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Proper edge coloring of BIBD$(v, 4, \lambda)$s

John Asplund   Melissa S. Keranen

Department of Mathematical Sciences
Michigan Technological University
Houghton, MI 49931-0402
U.S.A.

Abstract

A decomposition of λ copies of monochromatic K_v into copies of K_4 such that each copy of K_4 contains at most one edge from each K_v is called a proper edge coloring of a BIBD$(v, 4, \lambda)$. We show that the necessary conditions are sufficient for the existence of a BIBD$(v, 4, \lambda)$ which has such a proper edge coloring.

1 Introduction

A balanced incomplete block design, BIBD(v, k, λ) is a decomposition of λ copies of K_v into copies of K_k (see [1] and [2]). Although most coloring problems for designs deal with coloring the vertices ([1] and [2]), Hurd and Sarvate [6] studied the problem of coloring the pairs determined by the points. They posed the following problem in [6]. *"Is it possible to decompose λ copies of monochromatic K_v into copies of K_k such that each copy of K_k contains at most one edge from each K_v?"* By the phrase λ copies of monochromatic K_v, we mean there is a set of λ colors, and each edge of K_v has been repeated and colored once for each color. They call such a decomposition a proper edge coloring and were able to show that there exists a proper edge coloring for any BIBD(v, k, λ) with $\lambda = mk(k-1)/2$. They also showed that the necessary conditions are sufficient for the existence of a triple system BIBD$(v, 3, \lambda)$ that has a proper edge coloring. We focus on the case when $k = 4$.

If there exists a BIBD$(v, k, 1)$, then Hurd and Sarvate showed there is an easy solution to the proper edge coloring problem for a BIBD(v, k, λ). Take λ copies of each block in the BIBD$(v, k, 1)$. For each block in the BIBD$(v, k, 1)$ form a $\binom{k}{2} \times \lambda$ matrix. The rows of this matrix will be indexed by the $\binom{k}{2}$ pairs of points, and the columns will be indexed by the λ copies of the block. The entries will be the first $\binom{k}{2}$ rows of a Latin square of order λ. Thus, we have the following theorem.

Theorem 1 (Hurd, Sarvate, [6]) *If a* BIBD$(v, k, 1)$ *exists, then for each* $\lambda \geq k(k-1)/2$*, there exists a* BIBD(v, k, λ) *that has a proper edge coloring.*

Theorem 2 (Hurd, Sarvate, [6]) *Every* $\mathrm{BIBD}(v, k, mk(k-1)/2)$ *has a proper edge coloring.*

The necessary and sufficient conditions for the existence of a $\mathrm{BIBD}(v, 4, \lambda)$ were obtained by Hanani in [5]. They are as follows:

If $\lambda \equiv 1, 5 \pmod 6$, then $v \equiv 1, 4 \pmod{12}$;

if $\lambda \equiv 2, 4 \pmod 6$, then $v \equiv 1 \pmod 3$;

if $\lambda \equiv 3 \pmod 6$, then $v \equiv 0, 1 \pmod 4$; and

if $\lambda \equiv 0 \pmod 6$, then $v \geq 4$.

If $\lambda \equiv 1, 5 \pmod 6$, then we can properly edge color a $\mathrm{BIBD}(v, 4, \lambda)$ by applying Theorem 1, and if $\lambda \equiv 0 \pmod 6$, then we can properly edge color a $\mathrm{BIBD}(v, 4, \lambda)$ by applying Theorem 2. Therefore, in this article we need only consider $\mathrm{BIBD}(v, 4, \lambda)$ with $\lambda \equiv 2, 3, 4 \pmod 6$.

2 Direct Constructions

An edge coloring for a $\mathrm{BIBD}(v, 4, \lambda)$ $(\mathcal{X}, \mathcal{B})$ can be described by providing an edge coloring incidence matrix. This is the pair by block matrix M defined by

$$M = M[\{x, y\}, B] = \begin{cases} c_j & \text{if } \{x, y\} \in B \text{ has color } c_j, \\ 0 & \text{if } \{x, y\} \notin B \end{cases}$$

where the c_j represent the colors used for all $\{x, y\} \in \binom{\mathcal{X}}{2}$, $B \in \mathcal{B}$. For example, an edge coloring for a $\mathrm{BIBD}(5, 4, 3)$ can be represented by the edge coloring incidence matrix using 6 colors given in Figure 1.

Let A be the 5×5 circulant matrix whose first row is [01000]. Then we can represent this edge coloring incidence matrix in terms of A as follows.

$c_1A^2 + c_2A^3 + c_3A^4$
$c_4A + c_5A^3 + c_6A^4$

To form a properly colored $\mathrm{BIBD}(5, 4, \lambda)$ with $\lambda = 3k$, $k \geq 2$, we simply repeat the blocks of the $\mathrm{BIBD}(5, 4, 3)$ k times and follow the same coloring scheme with different colors. Define M_i as the following sub-matrix. Note that all subscripts are computed $\pmod \lambda$ where we identify c_0 with c_λ.

$$M_i = \begin{array}{|c|} \hline (c_{1+3i})A^2 + (c_{2+3i})A^3 + (c_{3+3i})A^4 \\ \hline (c_{4+3i})A + (c_{5+3i})A^3 + (c_{6+3i})A^4 \\ \hline \end{array}$$

Then the edge coloring incidence matrix of a properly colored $\mathrm{BIBD}(5, 4, \lambda)$ with $\lambda = 3k$, $k \geq 2$ is given by

	B_1	B_2	B_3	B_4	B_5
(1,2)	0	0	c_1	c_2	c_3
(2,3)	c_3	0	0	c_1	c_2
(3,4)	c_2	c_3	0	0	c_1
(4,5)	c_1	c_2	c_3	0	0
(5,1)	0	c_1	c_2	c_3	0
(1,3)	0	c_4	0	c_5	c_6
(2,4)	c_6	0	c_4	0	c_5
(3,5)	c_5	c_6	0	c_4	0
(4,1)	0	c_5	c_6	0	c_4
(5,2)	c_4	0	c_5	c_6	0

Figure 1: Edge-Coloring Incidence Matrix for a BIBD$(5, 4, 3)$

$$M = \begin{array}{|c|c|c|c|}\hline M_0 & M_1 & \cdots & M_{k-1} \\ \hline\end{array}\,.$$

The matrix M has the property that every color $c_1, c_2, \ldots, c_\lambda = c_0$ is seen exactly once in each row, and every color is seen at most once in every column. Therefore, M is an edge coloring incidence matrix of a properly colored BIBD$(5, 4, \lambda)$.

We state this result as a lemma.

Lemma 3 *There exists a* BIBD$(5, 4, \lambda)$ *which can be properly edge-colored for any* $\lambda = 3k$, $k \geq 2$.

As an example of this lemma, we provide the edge coloring incidence matrix of a properly colored BIBD$(5, 4, 9)$ in Figure 2.

