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Chromatic numbers of 6-regular graphs on the Klein bottle

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Abstract

In this paper, we determine chromatic numbers of all 6-regular loopless graphs on the Klein bottle. As a consequence, it follows that every simple 6-regular graph on the Klein bottle is 5-colorable.

1 Introduction

We consider only loopless graphs on a closed surface F^2. We call this an *embedding* on F^2. We always suppose that an embedding is *topologically simple*, that is, every cycle of length 2 does not bound a 2-cell on F^2.

For a graph G, let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. For $S \subset V(G)$, let $\langle S \rangle$ denote the subgraph of G induced by S. A graph G is said to be k*-regular* if every vertex of G has degree k. A *triangulation* on a closed surface F^2 is a fixed embedding of a graph on F^2 such that each face is triangular. We can see that every 6-regular graph on the Klein bottle is a triangulation by an easy computation of Euler's formula.

Let G be a graph. A map $c : V(G) \to \{1, 2, \dots, k\}$ is called a k*-coloring* of G (or a *coloring* of G) if for any $xy \in E(G)$, $c(x) \neq c(y)$. We say that G is k*-colorable* if G admits a k-coloring. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest integer k such that G is k-colorable. We say that G is k*-chromatic* if $\chi(G) = k$.

In general, the following is known for all closed surfaces F^2.

Theorem 1 (Ringle [7]) *Let F^2 be a closed surface and let G be a graph embeddable in F^2. Then*

$$\chi(G) \leq \left\lfloor \frac{7 + \sqrt{49 - 24\epsilon(F^2)}}{2} \right\rfloor,$$

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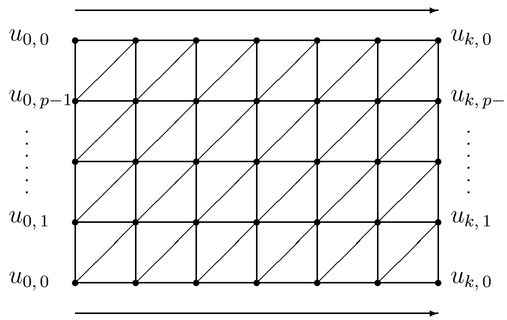

Figure 1: $H_{p,k}$

where $\epsilon(F^2)$ stands for the Euler characteristic of F^2. Moreover, this is best possible except for the Klein bottle. If G is embeddable in the Klein bottle, then $\chi(G) \leq 6$.

In this paper, we focus on 6-regular triangulations, which appear only on the torus and the Klein bottle, and consider the chromatic number of them.

Let p and k be natural numbers. Let $R_{p,k}$ be a $p \times k$ grid graph with additional edges from bottom left to top right. Let $H_{p,k}$ be the embedding on the annulus obtained from $R_{p,k}$ by identifying the top and the bottom, as shown in Figure 1. For $u_{i,j} \in V(H_{p,k})$, we take i modulo p. Let C_j be the cycle of $H_{p,k}$ passing through $u_{j,0}, u_{j,1}, \ldots, u_{j,p}$ in this order, for $j = 0, 1, \ldots, k$. We call these the *geodesic cycles* in $H_{p,k}$. Since $H_{p,k}$ has no loop, we have $p \geq 2$. The vertices on C_0 and C_k have degree 4 and others 6 in $H_{p,k}$.

Let q be a natural number. Let $G[p \times k, q]$ be the 6-regular graph on the torus obtained from $H_{p,k}$ by identifying $u_{0,j}$ with $u_{k,j+q-1}$ for each j. Note that the edges in C_0 and C_k are identified in the natural way, and that $G[p \times k, q]$ has exactly $p \times k$ vertices.

Theorem 2 (Altshuler [1]) *Every 6-regular torus graph is isomorphic to $G[p \times k, q]$ for some integers $p \geq 3$, $k \geq 3$, and $q \geq 1$.*

Yeh and Zhu [8] determined the 4-colorable 6-regular graphs on the torus, using the above classification of the 6-regular torus graphs, as follows.

