2) no element of G_2 occurs as a difference between two points in a block of S and each element of $G \backslash G_2$ occurs $k-1$ times as such a difference.

If Z is a subset of G, and $b \in G$, then we use the notation $Z + b$ to denote the set obtained by adding b to all elements of Z. If t, m, v, G and G_2 are defined as in the previous paragraph, then an *adder* $A = (a_1, a_2, \ldots, a_t)$ for a $(1, k-1; k)$ starter S of type m^v over G is a set $\{a_1, a_2, \ldots, a_t\}$ such that for all $i, j = 1, 2, \ldots, t$: (1) $a_i \in G \backslash G_2$, (2) $a_i \neq a_j$ if $i \neq j$ and (3) $\cup_{i=1}^{t}(S_i + a_i) = G \backslash G_2$.

If a starter adder pair (S, A) exists for a $(1, k-1; k)$-frame of type g^v then the frame can be obtained by developing the blocks of S over G. Partial parallel classes are obtained by adding each element of G to $S_1, S_2, \ldots, S_t$ for the first partial resolution, and to $(S_1 + a_1), (S_2 + a_2), \ldots, (S_t + a_t)$ for the orthogonal partial resolution.

For $k = 4$, various starters are already known for $(4t + 1, 4, 3)$-BIBDs or $(1, 3; 4)$-frames of type 1^v (see for instance [3, 4, 5, 6]). In the next few lemmas, we give starters and adders for $v \equiv 1 \pmod 4$ and either $21 \leq v \leq 105$, $v \in \{121, 125, 133, 169\}$ or v prime, $13 \leq v \leq 241$, $v \neq 17$. In several cases (usually for v prime or $v \geq 45$), we were able to use starters from [3], [4] or [5], and only had to find the adders.

Lemma 3.1. *There exists a* $(1, 3; 4)$*-frame of type* 1^v *for each* $v \in \{21, 25, 33, 45, 49, 57, 65, 69, 77, 81, 85, 93, 105\}$.

Proof. Tables 1 and 2 display suitable starter blocks and adders (over Z_v) for all the given values of v.

Lemma 3.2. *A cyclic* $(1, 3; 4)$*-frame of type* 1^v *exists for each* $v \in \{97, 121, 133, 169, 193\}$.

Proof. These are obtained like the designs in the previous lemma, except that a multiplier of order 3 or 5 is used. For each value, we give a multiplier w of order 5 (for $v = 121$) or order 3 for the other values. The required starter blocks and adders are then obtained by multiplying the given starter blocks and adders (in Table 3) by w^i for $0 \leq i \leq 4$ (when $v = 121$) or $0 \leq i \leq 2$ (in the other cases). ∎

Table 1: Starter blocks and adders for $(1, 3; 4)$-frames of type 1^v with $21 \le v \le 77$, v not prime

v	Starter	Adder	Starter	Adder	Starter	Adder
21	(12, 15, 13, 2)	2	(16, 11, 5, 9)	7	(20, 18, 3, 4)	6
	(6, 19, 10, 7)	1	(14, 8, 17, 1)	5		
25	(13, 11, 7, 16)	7	(23, 10, 8, 14)	18	(20, 6, 15, 19)	2
	(1, 9, 12, 2)	3	(21, 18, 4, 3)	6	(24, 5, 17, 22)	14
33	(2, 5, 3, 28)	8	(29, 24, 15, 4)	24	(20, 6, 9, 10)	20
	(7, 23, 14, 8)	11	(22, 31, 1, 18)	23	(26, 16, 11, 32)	6
	(21, 27, 13, 25)	10	(30, 19, 17, 12)	30		
45	(1, 3, 5, 21)	18	(4, 43, 6, 34)	28	(33, 41, 14, 29)	14
	(7, 25, 32, 10)	31	(44, 12, 31, 28)	21	(13, 17, 8, 18)	40
	(22, 2, 15, 39)	20	(40, 9, 19, 26)	35	(38, 37, 16, 27)	13
	(35, 30, 24, 36)	1	(20, 11, 23, 42)	4		
49	(1, 2, 3, 6)	1	(4, 8, 25, 40)	36	(7, 42, 16, 32)	26
	(12, 48, 13, 45)	17	(18, 36, 5, 10)	18	(23, 46, 9, 34)	11
	(28, 21, 43, 37)	27	(20, 31, 38, 26)	12	(30, 11, 41, 33)	30
	(22, 44, 15, 24)	2	(14, 35, 39, 29)	45	(17, 19, 47, 27)	20
57	(25, 14, 1, 56)	21	(10, 31, 26, 40)	49	(2, 35, 52, 39)	34
	(9, 47, 19, 11)	19	(29, 38, 4, 20)	52	(15, 27, 12, 8)	43
	(43, 37, 13, 42)	41	(41, 23, 33, 3)	11	(36, 46, 24, 44)	1
	(34, 51, 48, 45)	16	(6, 32, 7, 49)	56	(55, 5, 21, 16)	44
	(22, 54, 50, 28)	46	(30, 53, 17, 18)	23		
65	(56, 33, 9, 36)	35	(44, 12, 60, 2)	57	(57, 54, 45, 31)	29
	(39, 64, 5, 53)	25	(16, 59, 21, 34)	59	(26, 8, 63, 10)	24
	(4, 50, 18, 23)	31	(3, 19, 24, 25)	37	(49, 15, 17, 37)	55
	(55, 40, 48, 20)	3	(35, 41, 52, 28)	38	(46, 7, 38, 42)	56
	(1, 30, 27, 29)	18	(61, 51, 11, 32)	6	(43, 13, 14, 22)	63
	(47, 58, 6, 62)	49				
69	(21, 45, 30, 13)	33	(31, 53, 3, 24)	11	(44, 40, 58, 48)	28
	(17, 35, 62, 4)	17	(43, 68, 23, 49)	42	(52, 22, 42, 19)	40
	(56, 18, 29, 34)	15	(7, 6, 65, 50)	30	(47, 55, 32, 16)	34
	(57, 59, 11, 66)	18	(25, 54, 37, 28)	3	(63, 14, 67, 61)	53
	(51, 10, 15, 12)	43	(5, 38, 26, 39)	22	(9, 41, 27, 46)	47
	(64, 2, 1, 36)	37	(20, 8, 33, 60)	10		
77	(11, 31, 42, 71)	76	(26, 12, 33, 66)	20	(20, 58, 64, 62)	27
	(23, 75, 45, 3)	60	(56, 57, 9, 28)	64	(61, 35, 47, 53)	43
	(40, 52, 27, 54)	74	(39, 34, 49, 46)	30	(73, 60, 21, 76)	39
	(4, 51, 7, 17)	16	(68, 37, 69, 13)	13	(19, 43, 38, 15)	2
	(63, 2, 18, 22)	50	(24, 65, 74, 5)	37	(30, 67, 36, 29)	26
	(55, 70, 44, 1)	10	(59, 10, 8, 25)	49	(41, 50, 32, 14)	34
	(16, 48, 72, 6)	23				