	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}	B_{14}	B_{15}
(1,2)	0	0	c_1	c_2	c_3	0	0	c_4	c_5	c_6	0	0	c_7	c_8	c_9
(2,3)	c_3	0	0	c_1	c_2	c_6	0	0	c_4	c_5	c_9	0	0	c_7	c_8
(3,4)	c_2	c_3	0	0	c_1	c_5	c_6	0	0	c_4	c_8	c_9	0	0	c_7
(4,5)	c_1	c_2	c_3	0	0	c_4	c_5	c_6	0	0	c_7	c_8	c_9	0	0
(5,1)	0	c_1	c_2	c_3	0	0	c_4	c_5	c_6	0	0	c_7	c_8	c_9	0
(1,3)	0	c_4	0	c_5	c_6	0	c_7	0	c_8	c_9	0	c_1	0	c_2	c_3
(2,4)	c_6	0	c_4	0	c_5	c_9	0	c_7	0	c_8	c_3	0	c_1	0	c_2
(3,5)	c_5	c_6	0	c_4	0	c_8	c_9	0	c_7	0	c_2	c_3	0	c_1	0
(4,1)	0	c_5	c_6	0	c_4	0	c_8	c_9	0	c_7	0	c_2	c_3	0	c_1
(5,2)	c_4	0	c_5	c_6	0	c_7	0	c_8	c_9	0	c_1	0	c_2	c_3	0

Figure 2: Edge-Coloring Incidence Matrix for a BIBD$(5, 4, 9)$

Lemma 4 *There exists a* BIBD$(8,4,\lambda)$ *which can be properly edge-colored for any* $\lambda = 3k$, $k \geq 2$.

Proof.

Let A be the 8×8 circulant matrix whose first row is [01000000]. Then we can represent an edge coloring incidence matrix for a BIBD$(8,4,3)$ in terms of A on the colors $c_1, c_2, c_3, c_4, c_5, c_6$. This representation is given in Figure 3.

$c_1 I$	$c_2 I + c_3 A^6$
$c_2 A^6$	$c_1 I + c_4 A^5$
$c_3 I$	$c_5 A^3 + c_6 A^6$
$c_4 I + c_6 A^4 + c_5 A^6$	0

Figure 3: Edge-Coloring Incidence Matrix for a BIBD$(8,4,3)$

For any i, let $M_i^{(6)}$ be the edge coloring incidence matrix of a BIBD$(8,4,6)$ on the colors $c_{1+6i}, c_{2+6i}, c_{3+6i}, c_{4+6i}, c_{5+6i}, c_{6+6i}$ found in Figure 4.

$M_i^{(6)} =$

$(c_{1+6i})I$	$(c_{2+6i})I+(c_{3+6i})A^6$	$(c_{5+6i})I$	$(c_{4+6i})I+(c_{6+6i})A^6$
$(c_{2+6i})A^6$	$(c_{1+6i})I+(c_{4+6i})A^5$	$(c_{6+6i})A^6$	$(c_{3+6i})I+(c_{5+6i})A^5$
$(c_{3+6i})I$	$(c_{5+6i})A^3+(c_{6+6i})A^6$	$(c_{4+6i})I$	$(c_{1+6i})A^3+(c_{2+6i})A^6$
$(c_{4+6i})I$ $+(c_{6+6i})A^4$ $+(c_{5+6i})A^6$	0	$(c_{1+6i})I$ $+(c_{2+6i})A^4$ $+(c_{3+6i})A^6$	0

Figure 4: Edge-Coloring Incidence Matrix for a BIBD$(8,4,6)$

Denote $M_0^{(9)}$ as the edge coloring incidence matrix of a BIBD$(8,4,9)$ on the colors $c_{1'}, c_{2'}, \ldots, c_{9'}$ found in Figure 5.

$M_0^{(9)} =$

$c_{1'}I$	$c_{2'}I + c_{3'}A^6$	$c_{5'}I$	$c_{4'}I + c_{6'}A^6$	$c_{7'}I$	$c_{8'}I + c_{9'}A^6$
$c_{2'}A^6$	$c_{1'}I + c_{4'}A^5$	$c_{6'}A^6$	$c_{5'}I + c_{8'}A^5$	$c_{9'}A^6$	$c_{3'}I + c_{7'}A^5$
$c_{3'}I$	$c_{5'}A^3 + c_{6'}A^6$	$c_{4'}I$	$c_{7'}A^3 + c_{9'}A^6$	$c_{8'}I$	$c_{1'}A^3 + c_{2'}A^6$
$c_{4'}I$ $+c_{6'}A^4$ $+c_{5'}A^6$	0	$c_{7'}I$ $+c_{8'}A^4$ $+c_{9'}A^6$	0	$c_{1'}I$ $+c_{2'}A^4$ $+c_{3'}A^6$	0

Figure 5: Edge-Coloring Incidence Matrix for a BIBD$(8,4,9)$

We now give the edge coloring incidence matrix of a properly colored BIBD$(8,4,\lambda)$ with $\lambda = 3k$, $k \geq 2$. The subscripts are computed $\pmod{\lambda}$ where we identify c_0 with c_λ. In the case where $\lambda = 6k$, our matrix is

$M =$

$M_0^{(6)}$	$M_1^{(6)}$	$\cdots$	$M_{k-1}^{(6)}$

.

The set of colors used is $C_0, C_1, \ldots, C_{k-1}$ where
$C_i = \{c_{1+6i}, c_{2+6i}, c_{3+6i}, c_{4+6i}, c_{5+6i}, c_{6+6i}\}$ for $i = 0, 1, \ldots, k-1$. Because each $M_i^{(6)}$ is an edge coloring incidence matrix of a properly colored BIBD$(8, 4, 6)$ on a different set of 6 colors, it follows that M is an edge coloring incidence matrix of a properly colored BIBD$(8, 4, 6k)$.

In the case where $\lambda = 6k + 3$, our matrix is

$$M = \boxed{M_0^{(9)} \mid M_0^{(6)} \mid \cdots \mid M_{k-2}^{(6)}}\,.$$

The set of colors used is $C, C_0, C_1, \ldots, C_{k-2}$ where $C = \{c_{1'}, c_{2'}, \ldots, c_{9'}\}$ and $C_i = \{c_{1+6i}, c_{2+6i}, c_{3+6i}, c_{4+6i}, c_{5+6i}, c_{6+6i}\}$ for $i = 0, 1, \ldots, k-2$. Because $M_0^{(9)}$ is an edge coloring incidence matrix of a properly colored BIBD$(8, 4, 9)$ on 9 colors, and each $M_i^{(6)}$ is an edge coloring incidence matrix of a properly colored BIBD$(8, 4, 6)$ on a different set of 6 colors which are all disjoint from the colors in C; it follows that M is an edge coloring incidence matrix of a properly colored BIBD$(8, 4, 6k + 3)$.