Theorem 3 (Yeh and Zhu [8]) *Every 6-regular torus graph is 4-colorable, with the following exceptions:*

- $G \in \{G[3 \times 3, 2], G[3 \times 3, 3], G[5 \times 3, 2], G[5 \times 3, 3], G[5 \times 5, 3], G[5 \times 5, 4]\}$.
- $G = G[p \times 2, 1]$ *with p odd.*
- $G = G[p \times 1, r + 2]$ *such that $p = 2r + 2$, $2r + 3$, $3r + 1$ or $3r + 2$ and p is not divisible by 4.*

- $G = G[p \times 1, 4]$ *such that* p *is not divisible by* 4.
- $G = G[p \times 1, r + 2]$ *with* $(r, p) \in \{(3, 13), (3, 17), (3, 18), (3, 25), (4, 17), (6, 17), (6, 25), (6, 33), (7, 19), (7, 25), (7, 26), (9, 25), (10, 25), (10, 26), (10, 37), (14, 33)\}$.

In this paper, we shall determine the chromatic numbers of the 6-regular graphs on the Klein bottle. In the toroidal case mentioned above, they considered only simple graphs, but we deal with loopless 6-regular graphs on the Klein bottle, since some 6-regular graphs with multiple edges are worth considering.

The 6-regular graphs on the Klein bottle were classified by Negami [6], as follows: Recall the embedding $H_{p,k}$ on the annulus. A 6-regular graph on the Klein bottle can be obtained from $H_{p,k}$ by identifying each $u_{0,j}$ with $u_{k,-j}$ for each j. This graph is called a 6-regular Klein bottle graph of *handle type* and is denoted by $Kh(p, k)$. (Note that in order to get a toroidal 6-regular graph from $H_{p,k}$, we need a "twist" for q when C_0 and C_k are identified, but we do not need it in the Klein bottle case.)

Identify $u_{0,j}$ with $u_{0,j+p/2}$ and $u_{k,j}$ with $u_{k,j+p/2}$ in $H_{p,k}$ for each j, respectively, when p is even, then we can obtain a 6-regular graph on the Klein bottle such that two cycles of length $p/2$ arise from C_0 and C_k. Suppose that $p = 2m + 1$. Add a crosscap to each boundary component of $H_{p,k-1}$, and for each j, join $u_{0,j}$ to $u_{0,j+m}$ and $u_{0,j+m+1}$, and $u_{k-1,j}$ to $u_{k-1,j+m}$ and $u_{k-1,j+m+1}$ on the crosscaps added. The resulting graph is called a 6-regular Klein bottle graph of *crosscap type* and is denoted by $Kc(p, k)$, although the constructions of $Kc(p, k)$ are slightly different, depending on the parity of k.

By the following theorem, the 6-regular Klein bottle graphs can be classified into two types with two parameters.

Theorem 4 (Negami [6]) *A loopless* 6*-regular Klein bottle graph is equivalent to precisely one of*

$$Kh(p, k)\ (p \geq 2, k \geq 2), \quad \text{and} \quad Kc(p, k)\ (p \geq 3, k \geq 2).$$

In particular,

- $Kh(p, k)$ *is simple if and only if* $p \geq 3$ *and* $k \geq 3$,
- $Kc(p, k)$ *is simple if and only if* $p \geq 5$ *and* $k \geq 2$.

We shall determine chromatic numbers of the 6-regular Klein bottle graphs according to Theorem 4. The following is our main theorem.

Theorem 5 *Let* G *be a loopless* 6*-regular Klein bottle graph. Then* $\chi(G) = 4$*, with the following exceptions:*

- $G = Kc(3l, k)$ *for* $l \geq 1, k \geq 2$ *if and only if* $\chi(G) = 3$.
- $G = Kc(5, m + 1), Kh(3, 2), Kh(4, 2m + 1), Kh(2, 2m + 3)$ *for* $m \geq 1$ *if and only if* $\chi(G) = 5$.

- $G = Kh(2,3)$ *if and only if* $\chi(G) = 6$.

It is well-known that the complete graph K_7 cannot be embedded on the Klein bottle. Therefore, any 6-regular Klein bottle graph G is 6-colorable by *Brooks's theorem* [2] which states that every simple graph with maximum degree Δ is Δ-colorable unless G is isomorphic to a complete graph or a cycle of odd length. Hence the chromatic number of any 6-regular Klein bottle graph is at least 3 and at most 6.

The following is an immediate consequence of Theorem 5, since a unique 6-chromatic 6-regular graph $Kh(2,3)$ is non-simple and contains K_6 as a subgraph.

Corollary 6 *Every* 6*-regular simple graph on the Klein bottle is* 5*-colorable. Moreover, every* 6*-regular* 6*-chromatic loopless graph on the Klein bottle has* K_6 *as a subgraph.*

A triangulation is said to be *even* if each vertex has even degree. The following is known for even triangulations on the Klein bottle.