Table 2: Starter blocks and adders for $(1,3;4)$-frames of type 1^v with $81 \le v \le 105$, v not prime

v	Starter	Adder	Starter	Adder	Starter	Adder
81	(18, 53, 76, 40)	64	(29, 41, 4, 12)	79	(74, 52, 55, 59)	69
	(19, 22, 34, 39)	53	(25, 77, 5, 66)	29	(62, 9, 63, 78)	40
	(11, 51, 49, 6)	6	(24, 72, 15, 70)	27	(69, 58, 47, 37)	19
	(21, 3, 45, 31)	65	(16, 50, 57, 8)	57	(14, 27, 28, 35)	17
	(1, 10, 48, 54)	31	(43, 60, 46, 30)	18	(33, 56, 23, 2)	78
	(32, 13, 64, 7)	77	(42, 73, 26, 68)	32	(79, 80, 44, 17)	72
	(36, 61, 67, 65)	2	(75, 38, 20, 71)	56		
85	(75, 14, 69, 35)	62	(23, 36, 25, 18)	6	(1, 58, 43, 49)	25
	(37, 53, 70, 17)	53	(45, 15, 5, 42)	35	(21, 79, 50, 55)	78
	(62, 47, 63, 24)	31	(30, 44, 52, 7)	77	(4, 64, 77, 56)	23
	(9, 83, 12, 8)	51	(28, 54, 13, 67)	34	(3, 78, 74, 82)	68
	(60, 61, 40, 11)	42	(57, 16, 76, 33)	48	(32, 20, 2, 51)	5
	(10, 73, 38, 41)	57	(71, 80, 27, 29)	46	(65, 46, 19, 39)	50
	(68, 6, 84, 66)	52	(26, 59, 31, 48)	60	(81, 34, 22, 72)	32
93	(92, 25, 73, 66)	47	(10, 89, 76, 85)	88	(2, 50, 87, 45)	65
	(36, 20, 54, 34)	91	(28, 59, 68, 16)	46	(86, 46, 6, 80)	83
	(57, 79, 37, 62)	11	(1, 33, 23, 69)	28	(49, 83, 17, 13)	79
	(5, 65, 82, 77)	18	(40, 42, 38, 75)	3	(19, 55, 48, 4)	5
	(31, 8, 32, 47)	8	(3, 9, 72, 64)	78	(14, 52, 81, 15)	49
	(71, 60, 41, 11)	80	(53, 18, 88, 7)	24	(39, 29, 67, 91)	27
	(63, 84, 56, 74)	63	(70, 27, 24, 21)	61	(44, 43, 35, 22)	64
	(58, 30, 61, 78)	45	(90, 26, 12, 51)	53		
105	(8, 77, 75, 80)	4	(32, 30, 43, 74)	29	(78, 53, 88, 69)	2
	(33, 48, 38, 61)	68	(19, 35, 95, 7)	3	(20, 12, 50, 94)	24
	(9, 29, 51, 89)	49	(11, 54, 28, 58)	38	(14, 71, 23, 70)	37
	(10, 45, 41, 91)	87	(27, 87, 82, 16)	93	(17, 63, 81, 49)	1
	(42, 36, 22, 76)	12	(99, 4, 1, 72)	15	(92, 68, 104, 86)	58
	(56, 15, 103, 96)	78	(40, 31, 97, 93)	64	(83, 5, 34, 37)	57
	(26, 59, 79, 67)	63	(21, 100, 101, 64)	47	(3, 2, 57, 85)	28
	(65, 44, 98, 46)	21	(24, 13, 102, 39)	86	(47, 18, 62, 84)	84
	(90, 25, 55, 6)	22	(66, 52, 60, 73)	85		

Table 3: Starter blocks and adders for $(1, 3; 4)$-frames of type 1^v with $v \in \{97, 121, 133, 169, 193\}$

(v, w)	S	A	S	A	S	A
(97, 35)	(19, 73, 43, 94)	19	(29, 60, 95, 9)	61	(67, 66, 70, 44)	12
	(51, 18, 52, 62)	22	(20, 47, 37, 5)	30	(1, 53, 55, 15)	36
	(84, 90, 59, 38)	67	(75, 8, 41, 22)	87		
(121, 3)	(109, 59, 118, 12)	13	(1, 78, 107, 32)	1	(93, 25, 50, 58)	76
	(7, 23, 11, 52)	40	(103, 18, 62, 102)	96	(8, 45, 66, 10)	95
(133, 11)	(57, 103, 115, 66)	38	(12, 51, 86, 128)	10	(3, 96, 24, 77)	7
	(102, 36, 93, 107)	29	(47, 56, 79, 20)	61	(119, 129, 37, 42)	105
	(10, 121, 70, 52)	50	(34, 67, 21, 117)	128	(76, 120, 9, 80)	120
	(27, 85, 55, 74)	89	(17, 2, 39, 50)	98		
(169, 22)	(23, 63, 62, 59)	23	(82, 121, 96, 41)	80	(3, 36, 50, 45)	9
	(11, 54, 8, 128)	89	(130, 139, 21, 67)	14	(48, 55, 75, 140)	53
	(46, 76, 91, 15)	41	(152, 4, 32, 69)	136	(9, 105, 40, 30)	93
	(102, 74, 150, 97)	156	(25, 117, 108, 6)	132	(119, 56, 163, 92)	64
	(141, 61, 26, 68)	8	(22, 44, 93, 19)	51		
(193, 84)	(44, 73, 113, 110)	87	(128, 4, 142, 111)	125	(5, 68, 52, 47)	34
	(12, 69, 9, 132)	63	(2, 17, 41, 150)	76	(25, 39, 40, 27)	28
	(70, 100, 108, 3)	14	(70, 100, 108, 3)	82	(63, 124, 98, 114)	50
	(82, 65, 180, 144)	158	(127, 38, 11, 179)	160	(95, 117, 103, 183)	39
	(109, 174, 135, 190)	35	(16, 20, 14, 92)	149	(165, 24, 57, 176)	156
	(42, 139, 189, 106)	105				

Lemma 3.3. *There exists a $(1, 3; 4)$-frame of type 1^v if either (1) $v = 125$ or (2) v is prime, $v \equiv 1 \pmod 4$, $13 \le v \le 241$ and $v \ne 17$.*

Proof. For $v = 97$ and 193, see Lemma 3.2. For the other values of v, let x be a primitive element in $\mathrm{GF}(v)$.