□

Lemma 5 *There exists a* BIBD$(9, 4, \lambda)$ *which can be properly edge-colored for any* $\lambda = 3k,\ k \geq 2$.

Proof. Let A be the 9×9 circulant matrix whose first row is [010000000]. Then the edge coloring incidence matrix of a BIBD$(9, 4, 3)$ using 6 colors is given in Figure 6.

$M_i =$

$(c_{2+3i})I + (c_{1+3i})A^8$	$(c_{3+3i})I$
$(c_{3+3i})A^8$	$(c_{2+3i})A^3 + (c_{1+3i})A^8$
$(c_{4+3i})I + (c_{5+3i})A^7$	$(c_{6+3i})A^5$
$(c_{6+3i})I$	$(c_{4+3i})I + (c_{5+3i})A^3$

Figure 6: Edge-Coloring Incidence Matrix for a BIBD$(9, 4, 3)$

To form a properly colored BIBD$(9, 4, \lambda)$ with $\lambda = 3k,\ k \geq 2$, we simply repeat the blocks of the BIBD$(9, 4, 3)$ k times and follow the same coloring scheme with different colors. Thus the edge coloring incidence matrix of a properly colored BIBD$(9, 4, \lambda)$ can be given in terms of the M_i as follows. Note that the subscripts of the colors are all computed $\pmod{\lambda}$ where we identify c_0 with c_λ.

$$M = \boxed{M_0 \mid M_1 \mid \cdots \mid M_{k-1}}\,.$$

The proof that M is an edge coloring incidence matrix is similar to the argument given in Lemma 3.

□

Lemma 6 *There exists a* BIBD$(12, 4, \lambda)$ *which can be properly edge-colored for any* $\lambda = 3k,\ k \geq 2$.

$M_i =$

$(c_{1+3i})I$	$(c_{2+3i})A^2$	$(c_{3+3i})A^5$
$(c_{2+3i})A^{10}$	$(c_{3+3i})A^9$	$(c_{4+3i})A^4$
$(c_{3+3i})I$	$(c_{4+3i})A$	$(c_{5+3i})A^5$
$(c_{4+3i})A^4 + (c_{5+3i})A^8$	$(c_{6+3i})A^2$	0
$(c_{6+3i})A^4$	$(c_{1+3i})A + (c_{5+3i})A^7$	0
0	0	$(c_{1+3i})A^2 + (c_{2+3i})A^4 + (c_{6+3i})A^5$

Figure 7: Edge-Coloring Incidence Matrix for a $\mathrm{BIBD}(12, 4, 3)$

Proof. Let A be the 12×12 circulant matrix whose first row is [010000000000]. Then the edge coloring incidence matrix of a $\mathrm{BIBD}(12, 4, 3)$ using 6 colors is given in Figure 7.

To form a properly colored $\mathrm{BIBD}(12, 4, \lambda)$ with $\lambda = 3k$, $k \geq 2$, we simply repeat the blocks of the $\mathrm{BIBD}(12, 4, 3)$ k times and follow the same coloring scheme with different colors. The subscripts of the colors are all computed $\pmod{\lambda}$ where we identify c_0 with c_λ. Thus the edge coloring incidence matrix of a properly colored $\mathrm{BIBD}(12, 4, \lambda)$ can be given in terms of the M_i as

$$M = \begin{array}{|c|c|c|c|} \hline M_0 & M_1 & \cdots & M_{k-1} \\ \hline \end{array}\,.$$

□

Lemma 7 *There exists a properly edge-colored* $\mathrm{BIBD}(7, 4, \lambda)$ *for* $\lambda = 2k$, $k \geq 3$.

Proof. Let A be the 7×7 circulant matrix whose first row is [0100000]. Then the edge coloring incidence matrix of a $\mathrm{BIBD}(7, 4, 2)$ using 6 colors is given in Figure 8.

$$M_i = \begin{array}{|c|} \hline (c_{1+2i})I + (c_{2+2i})A \\ \hline (c_{3+2i})I + (c_{4+2i})A^2 \\ \hline (c_{5+2i})I + (c_{6+2i})A^4 \\ \hline \end{array}$$

Figure 8: Edge-Coloring Incidence Matrix for a $\mathrm{BIBD}(7, 4, 2)$

To form a properly colored $\mathrm{BIBD}(7, 4, \lambda)$ with $\lambda = 2k$, $k \geq 3$, we simply repeat the blocks of the $\mathrm{BIBD}(7, 4, 2)$ k times and follow the same coloring scheme with different colors. The subscripts of the colors are all computed $\pmod{\lambda}$ where we identify c_0 with c_λ. Thus the edge coloring incidence matrix of a properly colored $\mathrm{BIBD}(7, 4, \lambda)$ can be given in terms of the M_i as

$$M = \begin{array}{|c|c|c|c|} \hline M_0 & M_1 & \cdots & M_{k-1} \\ \hline \end{array}\,.$$

□

Lemma 8 *There exists a properly edge-colored* $\mathrm{BIBD}(19, 4, \lambda)$ *for* $\lambda = 2k$, $k \geq 3$.

$M_i =$

$(c_{1+2i})A^{16}$	0	$(c_{2+2i})I$
$(c_{2+2i})A^{18}$	0	$(c_{3+2i})A^{15}$
$(c_{3+2i})I$	0	$(c_{4+2i})A^{13}$
0	$(c_{4+2i})A^{18}$	$(c_{5+2i})I$
0	$(c_{5+2i})I$	$(c_{6+2i})A^{15}$
0	$(c_{6+2i})A^{6}$	$(c_{1+2i})I$
$(c_{4+2i})A^{7}$	$(c_{1+2i})A^{5}$	0
$(c_{5+2i})A^{7}$	$(c_{2+2i})A^{14}$	0
$(c_{6+2i})I$	$(c_{3+2i})I$	0

Figure 9: Edge-Coloring Incidence Matrix for a BIBD$(19, 4, 2)$

Proof. Let A be the 19×19 circulant matrix whose first row is [0100000000000000000]. Then the edge coloring incidence matrix of a BIBD$(19, 4, 2)$ using 6 colors is given in Figure 9.

To form a properly colored BIBD$(19, 4, \lambda)$ with $\lambda = 2k$, $k \geq 3$, we simply repeat the blocks of the BIBD$(19, 4, 2)$ k times and follow the same coloring scheme with different colors. The subscripts of the colors are all computed $\pmod{\lambda}$ where we identify c_0 with c_λ. Thus the edge coloring incidence matrix of a properly colored BIBD$(19, 4, \lambda)$ can be given in terms of the M_i as

$$M = \boxed{M_0 \mid M_1 \mid \cdots \mid M_{k-1}}\,.$$

□

Lemma 9 *There exists a properly colored* BIBD$(10, 4, \lambda)$ *for* $\lambda = 2k$*,* $k \geq 4$*.*

Proof. We form an edge coloring incidence matrix using 8 colors for a BIBD$(10, 4, 2)$. This is given in Figure 10.

