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Unitals and replaceable t-nests

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Abstract

Let $\overline{U}$ be a known unital of $PG(2,q^2)$, q odd, and let N be a t-nest in a regular spread $\mathcal{S}$. Suppose we replace N by a replacement set $\hat{N}$ to form the new spread $\mathcal{S}' = (\mathcal{S}\backslash N) \cup \hat{N})$. Using the Bruck-Bose correspondence, we look at the affine points of $\overline{U}$ in $\mathcal{P}(\mathcal{S}) \cong PG(2,q^2)$ and see whether the corresponding affine point set in the new plane $\mathcal{P}(\mathcal{S}')$ can be completed to a unital.

1 Introduction

We wish to investigate the effect that t-nest replacement has on unitals of the Desarguesian plane $PG(2,q^2)$ where q is odd. Let $\mathcal{P}(\mathcal{S}) \cong PG(2,q^2)$ be the projective plane corresponding to a regular spread $\mathcal{S}$ via the Bruck-Bose correspondence. Let $\overline{U}$ be a known unital in $\mathcal{P}(\mathcal{S})$.

In [3], [4], [11], [13], [15], [16] the authors investigate the effect of derivation on unitals of $PG(2,q^2)$ and the structure of the corresponding sets in the Hall plane. That is, suppose a regulus is replaced in the spread $\mathcal{S}$ to form a new spread $\mathcal{S}'$; can the corresponding affine point-sets of $\overline{U}$, in the new plane $\mathcal{P}(\mathcal{S}')$, be completed to unitals? It was shown that in some cases it is possible and in others it is not.

In this paper, we will look at a similar problem. Suppose we replace a t-nest in the spread $\mathcal{S}$. That is, if we replace the t-nest N to form a new spread $\mathcal{S}'$, in the new plane $\mathcal{P}(\mathcal{S}')$ can we complete the corresponding affine point-sets of $\overline{U}$ to unitals?

By considering the different known unitals of $PG(2,q^2)$, via the Bruck-Bose correspondence, we divide the analysis into different cases. We summarise the findings in Theorem 4.6. We conclude by remarking on unitals in the new plane $\mathcal{P}(\mathcal{S}')$ that are not inherited from $\mathcal{P}(\mathcal{S})$.

2 Background

A **spread** of $\Sigma_\infty = PG(3,q)$ is a collection of q^2+1 mutually skew lines which necessarily partition the points of Σ_∞. A **regulus** of Σ_∞ is a collection $\mathcal{R}$ of $q+1$ mutually skew lines in Σ_∞ with the property that any line meeting three lines of $\mathcal{R}$ necessarily meets all lines of $\mathcal{R}$. The transversals to a regulus $\mathcal{R}$ form another regulus $\mathcal{R}'$, called the **opposite regulus** to $\mathcal{R}$, where $\mathcal{R}'$ covers the same set of points as $\mathcal{R}$. Any three mutually skew lines of Σ_∞ uniquely determine a regulus containing them, and a spread of Σ_∞ is called **regular** if the regulus determined by any three of its lines is completely contained in the spread.

We will use the Bruck-Bose representation of a translation plane $\mathcal{P}$ of order q^2, with dimension at most 2 over its kernel, in $PG(4,q)$. Let Σ_∞ be a hyperplane of $PG(4,q)$ and let $\mathcal{S}$ be a spread of Σ_∞. We define a new incidence structure $\mathcal{A}(\mathcal{S})$ as follows: the points of $\mathcal{A}(\mathcal{S})$ are the points of $PG(4,q)\backslash\Sigma_\infty$, the lines of $\mathcal{A}(\mathcal{S})$ are the planes of $PG(4,q)$ that meet Σ_∞ in a line of $\mathcal{S}$ and incidence is the natural inclusion. From [7] or [1], it is known that $\mathcal{A}(\mathcal{S})$ is an affine translation plane of order q^2. We complete $\mathcal{A}(\mathcal{S})$ to a projective plane $\mathcal{P}(\mathcal{S})$ of order q^2 by letting the points of the line at infinity l_∞ of $\mathcal{P}(\mathcal{S})$ be the lines of the spread $\mathcal{S}$. The translation plane $\mathcal{P}(\mathcal{S})$ is Desarguesian if and only if $\mathcal{S}$ is regular [8].

A unital in a projective plane $\mathcal{P}$ of order q^2 is a set $\overline{U}$ of q^3+1 points such that every line of the plane meets $\overline{U}$ in 1 or $q+1$ points. A line of $\mathcal{P}$ is a **tangent line** or a **secant line** of $\overline{U}$ if it contains 1 or $q+1$ points of $\overline{U}$ respectively. Each point of $\overline{U}$ lies on 1 tangent and q^2 secant lines of $\overline{U}$. Each point of $\mathcal{P}$ not in $\overline{U}$ lies on $q+1$ tangent lines and q^2-q secant lines of $\overline{U}$. A known unital in $PG(2,q^2)$ is the **classical unital**, which consists of the absolute points and non-absolute lines of a unitary polarity. See [6] for more information about unitals.