For $v \equiv 5 \pmod 8$, the required starter blocks (S) and adders (A) are obtained by multiplying one initial block and its adder (given in Table 4) by x^{4t} for $0 \le t < (v-1)/4$. For $v = 125$, x is a primitive element satisfying $x^3 = x^2 + 2$.

For $v \equiv 9 \pmod{16}$, starter blocks and adders are obtained by multiplying two initial blocks and their adders (given in Table 5) by x^{8t} for $0 \le t < (v-1)/8$.

For $v \equiv 17 \pmod{32}$, starter blocks and adders are obtained by multiplying four initial blocks and their adders (given in Table 6) by x^{16t} for $0 \le t < (v-1)/16$.

∎

Summarizing the results of Lemmas 3.1–3.3, we have the following lemma:

Table 4: Starter blocks and adders for $(1,3;4)$-frames of type 1^v with v prime, $v \equiv 5 \pmod 8$

v	x	S	A	v	x	S	A
13	2	$(1,4,6,7)$	5	29	2	$(1,3,13,8)$	4
37	2	$(1,2,4,17)$	3	53	2	$(1,2,11,34)$	25
61	2	$(1,2,4,10)$	13	101	2	$(1,2,4,98)$	8
109	2	$(1,2,8,64)$	4	125		$(x,x^{22},x^{19},x^{116})$	x^2
149	2	$(1,2,4,18)$	7	157	18	$(1,2,4,116)$	46
173	2	$(1,2,4,11)$	9	181	2	$(1,2,12,63)$	11
197	2	$(1,2,6,18)$	14	229	50	$(1,2,4,145)$	4

Table 5: Starter blocks and adders for $(1,3;4)$-frames of type 1^v with v prime, $v \equiv 9 \pmod{16}$

v	x	S	A	v	x	S	A
41	7	$(1,2,4,17)$	1	73	5	$(1,3,9,14)$	15
		$(3,12,5,22)$	23			$(11,63,31,25)$	69
89	3	$(1,3,9,22)$	8	137	3	$(1,2,4,17)$	3
		$(5,54,41,13)$	69			$(3,47,89,116)$	9
233	3	$(1,3,9,14)$	9				
		$(5,35,84,159)$	10				

Table 6: Starter blocks and adders for $(1,3;4)$-frames of type 1^v with v prime, $v \equiv 17 \pmod{32}$

v	x	S	A	v	x	S	A
113	3	$(1,2,4,10)$	1	241	7	$(1,2,4,7)$	3
		$(3,6,13,23)$	7			$(3,8,19,25)$	18
		$(5,58,81,94)$	78			$(12,86,130,185)$	80
		$(9,59,100,63)$	68			$(10,17,139,202)$	98

Lemma 3.4. *There exists a* $(1,3;4)$*-frame of type* 1^v *for all* $v \equiv 1 \pmod 4$ *where* $13 \le v \le 241$*, with the possible exceptions of* $v \in \{17, 117, 129, 141, 145, 153, 161, 165, 177, 185, 189, 201, 205, 209, 213, 217, 221, 225, 237\}$.

Intransitive starters and adders can also be used to construct $(1,3;4)$-frames of type $(4g)^m(4u)^1$ (or type $(4g)^{m+1}$ if $g=u$). If Z_{4gm} is the cyclic group of order $4gm$, then let $Z^*_{4gm} = Z_{4gm} - \{0,\, m,\, 2m\ldots,(g-1)m\}$. An intransitive starter over Z_{4gm} for a $(1,3;4)$-frame of type $(4g)^m(4u)^1$ over $Z_{4gm} \cup (I = \{\infty_1,\infty_2,\ldots,\infty_{4u}\})$ and with groups I, $G_i = \{i, i+m, i+2m\ldots, i+(g-1)m\}$ for $i = 0,1,2,\ldots,m-1$ is a triple (S,R,C) where:

1. S consists of $g(m-1)$ blocks of size 4, $S_1, S_2, \ldots, S_{g(m-1)}$;
2. C, R both contain u blocks of size 4;
3. each block in $R \cup C$ contains four elements of Z^*_{4gm} which are distinct (mod 4);
4. no two points in any block of $S \cup R \cup C$ are congruent (mod m);
5. each element of $Z^*_{4gm} \cup I$ appears in exactly one block from $S \cup C$;
6. each element of Z^*_{4gm} appears three times as a difference between two points in a block in $S \cup R \cup C$.

An adder $A(S)$ for such a starter is a set $\{a_1, a_2, \ldots, a_{g(m-1)}\}$ of elements of Z^*_{4gm} such that:

1. Each element of $Z^*_{4gm} \cup I$ appears in exactly one block from $R \cup (S_1 + a_1) \cup (S_2 + a_2) \cup \ldots \cup (S_{g(m-1)} + a_{g(m-1)})$.
2. No element of Z^*_{4gm} occurs more than once in $A(S)$.

We now have the following theorem. Its proof is essentially the same as that for Theorem 2.2 in [2] and is not given here.