Let $A_i^{(j)}$ be the 5×5 matrices for $j = 1, 2, 3$ found in Figure 11.

Let $B_i^{(j)}$ be the 8×5 matrices for $j = 1, 2, 3, 4, 5, 6$ found in Figure 12.

Let $C_i^{(j)}$ be the 5×5 matrices for $j = 1, 2, 3$ found in Figure 13.

Now the edge coloring incidence matrix given in Figure 10 can be represented by the sub-matrices $A_i^{(j)}, B_i^{(j)}$, and $C_i^{(j)}$ along with the all 0 sub-matrix. This representation is given in Figure 14.

To form a properly edge colored BIBD$(10, 4, \lambda)$ with $\lambda = 2k$, $k \geq 4$, we simply repeat the blocks of the BIBD$(10, 4, 2)$ k times and follow the same coloring scheme with different colors. The subscripts of the colors will all be computed $\pmod{\lambda}$ where we identify c_0 with c_λ. Thus the edge coloring incidence matrix of a properly colored BIBD$(10, 4, \lambda)$ can be given in terms of the M_i as follows.

$$M = \boxed{M_0 \mid M_1 \mid \cdots \mid M_{k-1}}\,.$$

Careful checking of M reveals that each color $c_1, c_2, \ldots, c_\lambda$ occurs exactly once in every row of M, and each color occurs no more than once in every column.

□

	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}	B_{14}	B_{15}
$(0,2)$	c_1	0	c_2	0	0	0	0	0	0	0	0	0	0	0	0
$(0,3)$	c_2	0	0	c_1	0	0	0	0	0	0	0	0	0	0	0
$(0,4)$	0	c_2	c_1	0	0	0	0	0	0	0	0	0	0	0	0
$(0,5)$	0	c_1	0	0	c_2	0	0	0	0	0	0	0	0	0	0
$(7,0)$	0	0	0	c_2	c_1	0	0	0	0	0	0	0	0	0	0
$(3,4)$	0	0	0	0	0	0	0	0	0	0	0	0	0	c_2	c_1
$(2,5)$	0	0	0	0	0	0	0	0	0	0	c_1	0	c_2	0	0
$(4,8)$	0	0	0	0	0	0	0	0	0	0	0	c_2	0	c_1	0
$(9,2)$	0	0	0	0	0	0	0	0	0	0	c_2	c_1	0	0	0
$(6,7)$	0	0	0	0	0	0	0	0	0	0	0	0	c_1	0	c_2
$(6,8)$	0	0	0	0	0	c_2	0	0	0	c_1	0	0	0	0	0
$(9,1)$	0	0	0	0	0	0	0	c_1	c_2	0	0	0	0	0	0
$(6,9)$	0	0	0	0	0	c_1	0	c_2	0	0	0	0	0	0	0
$(8,1)$	0	0	0	0	0	0	c_1	0	0	c_2	0	0	0	0	0
$(7,1)$	0	0	0	0	0	0	c_2	0	c_1	0	0	0	0	0	0
$(1,2)$	c_3	0	0	0	0	0	c_4	0	0	0	0	0	0	0	0
$(1,3)$	c_4	0	0	0	0	0	0	c_3	0	0	0	0	0	0	0
$(8,0)$	0	0	0	c_3	0	c_4	0	0	0	0	0	0	0	0	0
$(1,4)$	0	c_3	0	0	0	0	0	0	c_4	0	0	0	0	0	0
$(1,5)$	0	c_4	0	0	0	0	0	0	0	c_3	0	0	0	0	0
$(6,0)$	0	0	c_4	0	0	c_3	0	0	0	0	0	0	0	0	0
$(7,8)$	0	0	0	c_4	0	0	c_3	0	0	0	0	0	0	0	0
$(7,9)$	0	0	0	0	c_4	0	0	0	c_3	0	0	0	0	0	0
$(2,3)$	c_5	0	0	0	0	0	0	0	0	0	c_6	0	0	0	0
$(4,5)$	0	c_5	0	0	0	0	0	0	0	0	0	0	0	c_6	0
$(2,4)$	0	0	c_5	0	0	0	0	0	0	0	0	c_6	0	0	0
$(5,7)$	0	0	0	0	c_5	0	0	0	0	0	0	0	c_6	0	0
$(2,6)$	0	0	c_6	0	0	0	0	0	0	0	0	0	c_5	0	0
$(3,7)$	0	0	0	c_5	0	0	0	0	0	0	0	0	0	0	c_6
$(5,9)$	0	0	0	0	c_6	0	0	0	0	0	c_5	0	0	0	0
$(3,8)$	0	0	0	c_6	0	0	0	0	0	0	0	0	0	c_5	0
$(5,6)$	0	0	0	0	0	0	0	0	0	c_7	0	0	c_8	0	0
$(8,9)$	0	0	0	0	0	c_7	0	0	0	0	0	c_8	0	0	0
$(3,6)$	0	0	0	0	0	0	0	c_8	0	0	0	0	0	0	c_7
$(4,7)$	0	0	0	0	0	0	0	0	c_7	0	0	0	0	0	c_8
$(5,8)$	0	0	0	0	0	0	0	0	0	c_8	0	0	0	c_7	0
$(9,3)$	0	0	0	0	0	0	0	c_7	0	0	c_8	0	0	0	0
$(2,7)$	0	0	0	0	0	0	c_8	0	0	0	0	0	c_7	0	0
$(4,9)$	0	0	0	0	0	0	0	0	c_8	0	0	c_7	0	0	0
$(0,1)$	c_7	c_8	0	0	0	0	0	0	0	0	0	0	0	0	0
$(3,5)$	0	0	0	0	0	0	0	0	0	0	c_3	0	0	c_4	0
$(1,6)$	0	0	0	0	0	0	0	c_5	0	c_6	0	0	0	0	0
$(9,0)$	0	0	0	0	c_7	c_6	0	0	0	0	0	0	0	0	0
$(4,6)$	0	0	c_8	0	0	0	0	0	0	0	0	0	0	0	c_3
$(8,2)$	0	0	0	0	0	0	c_6	0	0	0	0	c_3	0	0	0

Figure 10: Edge-Coloring Incidence Matrix of a BIBD$(10,4,2)$

$$A_i^{(1)} = \begin{array}{|ccccc|}\hline c_{1+2i} & 0 & c_{2+2i} & 0 & 0 \\ c_{2+2i} & 0 & 0 & c_{1+2i} & 0 \\ 0 & c_{2+2i} & c_{1+2i} & 0 & 0 \\ 0 & c_{1+2i} & 0 & 0 & c_{2+2i} \\ 0 & 0 & 0 & c_{2+2i} & c_{1+2i} \\ \hline \end{array} \quad A_i^{(2)} = \begin{array}{|ccccc|}\hline 0 & 0 & 0 & c_{2+2i} & c_{1+2i} \\ c_{1+2i} & 0 & c_{2+2i} & 0 & 0 \\ 0 & c_{2+2i} & 0 & c_{1+2i} & 0 \\ c_{2+2i} & c_{1+2i} & 0 & 0 & 0 \\ 0 & 0 & c_{1+2i} & 0 & c_{2+2i} \\ \hline \end{array}$$

$$A_i^{(3)} = \begin{array}{|ccccc|}\hline c_{2+2i} & 0 & 0 & 0 & c_{1+2i} \\ 0 & 0 & c_{1+2i} & c_{2+2i} & 0 \\ c_{1+2i} & 0 & c_{2+2i} & 0 & 0 \\ 0 & c_{1+2i} & 0 & 0 & c_{2+2i} \\ 0 & c_{2+2i} & 0 & c_{1+2i} & 0 \\ \hline \end{array}$$