Theorem 7 (Král' et al. [4]) *Let G be an even triangulation on the Klein bottle. Then G is* 6*-colorable. In particular, G is* 6*-chromatic if and only if G has* K_6 *as a subgraph.*

The complete graph K_6 plays an important role for the 6-chromaticity of not only 6-regular graphs on the Klein bottle but also of *even* triangulations. However, this does not necessarily hold for all graphs embeddable in the Klein bottle. In [3], it has been shown that the 6-chromatic graphs on the Klein bottle can be characterized by having one of nine graphs as subgraphs. This means that the Klein bottle admits precisely nice 6-critical graphs. (A graph is said to be *k-critical* if G is k-colorable but any proper subgraph of G is not k-colorable.)

2 Proof of the theorem

Let G be a graph and let H be a subgraph of G. Let $c : V(G) \to \{1, 2, \dots, k\}$ be a coloring. Then the coloring $c|_{V(H)} : V(H) \to \{1, 2, \dots, k\}$ is naturally obtained from c. Conversely, we say that a coloring d of H *extends* to a coloring $\tilde{d}$ of G if there exists a coloring $\tilde{d}$ such that $\tilde{d}(v) = d(v)$ for each $v \in V(H)$.

Consider the embedding of $H_{p,k}$ on the annulus which is defined in the previous section. Let $A_{i,j}$ be the connected, maximal subgraph of $H_{p,k}$ bounded by two geodesic cycles C_i and C_j, where $i < j$. The *$(u_{0,j}, u_{k,j})$-path* in $H_{p,k}$ means the path of length k passing through $u_{0,j}$, $u_{1,j}$, $\dots$, $u_{k,j}$ in this order in $H_{p,k}$.

Suppose that an n-coloring c of C_i is given. We say that a coloring c' of C_l is a *cyclic shift* of c if for each j, $c'(u_{l,j}) = c(u_{i,j+m})$ for some m.

Lemma 8 *Let $i \in \{0, 1, \dots, k-2\}$. If $A_{i,i+1}$ has an n-coloring c, then $A_{i,i+2}$ has an n-coloring c' such that $c'|_{V(C_{i+2})}$ is a cyclic shift of $c|_{V(C_i)}$.*

Proof. Let c' be the color-assignment of $A_{i,i+2}$ such that for any $v \in V(A_{i,i+1})$, $c'(v) = c(v)$, and $c'(u_{i+2,j}) = c(u_{i,j-1})$ for each j. Since any two adjacent vertices $u_{i+1,j-1}$ and $u_{i+1,j}$ on C_{i+1} are adjacent to $u_{i,j-1}$ on C_i and $u_{i+2,j}$ on C_{i+2} in common, and since c is an n-coloring of $A_{i,i+1}$, c' is also an n-coloring of $A_{i,i+2}$. By the definition of c', $c'|_{V(C_{i+2})}$ is a cyclic shift of $c|_{V(C_i)}$. ■

Lemma 9 *Let $C_0, C_1, \ldots, C_k$ be the geodesic cycles of $H_{p,k}$.*

(i) Suppose that $H_{p,k}$ has a 3-coloring c. Then for any i, $c(u_{i,j})$, $c(u_{i,j+1})$ and $c(u_{i,j+2})$ on C_i have three distinct colors for each j, and $c|_{V(C_i)}$ is a cyclic shift of $c|_{V(C_0)}$.

(ii) $H_{p,k}$ is 3-colorable if and only if $p \equiv 0 \pmod 3$.

Proof. (i) Suppose that $u_{i,j}$ and $u_{i,j+2}$ on C_i have the same color. Then both $u_{i+1,j+1}$ and $u_{i+1,j+2}$ on C_{i+1} must be colored by the third color, which is different from the two colors of $u_{i,j}$, $u_{i,j+1}$ and $u_{i,j+2}$. However, $u_{i+1,j+1}$ and $u_{i+1,j+2}$ cannot have the same color since they are adjacent on C_{i+1}, a contradiction.

Suppose that C_0 has a 3-coloring such that any consecutive three vertices on C_0 have distinct colors. Since the colorings of C_0 and C_1 must have no collision of colors, and since $u_{1,j}$ on C_1 is adjacent to $u_{0,j}$ and $u_{0,j-1}$ on C_0 for each j, $u_{1,j}$ on C_1 must be colored by the color used to color $u_{0,j+1}$ on C_0 for each j. Hence in $A_{0,1}$, the coloring of C_1 is a cyclic shift of that of C_0. Similarly, in $A_{1,2}$, the coloring of C_2 is a cyclic shift of that of C_1. Thus, (i) holds.