Theorem 3.5. *If there exists an intransitive starter (S,R,C) and adder $A(S)$ for a $(1,3;4)$-frame of type $(4g)^m(4u)^1$, then there is a $(1,3;4)$-frame of type $(4g)^m(4u)^1$.*

Lemma 3.6. *A $(1,3;4)$-frame of type 4^n over $Z_{4n-4} \cup \{\infty_1,\infty_2,\infty_3,\infty_4\}$ exists for $7 \le n \le 10$.*

Proof. Suitable starters (S,R,C) and adders $A(S)$ for these frames are given in Table 7. Each starter and its corresponding adder are defined on the set $Z_{4n-4} \cup (I = \{\infty_1,\infty_2,\infty_3,\infty_4\})$. Groups for these frames are I and $\{i, i+(n-1), i+2(n-1), i+3(n-1)\}$ for $0 \le i \le n-2$. ∎

Table 7: (S, C, R) and A for $(1, 3; 4)$-frames of type 4^n.

n	C		R			
	$(19, 4, 21, 14)$		$(2, 16, 15, 17)$			
7	S	A	$S + A$	S	A	$S + A$
	$(15, 23, 16, 2)$	5	$(20, 4, 21, 7)$	$(9, 22, 13, \infty_3)$	1	$(10, 23, 14, \infty_3)$
	$(11, 7, 3, \infty_1)$	22	$(9, 5, 1, \infty_1)$	$(17, 20, 1, \infty_4)$	2	$(19, 22, 3, \infty_4)$
	$(10, 8, 5, \infty_2)$	3	$(13, 11, 8, \infty_2)$			
	C		R			
	$(15, 24, 9, 6)$		$(4, 10, 13, 15)$			
8	S	A	$S + A$	S	A	$S + A$
	$(13, 23, 12, 25)$	27	$(12, 22, 11, 24)$	$(22, 18, 26, \infty_2)$	11	$(5, 1, 9, \infty_2)$
	$(16, 11, 8, 10)$	9	$(25, 20, 17, 19)$	$(19, 27, 4, \infty_3)$	4	$(23, 3, 8, \infty_3)$
	$(17, 1, 5, \infty_1)$	1	$(18, 2, 6, \infty_1)$	$(2, 3, 20, \infty_4)$	24	$(26, 27, 16, \infty_4)$
	C		R			
	$(31, 4, 17, 26)$		$(28, 10, 9, 7)$			
9	S	A	$S + A$	S	A	$S + A$
	$(13, 30, 9, 10)$	25	$(6, 23, 2, 3)$	$(29, 23, 14, \infty_2)$	22	$(19, 13, 4, \infty_2)$
	$(7, 11, 2, 22)$	15	$(22, 26, 17, 5)$	$(18, 25, 28, \infty_3)$	2	$(20, 27, 30, \infty_3)$
	$(5, 15, 1, 3)$	28	$(1, 11, 29, 31)$	$(21, 20, 27, \infty_4)$	26	$(15, 14, 21, \infty_4)$
	$(6, 19, 12, \infty_1)$	6	$(12, 25, 18, \infty_1)$			
	C		R			
	$(31, 8, 1, 34)$		$(34, 4, 5, 19)$			
10	S	A	$S + A$	S	A	$S + A$
	$(17, 6, 3, 22)$	8	$(25, 14, 11, 30)$	$(21, 14, 20, \infty_1)$	3	$(24, 17, 23, \infty_1)$
	$(4, 2, 16, 12)$	4	$(8, 6, 20, 16)$	$(7, 32, 33, \infty_2)$	32	$(3, 28, 29, \infty_2)$
	$(19, 35, 11, 30)$	2	$(21, 1, 13, 32)$	$(10, 15, 23, \infty_3)$	28	$(2, 7, 15, \infty_3)$
	$(28, 24, 26, 5)$	7	$(35, 31, 33, 12)$	$(29, 25, 13, \infty_4)$	33	$(26, 22, 10, \infty_4)$

Lemma 3.7. *There exist* $(1, 3; 4)$*-frames of types* $12^9 20^1$, $12^{10} 24^1$, $12^{11} 20^1$, $12^{12} 20^1$ *and* $12^{13} 20^1$.

Proof. We give intransitive starters (S, C, R) and adders $A(S)$ for these frames in Tables 8–12. For each frame of type $12^n w^1$, the starter and its corresponding adder are defined on the set $Z_{12n} \cup (I = \{\infty_1, \infty_2, \ldots, \infty_w\})$. Groups for these frames are I and $\{i, i + n, i + 2n, \ldots, i + 11n\}$ for $0 \le i \le n - 1$.

Table 8: (S, C, R) and A for a $(1, 3; 4)$-frame of type $12^9 20^1$.

C		R			
$(31, 32, 46, 57)$		$(1, 20, 26, 87)$			
$(43, 48, 53, 74)$		$(33, 59, 66, 4)$			
$(17, 30, 52, 59)$		$(34, 71, 84, 85)$			
$(97, 6, 8, 55)$		$(7, 41, 44, 74)$			
$(106, 19, 29, 68)$		$(106, 11, 13, 64)$			
S	A	S+A	S	A	S+A
$(40, 44, 56, 84)$	80	$(12, 16, 28, 56)$	$(82, 102, 26, 42)$	11	$(93, 5, 37, 53)$
$(47, 71, 79, 41)$	39	$(86, 2, 10, 80)$	$(50, 73, 93, 96)$	104	$(46, 69, 89, 92)$
$(\infty_1, 1, 35, 70)$	30	$(\infty_1, 31, 65, 100)$	$(\infty_2, 4, 21, 64)$	58	$(\infty_2, 62, 79, 14)$
$(\infty_3, 103, 7, 51)$	100	$(\infty_3, 95, 107, 43)$	$(\infty_4, 85, 89, 5)$	44	$(\infty_4, 21, 25, 49)$
$(\infty_5, 87, 95, 3)$	73	$(\infty_5, 52, 60, 76)$	$(\infty_6, 28, 39, 78)$	88	$(\infty_6, 8, 19, 58)$
$(\infty_7, 66, 83, 37)$	57	$(\infty_7, 15, 32, 94)$	$(\infty_8, 12, 92, 33)$	55	$(\infty_8, 67, 39, 88)$
$(\infty_9, 38, 88, 13)$	37	$(\infty_9, 75, 17, 50)$	$(\infty_{10}, 107, 2, 34)$	49	$(\infty_{10}, 48, 51, 83)$
$(\infty_{11}, 61, 65, 80)$	85	$(\infty_{11}, 38, 42, 57)$	$(\infty_{12}, 62, 67, 105)$	76	$(\infty_{12}, 30, 35, 73)$
$(\infty_{13}, 14, 20, 22)$	83	$(\infty_{13}, 97, 103, 105)$	$(\infty_{14}, 69, 76, 25)$	79	$(\infty_{14}, 40, 47, 104)$
$(\infty_{15}, 10, 11, 58)$	12	$(\infty_{15}, 22, 23, 70)$	$(\infty_{16}, 94, 104, 16)$	82	$(\infty_{16}, 68, 78, 98)$
$(\infty_{17}, 60, 100, 23)$	1	$(\infty_{17}, 61, 101, 24)$	$(\infty_{18}, 75, 86, 98)$	16	$(\infty_{18}, 91, 102, 6)$
$(\infty_{19}, 77, 91, 24)$	5	$(\infty_{19}, 82, 96, 29)$	$(\infty_{20}, 101, 15, 49)$	62	$(\infty_{20}, 55, 77, 3)$

Table 9: (S, C, R) and A for a $(1, 3; 4)$-frame of type $12^{10} 24^1$.