Figure 11: The 5×5 matrices $A_i^{(j)}$

3 Main Constructions

A *group divisible design* of index λ and order v, (k,λ)-GDD, is a triple $(\mathcal{V},\mathcal{G},\mathcal{B})$, where $\mathcal{V}$ is a set of order v, $\mathcal{G}$ is a partition of $\mathcal{V}$ into non-empty parts (*groups*), and $\mathcal{B}$ is a family of subsets (*blocks*) of $\mathcal{V}$ that satisfy the following.

1. If $B \in \mathcal{B}$ then $|B| \in K$.
2. Every pair of distinct elements of $\mathcal{V}$ occurs either in exactly λ blocks or a group, but not both.
3. $|\mathcal{G}| > 1$.

A GDD is *uniform* if all groups have the same size. We use exponential notation to denote the type of the GDD. For example, a GDD having u groups of size m would be referred to as a GDD(m^u), and a GDD having b_i groups of size a_i for $i = 1, 2, \ldots, k$ is referred to as a $GDD(a_1^{b_1} a_2^{b_2} \ldots a_k^{b_k})$ (see Lemma 14).

The necessary and sufficient conditions for the existence of a $(4,\lambda)$-GDD(m^u) were found by Zhu in [7].

Theorem 10 (Zhu, [7]) *The necessary and sufficient conditions for the existence of a* $(4,\lambda)$-GDD(m^u) *are*

1. $u \geq 4$,
2. $\lambda(u-1)m \equiv 0 \pmod 3$, *and*
3. $\lambda u(u-1)m^2 \equiv 0 \pmod{12}$,

with exception of $(m,u,\lambda) \in \{(2,4,1),(6,4,1)\}$, *in which case no such GDD exists.*

Non-uniform GDDs have also been studied. In [1], some results on non-uniform GDDS are enumerated. In particular, we find the following results.

Theorem 11 (Ling, Ge, [3]) *A* 4-GDD$(4^u m^1)$ *exists if and only if either* $u = 3$ *and* $m = 4$, *or* $u \geq 6$, $u \equiv 0 \pmod 3$ *and* $m \equiv 1 \pmod 3$ *with* $1 \leq m \leq 2(u-1)$.

$$B_i^{(1)} = \begin{bmatrix} c_{3+2i} & 0 & 0 & 0 & 0 \\ c_{4+2i} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{3+2i} & 0 \\ 0 & c_{3+2i} & 0 & 0 & 0 \\ 0 & c_{4+2i} & 0 & 0 & 0 \\ 0 & 0 & c_{4+2i} & 0 & 0 \\ 0 & 0 & 0 & c_{4+2i} & 0 \\ 0 & 0 & 0 & 0 & c_{4+2i} \end{bmatrix} \quad B_i^{(2)} = \begin{bmatrix} 0 & c_{4+2i} & 0 & 0 & 0 \\ 0 & 0 & c_{3+2i} & 0 & 0 \\ c_{4+2i} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{4+2i} & 0 \\ 0 & 0 & 0 & 0 & c_{3+2i} \\ c_{3+2i} & 0 & 0 & 0 & 0 \\ 0 & c_{3+2i} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{3+2i} & 0 \end{bmatrix}$$

$$B_i^{(3)} = \begin{bmatrix} c_{5+2i} & 0 & 0 & 0 & 0 \\ 0 & c_{5+2i} & 0 & 0 & 0 \\ 0 & 0 & c_{5+2i} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{5+2i} \\ 0 & 0 & c_{6+2i} & 0 & 0 \\ 0 & 0 & 0 & c_{5+2i} & 0 \\ 0 & 0 & 0 & 0 & c_{6+2i} \\ 0 & 0 & 0 & c_{6+2i} & 0 \end{bmatrix} \quad B_i^{(4)} = \begin{bmatrix} c_{6+2i} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{6+2i} & 0 \\ 0 & c_{6+2i} & 0 & 0 & 0 \\ 0 & 0 & c_{6+2i} & 0 & 0 \\ 0 & 0 & c_{5+2i} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{6+2i} \\ c_{5+2i} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{5+2i} & 0 \end{bmatrix}$$

$$B_i^{(5)} = \begin{bmatrix} 0 & 0 & 0 & 0 & c_{7+2i} \\ c_{7+2i} & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{8+2i} & 0 & 0 \\ 0 & 0 & 0 & c_{7+2i} & 0 \\ 0 & 0 & 0 & 0 & c_{8+2i} \\ 0 & 0 & c_{7+2i} & 0 & 0 \\ 0 & c_{8+2i} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{8+2i} & 0 \end{bmatrix} \quad B_i^{(6)} = \begin{bmatrix} 0 & 0 & c_{8+2i} & 0 & 0 \\ 0 & c_{8+2i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{7+2i} \\ 0 & 0 & 0 & 0 & c_{8+2i} \\ 0 & 0 & 0 & c_{7+2i} & 0 \\ c_{8+2i} & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{7+2i} & 0 & 0 \\ 0 & c_{7+2i} & 0 & 0 & 0 \end{bmatrix}$$

Figure 12: The 8×5 matrices $B_i^{(j)}$

Theorem 12 (Zhu, Ge, Rees, [4]) *A* 4-GDD$(1^u m^1)$ *exists if and only if* $u \geq 2m+1$ *and either* $m,\ u+m \equiv 1$ *or* $4 \pmod{12}$ *or* $m, u+m \equiv 7$ *or* $10 \pmod{12}$.

Let $\mathcal{B}$ be a set of blocks in a GDD. A *parallel class* is a collection of blocks that partition the point-set of the design. A GDD is called *resolvable* if the blocks of the design can be partitioned into parallel classes. A resolvable GDD is denoted by RGDD.