Clearly, (ii) follows from (i). ■

Lemma 10 *(i) There is an n-coloring of $Kh(p,k)$ if and only if there is an n-coloring c of $H_{p,k}$ such that $c(u_{0,j}) = c(u_{k,-j})$ for each j.*

(ii) There is an n-coloring of $Kc(p,k)$ if and only if, when p is even, there is an n-coloring c of $H_{p,k}$ such that $c(u_{0,j}) = c(u_{0,j+p/2})$ and $c(u_{k,j}) = c(u_{k,j+p/2})$ for each j, or, when $p = 2m+1$ is odd, there is an n-coloring c' of $H_{p,k-1}$ such that $c'(u_{0,j})$, $c'(u_{0,j+m})$ and $c'(u_{0,j+m+1})$ are all distinct, and $c'(u_{k-1,j})$, $c'(u_{k-1,j+m})$ and $c'(u_{k-1,j+m+1})$ are all distinct, for each j.

Proof. This is trivial by the constructions of $Kh(p,k)$ and $Kc(p,k)$ from $H_{p,k}$. ■

Lemma 11 *Let G be a 6-regular Klein bottle graph of crosscap type $Kc(p,k)$, where $p = 2m+1 \geq 3$ and $k \geq 1$. If the subgraph $\langle C_0 \rangle$ of G is n-colorable, then so is G.*

Proof. By Lemma 10, it suffices to construct an n-coloring c of $H_{p,k-1}$ such that for $i = 0, k-1$,

$$c(u_{i,j}),\ c(u_{i,j+m}) \text{ and } c(u_{i,j+m+1}) \text{ are all distinct for each } j. \qquad (*)$$

Suppose that an n-coloring of C_0 with the property $(*)$ is given in $H_{p,k-1}$. We can color each $u_{1,j+m+1}$ on C_1 by the color used to color $u_{0,j}$ on C_0 without collision of colors. Therefore $A_{0,1}$ has an n-coloring such that the n-coloring of C_1 is a cyclic shift of that of C_0. By Lemma 8, the n-coloring of $A_{0,1}$ can be extended to that of $H_{p,k-1}$ such that the coloring of C_{i+2} is a cyclic shift of that of C_i for $i = 0, 1, \ldots, k-2$. Thus, the colorings of all C_i are cyclic shifts of that of C_0. Since the property $(*)$ is preserved by cyclic shifts, we get an n-coloring of $H_{p,k-1}$ satisfying $(*)$. ■

The following lemma determines chromatic numbers of the 6-regular Klein bottle graphs of crosscap type.

Lemma 12 *Let G be a 6-regular Klein bottle graph of crosscap type $Kc(p, k)$, where $p \geq 3$ and $k \geq 1$. Then $\chi(G) = 4$, with the following exceptions:*

- *$G = Kc(5, k)$ if and only if $\chi(G) = 5$.*
- *$G = Kc(3l, k)$ for $l \geq 1$ if and only if $\chi(G) = 3$.*

Proof. We consider the following two cases, depending on the parity of p.

Case 1. p is an odd number.

We first consider the case when $p = 5$. Then $\langle C_0 \rangle = K_5$, and hence $\langle C_0 \rangle$ is 5-chromatic. Hence $Kc(5, k)$ is 5-chromatic, by Lemma 11. For other cases, suppose that $p = 2m + 1 \geq 3$. We have $\langle C_0 \rangle = K_3$, or $\langle C_0 \rangle$ is 4-regular but not complete. Therefore, by Brooks's theorem, it is 4-colorable, and hence $Kc(p, k)$ with $p \neq 5$ is 4-colorable, by Lemma 11.

We consider whether G is 3-colorable. If we put $p = 2m+1 \equiv 0 \pmod 3$, then C_0 has a 3-coloring such that $u_{0,j}$, $u_{0,j+m}$ and $u_{0,j+m+1}$ are colored by different colors in $H_{p,k-1}$ for each j, and for $i = 1, \ldots, k-1$, so are $u_{i,j}$, $u_{i,j+m}$ and $u_{i,j+m+1}$ for each j. Then we can get a 3-coloring of $H_{p,k-1}$ such that the coloring of C_i is a cyclic shift of that of C_0 for any i. Hence the 3-coloring of $H_{p,k-1}$ is also a 3-coloring of G.

Case 2. p is an even number.