C		R			
$(36, 43, 34, 117)$		$(44, 45, 78, 107)$			
$(69, 104, 31, 2)$		$(13, 18, 59, 72)$			
$(32, 95, 73, 98)$		$(6, 15, 29, 84)$			
$(8, 22, 7, 41)$		$(79, 81, 96, 118)$			
$(81, 58, 107, 64)$		$(3, 34, 41, 52)$			
$(5, 67, 94, 16)$		$(2, 28, 49, 55)$			
S	A	S+A	S+A	A	S+A
$(72, 75, 91, 116)$	37	$(109, 112, 8, 33)$	$(11, 46, 54, 99)$	12	$(23, 58, 66, 111)$
$(17, 45, 49, 1)$	8	$(25, 53, 57, 9)$	$(\infty_1, 35, 47, 83)$	4	$(\infty_1, 39, 51, 87)$
$(\infty_2, 23, 27, 79)$	42	$(\infty_2, 65, 69, 1)$	$(\infty_3, 56, 84, 108)$	7	$(\infty_3, 63, 91, 115)$
$(\infty_4, 42, 63, 55)$	34	$(\infty_4, 76, 97, 89)$	$(\infty_5, 61, 74, 9)$	73	$(\infty_5, 14, 27, 82)$
$(\infty_6, 88, 93, 29)$	115	$(\infty_6, 83, 88, 24)$	$(\infty_7, 21, 37, 82)$	17	$(\infty_7, 38, 54, 99)$
$(\infty_8, 13, 97, 109)$	85	$(\infty_8, 98, 62, 74)$	$(\infty_9, 18, 66, 101)$	1	$(\infty_9, 19, 67, 102)$
$(\infty_{10}, 118, 19, 52)$	23	$(\infty_{10}, 21, 42, 75)$	$(\infty_{11}, 33, 78, 114)$	38	$(\infty_{11}, 71, 116, 32)$
$(\infty_{12}, 62, 65, 96)$	101	$(\infty_{12}, 43, 46, 77)$	$(\infty_{13}, 53, 57, 71)$	48	$(\infty_{13}, 101, 115, 119)$
$(\infty_{14}, 119, 4, 28)$	57	$(\infty_{14}, 56, 61, 85)$	$(\infty_{15}, 86, 92, 103)$	22	$(\infty_{15}, 108, 114, 5)$
$(\infty_{16}, 44, 51, 106)$	62	$(\infty_{16}, 106, 113, 48)$	$(\infty_{17}, 24, 26, 68)$	68	$(\infty_{17}, 92, 94, 16)$
$(\infty_{18}, 6, 15, 38)$	89	$(\infty_{18}, 95, 104, 7)$	$(\infty_{19}, 111, 112, 48)$	45	$(\infty_{19}, 36, 37, 93)$
$(\infty_{20}, 77, 89, 115)$	78	$(\infty_{20}, 35, 47, 73)$	$(\infty_{21}, 105, 113, 12)$	19	$(\infty_{21}, 4, 12, 31)$
$(\infty_{22}, 102, 87, 59)$	44	$(\infty_{22}, 26, 11, 103)$	$(\infty_{23}, 25, 3, 76)$	61	$(\infty_{23}, 86, 64, 17)$
$(\infty_{24}, 85, 39, 14)$	103	$(\infty_{24}, 68, 22, 117)$			

Table 10: (S, C, R) and A for a $(1, 3; 4)$-frame of type $12^{11}20^1$.

C		R	C		R
(6, 8, 49, 111)		(20, 46, 51, 65)	(20, 21, 27, 58)		(29, 42, 120, 127)
(80, 97, 10, 23)		(18, 107, 32, 69)	(41, 56, 94, 31)		(43, 106, 72, 45)
(39, 42, 65, 124)		(41, 80, 115, 90)			
S	A	S+A	S	A	S+A
(7, 12, 26, 93)	63	(70, 75, 89, 24)	(14, 79, 129, 76)	5	(19, 84, 2, 81)
(74, 102, 106, 126)	84	(26, 54, 58, 78)	(17, 73, 89, 107)	40	(57, 113, 129, 15)
(16, 116, 36, 32)	87	(103, 71, 123, 119)	(47, 115, 59, 29)	69	(116, 52, 128, 98)
(62, 64, 92, 13)	47	(109, 111, 7, 60)	(51, 67, 72, 103)	45	(96, 112, 117, 16)
(75, 120, 123, 34)	71	(14, 59, 62, 105)	(78, 82, 95, 19)	85	(31, 35, 48, 104)
$(\infty_1, 25, 46, 86)$	113	$(\infty_1, 6, 27, 67)$	$(\infty_2, 84, 54, 45)$	127	$(\infty_2, 79, 49, 40)$
$(\infty_3, 37, 9, 101)$	131	$(\infty_3, 36, 8, 100)$	$(\infty_4, 69, 81, 104)$	14	$(\infty_4, 83, 95, 118)$
$(\infty_5, 125, 35, 71)$	98	$(\infty_5, 91, 1, 37)$	$(\infty_6, 40, 48, 98)$	46	$(\infty_6, 86, 94, 12)$
$(\infty_7, 52, 61, 53)$	21	$(\infty_7, 73, 82, 74)$	$(\infty_8, 2, 128, 113)$	32	$(\infty_8, 34, 28, 13)$
$(\infty_9, 4, 28, 100)$	35	$(\infty_9, 39, 63, 3)$	$(\infty_{10}, 63, 112, 122)$	107	$(\infty_{10}, 38, 87, 97)$
$(\infty_{11}, 60, 68, 108)$	125	$(\infty_{11}, 53, 61, 101)$	$(\infty_{12}, 18, 24, 43)$	106	$(\infty_{12}, 124, 130, 17)$
$(\infty_{13}, 130, 5, 83)$	25	$(\infty_{13}, 23, 30, 108)$	$(\infty_{14}, 87, 96, 119)$	6	$(\infty_{14}, 93, 102, 125)$
$(\infty_{15}, 50, 114, 85)$	103	$(\infty_{15}, 21, 85, 56)$	$(\infty_{16}, 57, 118, 117)$	19	$(\infty_{16}, 76, 5, 4)$
$(\infty_{17}, 3, 15, 30)$	111	$(\infty_{17}, 114, 126, 9)$	$(\infty_{18}, 91, 109, 1)$	91	$(\infty_{18}, 50, 68, 92)$
$(\infty_{19}, 70, 127, 90)$	52	$(\infty_{19}, 122, 47, 10)$	$(\infty_{20}, 38, 131, 105)$	26	$(\infty_{20}, 64, 25, 131)$