Theorem 13 (Ling, Ge, [3]) *The necessary conditions for the existence of a* 4-RGDD *of type* h^u, *namely,* $u \geq 4$, $hu \equiv 0 \pmod 4$ *and* $h(u-1) \equiv 0 \pmod 3$, *are also sufficient except for* $(h,u) \in \{(2,4),(2,10),(3,4),(6,4)\}$ *and possibly excepting:* $h = 2$ *and* $u \in \{34,46,52,70,82,94,100,118,130,142,178,184,202,214,238,250,334,346\}$*;* $h = 10$ *and* $u \in \{4,34,52,94\}$*;* $h \in [14,454] \cup \{478,502,514,526,614,626,686\}$ *and* $u \in \{10,70,82\}$*;* $h = 6$ *and* $u \in \{6,54,68\}$*;* $h = 18$ *and* $u \in \{18,38,62\}$*;* $h = 9$ *and* $u = 44$*;* $h = 12$ *and* $u = 27$*;* $h = 24$ *and* $u = 23$*; and* $h = 36$ *and* $u \in \{11,14,15,18,23\}$.

We can use 4-GDDs and 4-RGDDs to build our BIBD$(v,4,\lambda)$s in a way that will allow us to properly color the edges. We now give some recursive constructions which are based on this idea.

Lemma 14 *If there exists a* 4-GDD$(a_1^{b_1} a_2^{b_2} \ldots a_x^{b_x})$, *and a properly colored* BIBD $(a_i,4,\lambda)$ *where* $\lambda \geq 6$ *for all* $i = 1,2,\ldots,x$, *then there exists a properly colored* BIBD $\left(\sum_{i=1}^{x} a_i b_i, 4, \lambda\right)$.

$$C_i^{(1)} = \begin{array}{|ccccc|} \hline c_{7+2i} & c_{8+2i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{7+2i} \\ 0 & 0 & c_{8+2i} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \quad C_i^{(2)} = \begin{array}{|ccccc|} \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{5+2i} & 0 & c_{6+2i} \\ c_{6+2i} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & c_{6+2i} & 0 & 0 & 0 \\ \hline \end{array}$$

$$C_i^{(3)} = \begin{array}{|ccccc|} \hline 0 & 0 & 0 & 0 & 0 \\ c_{3+2i} & 0 & 0 & c_{4+2i} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{3+2i} \\ 0 & c_{3+2i} & 0 & 0 & 0 \\ \hline \end{array}$$

Figure 13: The 5×5 matrices $C_i^{(j)}$

$$M_i = \begin{array}{|c|c|c|} \hline A_i^{(1)} & 0 & 0 \\ \hline 0 & 0 & A_i^{(2)} \\ \hline 0 & A_i^{(3)} & 0 \\ \hline B_i^{(1)} & B_i^{(2)} & 0 \\ \hline B_i^{(3)} & 0 & B_i^{(4)} \\ \hline 0 & B_i^{(5)} & B_i^{(6)} \\ \hline C_i^{(1)} & C_i^{(2)} & C_i^{(3)} \\ \hline \end{array}$$

Figure 14: Representation of an Edge-Colored BIBD$(10, 4, 2)$

Proof. Repeat each of the blocks in the 4-GDD$(a_1^{b_1} a_2^{b_2} \ldots a_x^{b_x})$ λ times. For each block, we must color each edge a different color, using the colors $c_i \in \{c_1, \ldots, c_\lambda\}$. Each corresponding edge in the λ copies of the blocks must also be a different color. So we color the edges in the λ copies of each block as follows. Form a $6 \times \lambda$ matrix. The rows of the matrix will be indexed by the 6 edges of K_4, and the columns will be indexed by the λ copies of the block. The entries of the matrix will be the first 6 rows of an LS(λ). Now the only pairs that have not been covered are the pairs which lie within the groups. So on each group, we place a properly colored BIBD$(a_i, 4, \lambda)$ for each $i = 1, 2, \ldots, x$. This forms a properly colored BIBD $\left(\sum_{i=1}^{x} a_i b_i, 4, \lambda\right)$. □

Lemma 15 *If there exists a* 4-GDD(m^u) *and a properly colored* BIBD$(m+1, 4, \lambda)$*, where* $\lambda \geq 6$*, then there exists a properly colored* BIBD$(mu+1, 4, \lambda)$*.*

Proof. For $i = 1, \ldots, u$, let G_i denote the i^{th} group of size m in the 4-GDD(m^u). Repeat each of the blocks in the 4-GDD(m^u) λ times. For each block, we must color each edge a different color, using the colors $c_i \in \{c_1, \ldots, c_\lambda\}$. Each corresponding edge in the λ copies of the block must also be a different color. So we color the edges in the λ copies of each block as in the previous proof. Form a $6 \times \lambda$ matrix. The rows of the matrix will be indexed by the 6 edges of K_4, and the columns will be indexed by the λ copies of the block. The entries of the matrix will be the first 6 rows of an LS(λ). Now the only pairs that have not been covered are the pairs which lie within the groups and pairs which contain the point $\{\infty\}$. So we place a properly colored BIBD$(m+1, 4, \lambda)$ on each $G_i \cup \{\infty\}$ for all $i = 1, 2, \ldots, u$. This forms a properly colored BIBD $(mu+1, 4, \lambda)$. □

Lemma 16 *For $\lambda \geq 6$, if there exists a* 4-RGDD(m^u)*, a properly colored* BIBD $(5, 4, \lambda)$*, a properly colored* BIBD$(m, 4, \lambda)$*, and a properly colored* BIBD$(t, 4, \lambda)$ *for some $t \leq \dfrac{m(u-1)}{3}$, then there exists a properly colored* BIBD$(mu+t, 4, \lambda)$.

Proof. Let P_i for $i = 1, \ldots, \dfrac{m(u-1)}{3}$ denote the parallel classes in the 4-RGDD(m^u). Also let $\{\infty_1, \infty_2, \ldots, \infty_t\}$ be t new points where $0 \leq t \leq \dfrac{m(u-1)}{3}$. Consider each parallel class P_i for $i = 1, \ldots, t$. Place a properly colored BIBD$(5, 4, \lambda)$ on each block of $P_i \cup \{\infty_i\}$. Repeat each block in P_i for $i = t+1, \ldots, \dfrac{m(u-1)}{3}$ λ times. For each of these blocks, we must color each edge of each block a different color, using the colors $c_i \in \{c_1, \ldots, c_\lambda\}$. Each corresponding edge in the λ copies of the blocks must also be a different color. So we color the edges in the λ copies of each block as follows. Form a $6 \times \lambda$ matrix. The rows of the matrix will be indexed by the 6 edges of K_4, and the columns will be indexed by the λ copies of the block. The entries of the matrix will be the first 6 rows of a LS(λ). Now the only pairs that have not been covered are the pairs which lie within the groups and the pairs of the form $\{\{\infty_i, \infty_j\} : i, j \in \{1, \ldots, t\}\}$. So we place a properly colored BIBD$(m, 4, \lambda)$ on each group, G_i, for all $i = 1, 2, \ldots, u$ and we place a properly colored BIBD$(t, 4, \lambda)$ on the set of points $\{\infty_1, \ldots, \infty_t\}$. This forms a properly colored BIBD$(mu+t, 4, \lambda)$. □

The following theorem illustrates the use of these lemmas.