By Lemma 10, it suffices to construct a coloring c of $H_{p,k}$ such that for any i,

$$c(u_{i,j}) = c(u_{i,j+p/2}) \text{ for each } j. \qquad (**)$$

For doing so, we shall construct a coloring c' of $R_{p/2,k}$ such that for $i = 0, 1, \ldots, k$, $c'(u_{i,0}) = c'(u_{i,p/2})$ (where $R_{p/2,k}$ is the grid graph shown in Figure 1). By pasting two copies of $R_{p/2,k}$ with the same coloring, we can get a coloring of $H_{p,k}$ satisfying $(**)$.

Suppose that $k = 1$. When $p/2 = 2, 3$, $R_{p/2,2}$ can be colored as shown in Figure 2. In particular, if $p/2 = 3$, then it is 3-chromatic. Otherwise, it is 4-chromatic. When $p/2 \geq 4$ and $p/2 \equiv 0 \pmod 3$, we can get a 3-coloring of $R_{p/2,2}$ by pasting several copies of (B). Otherwise, we can get a 4-coloring of $R_{p/2,2}$ by combining (A) and (B) suitably. Therefore if $p \equiv 0 \pmod 3$, then $Kc(p, 2)$ is 3-chromatic. Otherwise, $Kc(p, 2)$ is 4-chromatic.

Suppose that $k \geq 2$. By Lemma 8, a coloring of $H_{p,k}$ is obtained by extending the coloring of $H_{p,1}$. Since for $i \geq 2$, C_i is a cyclic shift of either C_0 or C_1, the coloring of C_k satisfies $(**)$. Thus, we can get a coloring of $H_{p,k}$ satisfying $(**)$. ■

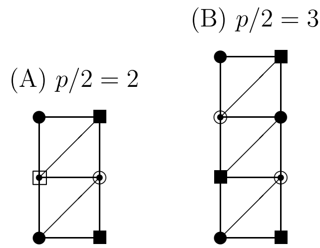

Figure 2: Colorings of two $R_{p/2,\,1}$

We next consider the case for handle type.

Lemma 13 *If a 6-regular Klein bottle graph of handle type $Kh(p, k)$ is n-colorable, then $Kh(p, k + 2m)$ is also n-colorable for any natural number m.*

Proof. It suffices to prove that $Kh(p, k+2)$ is n-chromatic since we can then proceed by induction on m. By Lemma 10, there is an n-coloring c of $H_{p,\,k}$ with the property described. Let $C_{-1}, C_0, C_1, \ldots, C_{k+1}$ $(= C_{-1})$ be the geodesic cycles of $Kh(p, k + 2)$ shown in Figure 3. To get an n-coloring of $Kh(p, k + 2)$, we shall make an n-coloring c' of $H_{p,\,k+2}$ such that $c'(u_{-1,\,j}) = c'(u_{k+1,\,-j})$ for each j, and $c'(v) = c(v)$ for each $v \in V(H_{p,\,k}) = V(C_0 \cup C_1 \cup \cdots \cup C_k)$.

We shall use another representation of $Kh(p, k + 2)$ shown in Figure 4, where $C_{k+2} = C_0$. Then, in this new representation, we have $c(u_{k,\,j}) = c(u_{k+2,\,j})$ for each j. Thus, $Kh(p, k + 2)$ is n-colorable. ■

The following lemma determines the chromatic numbers of the 6-regular Klein bottle graphs of handle type.

Lemma 14 *Let G be a 6-regular Klein bottle graph of handle type $Kh(p, k)$, where $p \geq 2$ and $k \geq 2$. Then $\chi(G) = 4$, with the following exceptions:*

- $G = Kh(2, 3)$ *if and only if* $\chi(G) = 6$
- $G = Kh(3, 2), Kh(2, 2m+3), Kh(4, 2m+1)$ *for* $m \geq 1$ *if and only if* $\chi(G) = 5$.

Proof. Suppose that G has a 3-coloring. By Lemmas 9 and 10, since we must have $p \geq 3$ and the coloring of C_k in the corresponding 3-coloring of $H_{p,\,k}$ is a cyclic shift of that of C_0, we cannot identify C_0 with C_k of $H_{p,\,k}$ incoherently to get a Klein bottle. Hence $G = Kh(p, k)$ is not 3-colorable for any $p \geq 2$ and $k \geq 2$.

We consider the following two cases, depending on the parity of k.

Case 1. k is an even number.