Table 11: (S, C, R) and A for a $(1, 3; 4)$-frame of type $12^{12}20^1$.

C		R	C		R
(82, 83, 125, 44)		(73, 80, 86, 95)	(25, 39, 58, 124)		(11, 109, 98, 128)
(61, 78, 131, 140)		(4, 123, 62, 125)	(43, 46, 93, 20)		(13, 34, 75, 140)
(45, 47, 86, 128)		(18, 33, 51, 68)			
S	A	S+A	S	A	S+A
(2, 34, 69, 109)	5	(7, 39, 74, 114)	(11, 22, 40, 98)	31	(42, 53, 71, 129)
(87, 117, 138, 4)	4	(91, 121, 142, 8)	(92, 94, 114, 119)	129	(77, 79, 99, 104)
(23, 30, 38, 79)	40	(63, 70, 78, 119)	(7, 13, 41, 105)	28	(35, 41, 69, 133)
(3, 67, 110, 118)	49	(52, 116, 15, 23)	(62, 139, 121, 95)	43	(105, 38, 20, 138)
(19, 85, 88, 14)	30	(49, 115, 118, 44)	(89, 103, 99, 134)	71	(16, 30, 26, 61)
(104, 9, 65, 64)	134	(94, 143, 55, 54)	(33, 75, 127, 143)	128	(17, 59, 111, 127)
(115, 16, 50, 27)	85	(56, 101, 135, 112)	$(\infty_1, 129, 133, 29)$	118	$(\infty_1, 103, 107, 3)$
$(\infty_2, 136, 123, 107)$	27	$(\infty_2, 19, 6, 134)$	$(\infty_3, 56, 55, 1)$	116	$(\infty_3, 28, 27, 117)$
$(\infty_4, 122, 6, 142)$	59	$(\infty_4, 37, 65, 57)$	$(\infty_5, 71, 18, 130)$	11	$(\infty_5, 82, 29, 141)$
$(\infty_6, 52, 59, 5)$	78	$(\infty_6, 130, 137, 83)$	$(\infty_7, 28, 32, 53)$	57	$(\infty_7, 85, 89, 110)$
$(\infty_8, 66, 113, 97)$	56	$(\infty_8, 122, 25, 9)$	$(\infty_9, 31, 37, 57)$	69	$(\infty_9, 100, 106, 126)$
$(\infty_{10}, 8, 17, 70)$	23	$(\infty_{10}, 31, 40, 93)$	$(\infty_{11}, 63, 73, 100)$	76	$(\infty_{11}, 139, 5, 32)$
$(\infty_{12}, 68, 81, 111)$	121	$(\infty_{12}, 45, 58, 88)$	$(\infty_{13}, 49, 116, 135)$	75	$(\infty_{13}, 124, 47, 66)$
$(\infty_{14}, 10, 74, 102)$	92	$(\infty_{14}, 102, 22, 50)$	$(\infty_{15}, 90, 21, 76)$	55	$(\infty_{15}, 1, 76, 131)$
$(\infty_{16}, 35, 106, 101)$	130	$(\infty_{16}, 21, 92, 87)$	$(\infty_{17}, 112, 42, 80)$	1	$(\infty_{17}, 113, 43, 81)$
$(\infty_{18}, 141, 51, 54)$	13	$(\infty_{18}, 10, 64, 67)$	$(\infty_{19}, 15, 137, 91)$	143	$(\infty_{19}, 14, 136, 90)$
$(\infty_{20}, 26, 126, 77)$	20	$(\infty_{20}, 46, 2, 97)$			

Table 12: (S, C, R) and A for a $(1, 3; 4)$-frame of type $12^{13}20^1$.