Theorem 17 *There exists a properly colored* BIBD$(v, 4, \lambda)$ *for* $v \equiv 0 \pmod{12}$ *where* $\lambda = 3k$, $k \geq 2$.

Proof. We can properly edge color all BIBD$(12, 4, \lambda)s$ for $\lambda = 3k$, $k \geq 2$ by Lemma 6. Let $v = 24$. By Theorem 13 a $4 -$ RGDD(5^4) exists with 5 parallel classes, and Lemma 3 allows us to properly color a BIBD$(5, 4, \lambda)$ for $\lambda = 3k$, $k \geq 2$. Therefore we can apply Lemma 16 with $m = 5$, $u = 4$, and $t = 4$ to obtain a properly colored BIBD$(24, 4, \lambda)$ for $\lambda = 3k$ $k \geq 2$.

Let $v = 36$. By Theorem 10, there exists a $4 -$ GDD(9^4). From Lemma 5 we have a properly colored BIBD$(9, 4, \lambda)$ for $\lambda = 3k$, $k \geq 2$. Hence, we apply Lemma 14 with $x = 1$, $a_1 = 9$, and $b_1 = 4$ to properly color a BIBD$(36, 4, \lambda)$ for $\lambda = 3k$, $k \geq 2$.

Now suppose $v = 12u$ where $u \geq 4$. There exists a 4-GDD(12^u) for $u \geq 4$ by Theorem 10. By Lemma 6, we can properly color a BIBD$(12, 4, \lambda)$ for $\lambda = 3k$, $k \geq 2$.

Thus, we can let $x = 1$, $a_1 = 12$, $b_1 = u$, so it follows by Theorem 14 that we can properly color a BIBD$(v, 4, \lambda)$ for $\lambda = 3k$, $k \geq 2$. □

4 $\lambda \equiv 3 \pmod 6$

In this section, we properly color all BIBD$(v, 4, \lambda)$s where $\lambda \equiv 3 \pmod 6$. In this case, the necessary and sufficient conditions for the existence of a BIBD$(v, 4, \lambda)$ are that $v \equiv 0, 1 \pmod 4$. Note that when $v \equiv 0, 1 \pmod 4$ and $\lambda \equiv 0 \pmod 6$ these are already covered by Theorem 2, but the results in this section will also cover this case.

Theorem 18 *There exists a proper coloring for every* BIBD$(v, 4, \lambda)$ *for* $\lambda = 3k$*,* $k \geq 2$*, where* $v \equiv 0, 1 \pmod 4$.

Proof. We break this problem up into two main cases, $v \equiv 1 \pmod 4$, and $v \equiv 0 \pmod 4$.

Case 1: $v \equiv 1 \pmod 4$

We consider three subcases $\pmod{12}$.

Case 1.1: $v \equiv 1 \pmod{12}$

By Theorem 1, we can properly color a BIBD$(v, 4, \lambda)$ where $v \equiv 1, 4 \pmod{12}$, $\lambda \geq 6 = \binom{4}{2}$.

Case 1.2: $v \equiv 5 \pmod{12}$

Let $v \equiv 5 (\text{mod } 12)$. So $v = 5 + 12x = 1 + 4(1 + 3x)$ where $x \in \mathbb{Z}^+$. We construct a 4-GDD(4^u) where $u = 1 + 3x$ and $x \geq 1$. This exists by Theorem 10. We also have that a properly colored BIBD$(5, 4, \lambda)$ exists for $\lambda = 3k$, $k \geq 2$ by Lemma 3. So we apply Lemma 15 with $m = 4$.

Case 1.3: $v \equiv 9 \pmod{12}$

If $v \equiv 9 \pmod{12}$, then we have that either $v \equiv 9 \pmod{24}$ or $v \equiv 21 \pmod{24}$.

Case 1.3.1: $v \equiv 9 \pmod{24}$

If $v = 9$, these are covered by Lemma 5.

For $v > 9$, let $v = 24x + 9 = 8(3x + 1) + 1$, $x \geq 1$. Let $u = 3x + 1$, so that $v = 8u + 1$. Theorem 10 says that there exists a $4 - \text{GDD}(8^u)$. We also have by Lemma 5 that there exists a properly colored BIBD$(9, 4, \lambda)$ for $\lambda = 3k$, $k \geq 2$. So apply Lemma 15 with $m = 8$.

Case 1.3.2: $v \equiv 21 \pmod{24}$

If $v \equiv 21 \pmod{24}$, then we can write v as $v \equiv 21 \pmod{48}$ or $v \equiv 45 \pmod{48}$.

Case 1.3.2.1: $v \equiv 45 \pmod{48}$

Suppose $v = 48x + 45 = 4(12x + 11) + 1$. Now let $m = 12x + 11, x \geq 0$, so that $v = 4m + 1$. We can construct a 4-GDD(m^4) for $m \equiv 11 \pmod{12}$ by Theorem 10; and by Theorem 17, we have that we can properly color a BIBD$(m + 1, 4, \lambda)$. So apply Lemma 15 to obtain a properly colored BIBD$(4m + 1, 4, \lambda)$.

Case 1.3.2.2: $v \equiv 21 \pmod{48}$

If $v = 21$, then we can write v as $v = 4(5) + 1$. We can construct a 4-RGDD(5^4) by Theorem 13. This has 5 parallel classes. Let $\{\infty\}$ be a new point, and let P_i denote the i^{th} parallel classes. We take each block of P_1 and join it with $\{\infty\}$. Place a properly colored BIBD$(5, 4, \lambda)$ design on each block of $P_1 \cup \{\infty\}$. Now repeat each block in P_i λ times for $i = 2, 3, 4, 5$. We must color each edge of each block a different color, using the colors $c_i \in \{c_1, \dots, c_\lambda\}$. Each corresponding edge in the λ copies of the block must also be a different color. So color the edges in the λ copies of each block as follows. Form a $6 \times \lambda$ matrix. The rows of the matrix will be indexed by the 6 edges of K_4, and the columns will be indexed by the λ copies of the block. The entries of the matrix will be the first 6 rows of a LS(λ). Now the only pairs that have not been covered are the pairs which lie within the groups. So we place a properly colored BIBD$(5, 4, \lambda)$ on each group G_i for $i = 1, 2, 3, 4$. This forms a properly colored 2-$(21, 4, \lambda)$ design.

Now suppose $v = 21 + 48x$ for $x \geq 1$. Since $v = 21 + 48x = 4(12x + 4) + 5$, we can construct a 4-RGDD(m^u) with $m = 12x + 5$ and $u = 4$, by Theorem 13. We also have that a properly colored BIBD$(5, 4, \lambda)$ design exists by Lemma 3, and a properly colored BIBD$(12x + 4, 4, \lambda)$ exists by Theorem 1. Since $5 \leq 12x + 4$ for all $x > 0$, we can apply Lemma 16.

Therefore, we can properly color a BIBD$(v, 4, \lambda)$ for $v \equiv 1 \pmod{4}$ and $\lambda = 3k$, $k \geq 2$.