We first consider the case when $p \geq 2$ is even. Observe that a 4-coloring of $Kh(p, 2)$ can be constructed from the 4-coloring of $R_{2,2}$ by pasting them suitably.

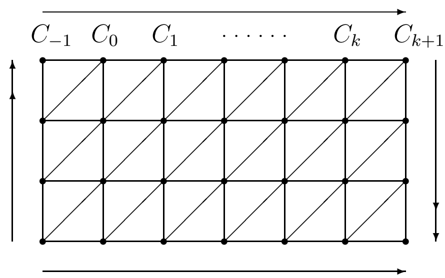

Figure 3: $Kh(p, k+2)$

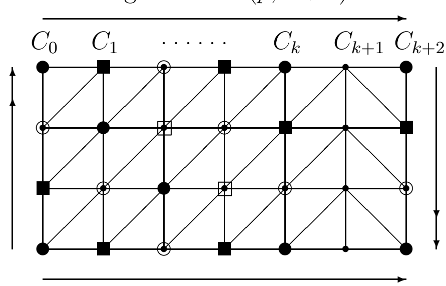

Figure 4: Another representation of $Kh(p, k+2)$

(For example, Figure 5 shows that a 4-coloring of $Kh(4, 2)$ is obtained from two $R_{2,2}$'s.) Hence, $Kh(p, 2)$ with p even is 4-chromatic, since the handle type admits no 3-coloring, as mentioned in the first paragraph. Then we can construct a 4-coloring of all $Kh(p, k)$ for all even integers $p, k \geq 2$, by Lemma 13.

Now consider the case when $p \geq 3$ is odd. Similarly to the previous case, we use Lemma 13, but we have to deal with the case when $p = 3$ and $k = 2$ independently, since $Kh(3, 2)$ exceptionally has chromatic number 5. Observe that $Kh(3, 2)$ is 5-colorable, as shown in the left of Figure 6. Since $Kh(3, 2)$ has K_5 as a subgraph, it is 5-chromatic. However, $Kh(3, 4)$ is 4-colorable, as shown in the middle of Figure 6, and hence $Kh(3, k)$ is 4-chromatic for all even $k \geq 4$, by Lemma 13.

We next consider $Kh(p, 2)$ with $p \geq 5$ odd. As shown in the right of Figure 6, $Kh(5, 2)$ has a 4-coloring. For $p \geq 7$, a 4-coloring of $Kh(p, 2)$ can be constructed, as follows. Regard $Kh(5, 2)$ with the 4-coloring shown in Figure 6 as a strip $R_{5,2}$, and take the strip $R_{p-5,2}$ constructed from $R_{2,2}$'s with the 4-colorings as in Figure 5. Then pasting the two strips to obtain an annulus $H_{p,2}$ with a 4-coloring. The two boundary cycles can be pasted incoherently so that vertices with the same color are identified. Thus we get a 4-coloring of $Kh(p, 2)$ for all even integer $p \geq 5$, and hence, by Lemma 13, $Kh(p, k)$ is 4-chromatic for any odd integer $p \geq 5$ and any even integer $k \geq 2$.

Case 2. k is an odd number.