C		R	C		R
(41, 47, 100, 114)		(14, 31, 61, 84)	(53, 60, 82, 103)		(8, 67, 118, 129)
(71, 105, 122, 152)		(4, 18, 97, 103)	(6, 9, 24, 55)		(55, 70, 144, 145)
(22, 93, 120, 155)		(34, 56, 75, 77)			
S	A	S+A	S	A	S+A
(138, 149, 150, 38)	4	(142, 153, 154, 42)	(11, 16, 35, 98)	77	(88, 93, 112, 19)
(75, 83, 116, 15)	7	(82, 90, 123, 22)	(54, 108, 140, 144)	57	(111, 9, 41, 45)
(28, 48, 76, 85)	129	(1, 21, 49, 58)	(51, 106, 111, 19)	73	(124, 23, 28, 92)
(4, 29, 124, 127)	34	(38, 63, 2, 5)	(50, 90, 99, 62)	36	(86, 126, 135, 98)
(7, 121, 125, 145)	47	(54, 12, 16, 36)	(92, 148, 8, 119)	133	(69, 125, 141, 96)
(67, 131, 141, 101)	6	(73, 137, 147, 107)	(2, 74, 23, 17)	72	(74, 146, 95, 89)
(89, 81, 10, 43)	38	(127, 119, 48, 81)	(84, 36, 133, 147)	66	(150, 102, 43, 57)
(59, 63, 86, 3)	24	(83, 87, 110, 27)	(110, 1, 139, 102)	79	(33, 80, 62, 25)
$(\infty_1, 18, 34, 95)$	114	$(\infty_1, 132, 148, 53)$	$(\infty_2, 126, 88, 46)$	25	$(\infty_2, 151, 113, 71)$
$(\infty_3, 136, 134, 94)$	128	$(\infty_3, 108, 106, 66)$	$(\infty_4, 77, 87, 112)$	51	$(\infty_4, 128, 138, 7)$
$(\infty_5, 113, 37, 30)$	42	$(\infty_5, 155, 79, 72)$	$(\infty_6, 27, 61, 115)$	135	$(\infty_6, 6, 40, 94)$
$(\infty_7, 151, 5, 49)$	154	$(\infty_7, 149, 3, 47)$	$(\infty_8, 118, 12, 137)$	3	$(\infty_8, 121, 15, 140)$
$(\infty_9, 135, 142, 73)$	58	$(\infty_9, 37, 44, 131)$	$(\infty_{10}, 79, 80, 96)$	106	$(\infty_{10}, 29, 30, 46)$
$(\infty_{11}, 66, 69, 97)$	70	$(\infty_{11}, 136, 139, 11)$	$(\infty_{12}, 31, 40, 72)$	28	$(\infty_{12}, 59, 68, 100)$
$(\infty_{13}, 109, 129, 45)$	5	$(\infty_{13}, 114, 134, 50)$	$(\infty_{14}, 146, 14, 57)$	119	$(\infty_{14}, 109, 133, 20)$
$(\infty_{15}, 58, 33, 153)$	27	$(\infty_{15}, 85, 60, 24)$	$(\infty_{16}, 42, 154, 64)$	124	$(\infty_{16}, 10, 122, 32)$
$(\infty_{17}, 20, 68, 70)$	31	$(\infty_{17}, 51, 99, 101)$	$(\infty_{18}, 107, 32, 21)$	84	$(\infty_{18}, 35, 116, 105)$
$(\infty_{19}, 132, 44, 56)$	20	$(\infty_{19}, 152, 64, 76)$	$(\infty_{20}, 123, 128, 25)$	148	$(\infty_{20}, 115, 120, 17)$

The next lemma gives two more useful $(1, 3; 4)$-frames of type g^u with $g > 1$ obtainable by the starter-adder method.

Lemma 3.8. *There exist* $(1, 3; 4)$*-frames of types* 3^9 *and* 13^9.

Proof. For type 3^9 suitable starter blocks and adders over Z_{27} are given in the Table 13. For a frame of type 13^9 over Z_{117}, we multiply the starter blocks given in Table 13 (except the last two) and their adders by 1, 61 and $61^2 = 94$. The last two starter blocks (with adders 78, 39) and their adders remain invariant when multiplied by these values and should included without any extra multiples.

Table 13: Starter blocks and adders for $(1, 3; 4)$-frames of type 3^9 and 13^9.

Type	S	A	S	A	S	A
3^9	(14, 2, 13, 16)	1	(5, 26, 20, 12)	20	(19, 8, 21, 4)	16
	(25, 17, 22, 15)	6	(7, 23, 24, 1)	15	(3, 10, 6, 11)	23
13^9	(69, 26, 1, 49)	56	(30, 106, 74, 59)	80	(83, 13, 44, 25)	48
	(21, 67, 15, 17)	115	(86, 66, 29, 85)	35	(84, 46, 2, 42)	73
	(79, 82, 33, 95)	105	(8, 73, 40, 70)	16	(78, 113, 107, 92)	78
	(39, 116, 56, 23)	39				

□

4 Recursive constructions

In Section 3, direct constructions were given for $(1,3;4)$-frames of type 1^v for $v \equiv 1 \pmod 4$ for (1) $21 \le v \le 105$, (2) v prime, $v \le 241$, $v \notin \{5,17\}$ and (3) $v \in \{121,125,133,169\}$. With these designs and several frames from the previous section, we can provide recursive constructions for most of the remaining cases.

Lemma 4.1. *A $(1,3;4)$-frame of type 1^v exists for $v \in \{117, 129, 141, 145, 153, 161, 165, 177, 189, 201, 209, 217, 221, 225, 237, 245, 249, 253\}$.*

Proof. For $v = 129$, 145, 153, 165, 177, apply Lemma 2.5 with $b = 1$ to the $(1,3;4)$-frames of types $12^9 20^1$, $12^{10} 24^1$, $12^{11} 20^1$, $12^{12} 20^1$ and $12^{13} 20^1$ in Lemma 3.7. For $v = 141$, 161, 201, we apply Construction 2.4 with $m = 5$ to the $(1,3;4)$-frames of type 4^n for $n = 7,8,10$ from Lemma 3.6. This gives $(1,3;4)$-frames of type 20^n, to which we can apply Lemma 2.5 with $b = 1$. Similarly, for $v = 117, 189$, we have $(1,3;4)$-frames of types 13^9 and 21^9; the first was given in Lemma 3.8, and the second is obtained by Construction 2.4 with $m = 7$, inflating a $(1,3;4)$-frame of type 3^9 (given in Lemma 3.8). Now apply Construction 2.5 with $b = 0$, filling in the groups with $(1,3;4)$-frames of types 1^{13} and 1^{21}.

For $v \in \{209, 217, 221, 225\}$, let $v = 196 + 4t + 1$ with $t \in \{3,5,6,7\}$, and for $v \in \{237, 245, 249, 253\}$, let $v = 224 + 4t + 1$. For the first four values, truncate one group of a $\mathrm{TD}(8,7)$ to size t, and for the second four, truncate one group of a $\mathrm{TD}(8,8)$ to size t. We now apply Construction 2.3 to this truncated TD, giving all points in the given truncated TD a weight of four, and using $(1,3;4)$-frames of types 4^7 and 4^8 in Lemma 3.6 for inflation. This way, we obtain a $(1,3;4)$-frame of type $28^7(4t)^1$ or $32^7(4t)^1$ on $v-1$ points; in each case, applying Lemma 2.5 with $b = 1$ to this frame now gives the required $(1,3;4)$ frame of type 1^v. ∎

Lemma 4.2. *If $v \equiv 1 \pmod 4$ and $v \ge 257$, then a $(1,3;4)$-frame of type 1^v exists.*

Proof. For $257 \le v \le 289$, we use a construction similar to the one for $209 \le v \le 253$ in the last example. Truncate up to two groups of a $\mathrm{TD}(9,8)$ to sizes in $\{0,3,5,6,7,8\}$ so that the total number of points in the truncated TD is $(v-1)/4$. We then inflate the resulting truncated TD by 4, using Construction 2.3 and the $(1,3;4)$-frames of type 4^n in Lemma 3.6. Finally we apply Construction 2.5 with $b = 1$.