Case 2: $v \equiv 0 \pmod{4}$

We consider three subcases $\pmod{12}$.

Case 2.1: $v \equiv 0 \pmod{12}$

We can properly color a BIBD$(v, 4, \lambda)$ for $v \equiv 0 \pmod{12}$ and $\lambda = 3k$ $k \geq 2$ by Theorem 17.

Case 2.2: $v \equiv 4 \pmod{12}$

By Theorem 1, we can properly color a BIBD$(v, 4, \lambda)$ for $v \equiv 4 \pmod{12}$ and $\lambda = 3k$ $k \geq 2$.

Case 2.3: $v \equiv 8 \pmod{12}$

If $v \equiv 8 \pmod{12}$, then we can write v as $v \equiv 8 \pmod{24}$ or $v \equiv 20 \pmod{24}$.

Case 2.3.1: $v \equiv 8 \pmod{24}$

We can properly color a BIBD$(8, 4, \lambda)$ for all $\lambda = 3k$, $k \geq 2$ by Lemma 4. There exists a $4 - \mathrm{GDD}(8^u)$ for $u = 3x + 1$ and $x \geq 1$ by Theorem 10. Therefore, we use Lemma 14 with $x = 1$, $a_1 = 8$, and $b_1 = u$.

Case 2.3.2: $v \equiv 20 \pmod{24}$

When $v \equiv 20 \pmod{24}$, we break this case into two subcases, $v \equiv 20 \pmod{48}$ and $v \equiv 44 \pmod{48}$.

Case 2.3.2.1: $v \equiv 20 \pmod{48}$

Let $v = 48x + 20 = 4(12x + 5)$ for $x \geq 0$. There exists a 4-GDD(m^4) where $m = 12x + 5$ by Theorem 10. We properly color all BIBD$(m, 4, \lambda)$s for each $\lambda = 3k$, $k \geq 2$ in Case 1.2 and Lemma 3. Thus we can apply Lemma 14 with $x = 1$, $a_1 = m$, and $b_1 = 4$.

Case 2.3.2.2: $v \equiv 44 \pmod{48}$

Let $v = 44 + 48x = 4(12x + 9) + 8$ for $x \geq 0$. There exists a 4-RGDD$((12x + 9)^4)$ by Theorem 13. This has $12x + 9$ parallel classes. We can properly color each BIBD$(v, 4, \lambda)$ for $v \equiv 9 \pmod{12}$ by Case 1.3, and we can properly color a $2 - (5, 4, \lambda)$ design using Lemma 3. Also, we can properly color a BIBD$(8, 4, \lambda)$ by Lemma 4. So we apply Lemma 16 with $m = 12x + 9$, $u = 4$, and $t = 8$. □

5 $\lambda \equiv 2, 4 \pmod{6}$

In this section, we properly color all BIBD$(v, 4, \lambda)$s where $\lambda \equiv 2$ or $4 \pmod{6}$. In this case, the necessary and sufficient conditions for the existence of a BIBD$(v, 4, \lambda)$ are that $v \equiv 1 \pmod{3}$. Note that when $v \equiv 1 \pmod{3}$ and $\lambda \equiv 0 \pmod{6}$, we could also use Theorem 2.

Theorem 19 *There exists a proper coloring for every* BIBD$(v, 4, \lambda)$ *design for* $\lambda = 2k$, $k \geq 3$, *where* $v \equiv 1 \pmod{3}$.

Proof. We consider three cases $\pmod{12}$.

Case 1: $v \equiv 1, 4 \pmod{12}$

By Theorem 1, we can properly color a BIBD$(v, 4, \lambda)$ for $v \equiv 1, 4 \pmod{12}$ and $\lambda = 2k, k \geq 3$.

Case 2: $v \equiv 7 \pmod{12}$

We can properly color a $\mathrm{BIBD}(7,4,\lambda)$ for $\lambda = 2k, k \geq 3$ by Lemma 7. We can also properly color a $\mathrm{BIBD}(19,4,\lambda)$ for all such λ by Lemma 8. Let $v = 12x + 7 = 6(2x+1)+1$. By Theorem 10, there exists a $4-\mathrm{GDD}(6^{2x+1})$ for all $x > 1$. So we can apply Lemma 15 with $m = 6$ and $u = 2x + 1$.

Case 3: $v \equiv 10 \pmod{12}$

If $v = 10$, we can properly color a $\mathrm{BIBD}(10,4,\lambda)$ with $\lambda = 6$ by Theorem 2. Then for all $\lambda = 2k$, $k \geq 4$, we apply Lemma 9.

If $v = 22$, we can apply Lemma 14 with $x = 2$, $a_1 = 1$, $b_1 = 15$, $a_2 = 7$, and $b_2 = 1$. Note that the required $4-\mathrm{GDD}(1^{15}7^1)$ exists by Theorem 12, and a properly colored $\mathrm{BIBD}(7,4,\lambda)$ exists by Lemma 7.

Now let $v = 12x + 10$, with $x \geq 2$. There exists a 4-GDD$(4^u m^1)$ where $m = 10$ and $u = 3x$ for all $x \geq 2$ by Theorem 11. So we let $x = 2$, $a_1 = 4$, $b_1 = u$, $a_2 = m$, and $b_2 = 1$; and we apply Lemma 14. The required properly colored $\mathrm{BIBD}(10,4,\lambda)$ exists by Lemma 9. □

6 Conclusion

We are now in a position to prove the main theorem.

Theorem 20 *There is a proper edge coloring for every* $\mathrm{BIBD}(v,4,\lambda)$, $\lambda \geq 6$.

Proof. Recall the necessary and sufficient conditions for the existence of a BIBD $(v,4,\lambda)$.

If $\lambda \equiv 1,5 \pmod 6$, then $v \equiv 1,4 \pmod{12}$;

if $\lambda \equiv 2,4 \pmod 6$, then $v \equiv 1 \pmod 3$;

if $\lambda \equiv 3 \pmod 6$, then $v \equiv 0,1 \pmod 4$; and

if $\lambda \equiv 0 \pmod 6$, then $v \geq 4$.

If $\lambda \equiv 1,5 \pmod 6$, then $v \equiv 1,4 \pmod{12}$ and we can properly color a BIBD $(v,4,\lambda)$ by applying Theorem 1. If $\lambda \equiv 0 \pmod 6$, then $v \geq 4$ and we can properly color a $\mathrm{BIBD}(v,4,\lambda)$ by applying Theorem 2. If $\lambda \equiv 3 \pmod 6$, then $v \equiv 0,1 \pmod 4$ and we can apply Theorem 18. Finally, if $\lambda \equiv 2,4 \pmod 6$, then $v \equiv 1 \pmod 3$ and we apply Theorem 19. Thus there is a proper edge coloring for every $\mathrm{BIBD}(v,4,\lambda)$, $\lambda \geq 6$. □

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