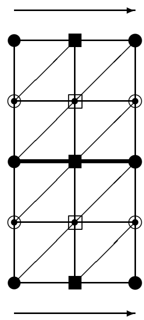

Figure 5: 4-colorings of $H_{4,2}$ obtained from two $R_{2,2}$

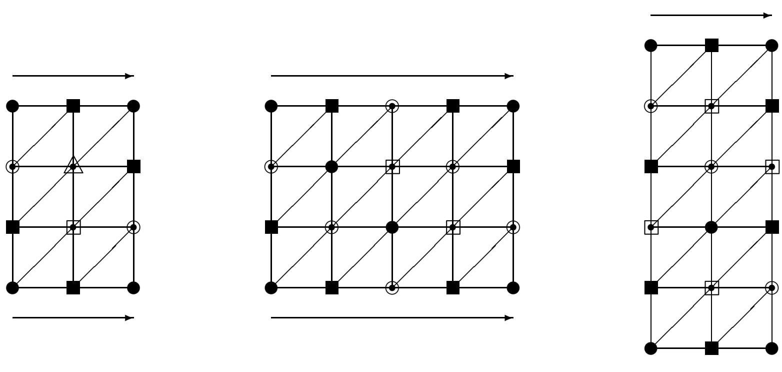

Figure 6: 5-coloring of $H_{3,2}$ and 4-colorings of $H_{3,4}$ and $H_{5,2}$

Suppose that p is odd. By Lemma 13, it suffices to prove that $Kh(p,3)$ is 4-colorable. By Lemma 10, we can construct a 4-coloring of $Kh(3,3)$ from the 4-coloring of $H_{3,3}$ shown in Figure 7. So, suppose that $p = 2m+1 \geq 5$. To get a 4-coloring c' of $H_{p,3}$ such that $c'(u_{0,j}) = c'(u_{3,-j})$ for each j, cut the 4-colored $H_{3,3}$ along the $(u_{0,2}, u_{3,2})$-path and the $(u_{0,1}, u_{3,1})$-path, and replace the center strip bounded by the two paths with the 4-colored $R_{p-2,3}$ bounded by $T_{\frac{p-1}{2}}$ and $B_{\frac{p-1}{2}}$ shown in Figure 8, after re-coloring suitably if necessary. Thus, we can get a 4-coloring of $H_{p,3}$ with the property described, and hence $Kh(p,3)$ with p (≥ 3) odd is 4-colorable.

Now consider the case when $p \geq 2$ is even. Observe that $Kh(2,3)$ is isomorphic to K_6 with 1-factor doubled (the left of Figure 9), and hence $Kh(2,3)$ is 6-chromatic. However, $Kh(2,5)$ is 5-colorable as shown in the right of Figure 9, and so is $Kh(4,3)$ as shown in Figure 10. Hence, by Lemma 13, $Kh(2,2m+3)$ and $Kh(4,2m+1)$ are 5-colorable for any $m \geq 1$. In the following, we shall prove that they are all 5-chromatic.

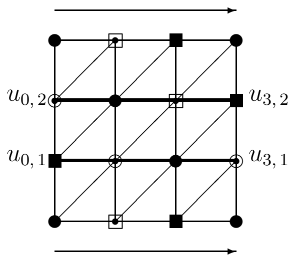

Figure 7: 4-coloring of $H_{3,3}$

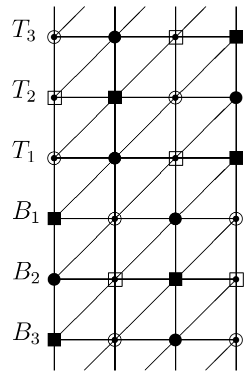

Figure 8: 4-coloring of $R_{2i-1,3}$ bounded by T_i and B_i

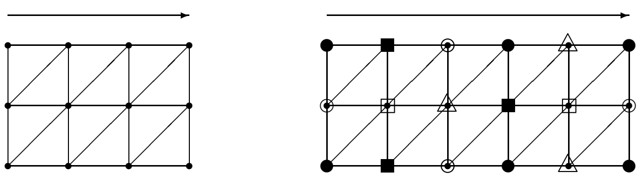

Figure 9: $Kh(2,3) = K_6$ and a 5-colorings of $Kh(2,5)$

For contradictions, we suppose that $Kh(p,k)$ is 4-colorable, where $k \geq 2m+3$ when $p = 2$, and $k \geq 2m+1$ when $p \geq 4$. We consider three cases for 4-colorings of C_0 in $H_{4,k}$ shown in Figure 11, depending on the number of colors appearing on C_0,

If C_0 is 2-colored, then C_1 must be colored by the two colors not appearing on C_0. (The 4-coloring of C_0 in $Kh(2,k)$ must be dealt with here. For $Kh(4,k)$, see Figure 11 (A).) Hence, if C_i is 2-colored, then C_{i+1} must be colored by the other two colors, for each i. Therefore, since k is odd, C_0 and C_k cannot have the same coloring in $H_{4,k}$. Therefore, we cannot identify C_0 and C_k of $H_{4,k}$ to construct the Klein bottle, a contradiction.

If C_0 is colored by exactly three colors, then we can conclude that for any coloring of C_1 by at most four colors, the color not appearing on C_0 is used twice on C_1, and the color appearing twice on C_0 is not used on C_1. (See Figure 11 (B).) Similarly to the above case, since k is odd, C_0 and C_k cannot have the same coloring.