If v is in one of the ranges $[273,361]$, $[329,440]$, $[385,521]$, $[497,681]$, $[665,921]$, $[893,1169]$, $[1061,1337]$, $[1229,1505]$, $[1397,1673]$ or $[1677,1953]$, we can apply a similar construction, truncating up to three groups in a $\mathrm{TD}(10,m)$ (for $m = 9$, 11, 13, 17, 23, 31, 37, 43, 49 or 59) either to size 0 or to sizes in the range $[5, \mathrm{Min}(m,25)]$. If $v > 1953$, then we can write $v = 4(7m + u_1 + u_2 + u_3) + 1$ where m is any one of five consecutive odd integers > 59, $5 \le u_1 + u_2 + u_3 \le 75$, and either $u_i = 0$ or $5 \le u_i \le 25$ for $i = 1,2,3$. At most two of the five feasible values of m can be divisible by 3, one by 5 and one by 7, so by Lemma 2.2(2), a $\mathrm{TD}(10,m)$ exists for at least one of these five values. We truncate three groups in this $\mathrm{TD}(10,m)$ to sizes u_1, u_2, u_3 and again inflate the resulting design by 4, using $(1,3;4)$ frames of type 4^n

for $7 \leq n \leq 10$. This gives a $(1, 3; 4)$-frame of type $(4m)^7(4u_1)^1(4u_2)^1(4u_3)^1$ on $v-1$ points. For all m, t_1, t_2, t_3 in the ranges given, $(1, 3; 4)$-frames of types 1^{4m+1}, 1^{4u_1+1}, 1^{4u_2+1}, 1^{4u_3+1} exist, so we can apply Lemma 2.5 with $b = 1$ to the $(1, 3; 4)$-frame of type $(4m)^7(4u_1)^1(4u_2)^1(4u_3)^1$. ∎

Summarizing the results of this section, we have now established the following improvement on Lemma 3.4:

Theorem 4.3. *There exists a* $(1, 3; 4)$*-frame of type* 1^v *for all* $v \equiv 1 \pmod 4$*, except for* $v \in \{5, 9\}$ *and possibly for* $v \in \{17, 185, 205, 213\}$*.*

We conclude by mentioning a few examples of doubly near resolvable $(v, 4, 3)$-BIBDs which are not $(1, 3; 4)$-frames. A DNR$(5, 4, 3)$-BIBD can be obtained by putting the five blocks of a $(5, 4, 3)$-BIBD on the main diagonal of a 5×5 square. When $v \in \{185, 205\}$, we can inflate a $(1, 3; 4)$-frame of type $1^{v/5}$ using Construction 2.4 with $m = 5$ to obtain a $(1, 3; 4)$-frame of type $5^{v/5}$. Filling in the groups with a DNR$(5, 4, 3)$-BIBD then gives the required result. Thus we have:

Theorem 4.4. *There exists a DNR*$(v, 4, 3)$ *BIBD for* $v \equiv 1 \pmod 4$*, except for* $v = 9$ *and possibly for* $v \in \{17, 213\}$*.*

References

[1] R.J.R. Abel, C.J. Colbourn and J.H. Dinitz, Mutually Orthogonal Latin Squares (MOLS), in: *The CRC Handbook of Combinatorial Designs, Second Ed.* (C.J. Colbourn and J.H. Dinitz, eds.), CRC Press, Boca Raton, FL, (2006), 160–193.

[2] R.J.R. Abel, E. Lamken and J. Wang, A few more Kirkman Squares and doubly resolvable BIBDs with block size 3, *Discrete Math.* **308** (2008), 1102–1123.

[3] R.J.R. Abel, F.E. Bennett and G. Ge, Existence of directed triplewhist tournaments with the three person property, 3PDTWh(v) *Discrete Appl. Math.* **156** (2008), 2655–2665.

[4] R.J.R. Abel and G. Ge, Some difference matrix constructions and an almost completion for the existence of triplewhist tournaments TWh(v), *European J. Combin.* **26** (2005), 1094–1104.

[5] R.J.R. Abel and G. Ge, Ordered tournaments and ordered triplewhist tournements with the three person property, *J. Combin. Des.* **17** (2009), 39–52.

[6] F. E. Bennett and G. Ge, Existence of directedwhist tournaments with the three person property, *Discrete Appl. Math.* **154** (2009), 139–146.

[7] C.J. Colbourn, E.R. Lamken, A.C.H. Ling and W.H. Mills, The existence of Kirkman squares — Doubly resolvable $(v, 3, 1)$-BIBDs, *Des. Codes Cryptogr.* **26** (2002), 169–196.

[8] E.R. Lamken, 3-Complementary frames and doubly near resolvable $(v, 3, 2)$-BIBDs, *Discrete Math.* **88** (1991), 59–78.

[9] E.R. Lamken, Constructions for generalized balanced tournament designs, *Discr. Math.* **131** (1994), 127–151.

[10] E.R. Lamken, The existence of doubly near resolvable $(v, 3, 2)$-BIBDs, *J. Combin. Des.* **2** (1994), 427–440.

[11] R.C. Mullin and W.D. Wallis, The existence of Room squares, *Aequationes Math.* **13** (1975), 1–7.

[12] R. Mathon and A. Rosa, 2-(v, k, λ) designs of small order, in: *The CRC Handbook of Combinatorial Designs*, 2nd ed., (eds. C.J. Colbourn and J.H. Dinitz), CRC Press, Boca Raton, FL, 2007, 25–58.

[13] A. Rosa and S.A. Vanstone, Starter-adder techniques for Kirkman squares and Kirkman cubes of small sides, *Ars Combin.* **14** (1982), 199–212.

[14] R.M. Wilson, Constructions and uses of pairwise balanced designs, *Math. Centre Tracts* **55** (1974), 18–41.

[15] D. Wu, Q. Shu, R. Fuji-Hara, D. Li and S. Chen, The existence of DNR$(mt + 1, m, m - 1)$-BIBDs, preprint.

[16] J. Yin, J. Yan and C. Wang, Generalized balanced tournament designs and related codes, *Des. Codes Cryptogr.* **46** (2008), 211–230.

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