If C_0 is colored by four distinct colors, then the coloring of C_1 must be a cyclic shift of that of C_0. (See Figure 11 (C).) Therefore, the coloring of C_k is obtained from doing so, too. Hence we cannot identify C_0 and C_k of $H_{p,k}$ incoherently. Thus,

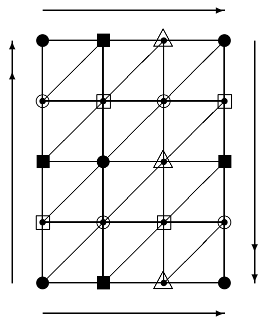

Figure 10: 5-coloring of $Kh(4,3)$

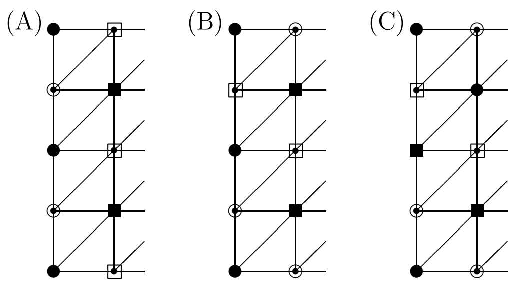

Figure 11: Three cases of coloring of $H_{4,k}$

G is 5-chromatic.

Suppose that p is even and at least six. Then p can be expressed in one of two forms: $p = 2^m$ for some $m \geq 3$ or $p = 2^m(2l+1)$ for some $l \geq 1$ and $m \geq 1$. For the former case, it suffices to check that $Kh(8,3)$ is 4-colorable. (See Figure 12.) For the latter case, $Kh(2l+1,3)$ is 4-colorable as shown above. By pasting $p = 2^m$ copies of 4-colored $H_{2l+1,3}$, we can get a 4-coloring of $H_{2^m(2l+1),3}$, similarly to Case 1.

Thus, by Lemmas 10 and 13, we can get a 4-coloring of $Kh(p,k)$ for any $p \geq 3$ and any odd $k \geq 3$. ■

Theorem 5 follows from Lemmas 12 and 14.

3 Observation

In this section, we consider a relationship between the chromatic number and the "representativity" of 6-regular Klein bottle graphs.

Let F^2 be a closed surface and let ℓ be a simple closed curve. We say that ℓ is *essential* if ℓ does not bound a 2-cell on F^2. (Similarly, we say that a cycle C of a graph G on F^2 is *essential* if C does not bound a 2-cell on F^2.) The *representativity* of an embedding G on a closed surface F^2, denoted by $r(G)$, is defined as the minimum

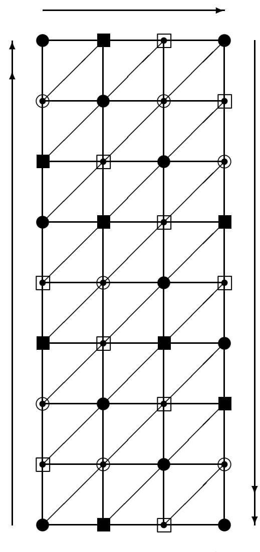

Figure 12: 4-coloring of $Kh(8,3)$

number of intersecting points of G and a closed curve ℓ, where ℓ ranges over all essential closed curves on F^2. In particular, the representativity of a triangulation G coincides with the length of a shortest essential cycle of G.

An *even embedding* is an embedding such that each face is bounded by an even cycle. A *quadrangulation* is an even embedding such that each face is quadrilateral. Let H be an even embedding of a 2-connected graph on a closed surface F^2 with representativity at least 2. Put one vertex into each face of H and join it to all vertices on the boundary. Then the resulting graph G is an even triangulation on F^2. We call G a *face subdivision* of H.

The following is an immediate consequence of Theorem 5, since every non-4-colorable 6-regular graph has an essential cycle of length at most 4.

Corollary 15 *Let G be a 6-regular Klein bottle graph. If G has representativity at least 5, then $\chi(G) \leq 4$.* ■

The following is a result on chromatic numbers of even triangulations with large representativity [5].

Theorem 16 (Nakamoto [5]) *There exists a positive integer N satisfying: Let G be an even triangulation on the Klein bottle with representativity at least N. Then G is 5-chromatic if and only if G is a face subdivision of H, where H is an even embedding including a quadrangulation H' which has an odd cycle cutting open the Klein bottle into an annulus.*

Theorem 16 implies that if an even triangulation G on the Klein bottle with sufficiently large representativity is 5-chromatic, then G is not 5-connected. (Since each quadrilateral face f of H' is subdivided by vertices to get G, the set of four vertices on the boundary cycle of f separates G.) On the other hand, Corollary 15 asserts that every simple 6-regular Klein bottle graph G with $\chi(G) \geq 5$ has representativity at most 4. Since a 6-regular Klein bottle graph can easily be checked to be 5-connected, the value of N in Theorem 16 is at least 5.

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