Let $\mathcal{S}$ be a regular spread of Σ_∞ and let $\mathcal{S}$ correspond to the line at infinity l_∞ of $\mathcal{P}(\mathcal{S}) \cong PG(2,q^2)$. Let $\overline{U}$ be a classical unital in $PG(2,q^2)$. In [9], it was shown that if $\overline{U}$ is secant to l_∞, then $\overline{U}$ corresponds to a non-singular quadric of $PG(4,q)$ that meets Σ_∞ in a regulus of the spread $\mathcal{S}$. It was also shown that if the classical unital $\overline{U}$ is tangent to l_∞ at the point P_∞ then $\overline{U}$ corresponds to an elliptic cone in $PG(4,q)$ that meets Σ_∞ in the corresponding line p_∞ of the spread $\mathcal{S}$.

Let $\mathcal{S}$ be a spread of Σ_∞, not necessarily regular. Let $\overline{\mathcal{U}}$ be an ovoidal cone of $PG(4,q)$ that meets Σ_∞ in the line p_∞ of $\mathcal{S}$. The vertex of $\overline{\mathcal{U}}$ is a point V which is necessarily on the line p_∞. In [9] it was shown that $\overline{\mathcal{U}}$ corresponds to a unital in $\mathcal{P}(\mathcal{S})$ which is tangent to l_∞ at the point P_∞ which corresponds to the line p_∞ of $\mathcal{S}$. Also, as shown in [9], if $\overline{\mathcal{U}}$ is a non-singular quadric in $PG(4,q)$ that meets Σ_∞ in a regulus of the spread $\mathcal{S}$, then $\overline{\mathcal{U}}$ corresponds to a unital of $\mathcal{P}(\mathcal{S})$ which is secant to l_∞. We call a unital of $\mathcal{P}(\mathcal{S})$ that corresponds to a non-singular quadric in $PG(4,q)$, a **non-singular-Buekenhout unital**. A unital of $\mathcal{P}(\mathcal{S})$ that corresponds to an ovoidal cone in $PG(4,q)$ we call an **ovoidal-Buekenhout-Metz unital**. In [5] it was shown that in $PG(2,q^2)$ every non-singular-Buekenhout unital is classical. In [9] and [14] it was shown that there exist ovoidal-Buekenhout-Metz unitals of $PG(2,q^2)$ that are not classical. All known unitals in $PG(2,q^2)$ are either non-singular-Buekenhout unitals or ovoidal-Buekenhout-Metz unitals [6].

We can use the Bruck-Bose setting to form new translation planes from existing translation planes using the technique of *net replacement.* The method of net replacement, in a spread $\mathcal{S}$, is to replace some subset of the lines of $\mathcal{S}$, say V, by another set V' of mutually skew lines in Σ_∞ which cover exactly the same points as V. As long as the resulting spread $\mathcal{S}' = (\mathcal{S}\backslash V) \cup V'$ is not regular, we have constructed a non-Desarguesian translation plane $\mathcal{P}(\mathcal{S}')$ of order q^2 with kernel $GF(q)$. One example of net replacement is derivation, where V is a regulus in the spread and V' is the opposite regulus to V.

3 t-nest replacement

In [2], a potential form of net replacement is defined, that is, *nest replacement.* Let $\mathcal{S}$ be a regular spread, so $\mathcal{P}(\mathcal{S}) \cong PG(2,q^2)$, and let q be odd. A nest of reguli in the spread $\mathcal{S}$, is defined to be a set N of reguli contained in $\mathcal{S}$ such that every line of $\mathcal{S}$ is contained in precisely 0 or 2 reguli of N. If N contains t reguli then N is called a **t-nest**. For the existence of t-nests and a survey of various results, see [12]. It is known that t-nests exist for $\frac{q+3}{2} \leq t \leq 2(q-1)$ and that no other size is possible.

Denote the reguli of a t-nest N by $R_1,\ldots,R_t$. A t-nest N contains $t\frac{q+1}{2}$ lines of the spread $\mathcal{S}$ and we call these lines, the *lines of* N. The points in $\mathcal{P}(\mathcal{S})$ that correspond to the lines of N will be denoted by P_N.

Let $N = \{R_1,\ldots,R_t\}$ be a t-nest. Then N is called a **half-regulus replaceable** t-nest, if for each regulus R_i, a set $\hat{R}_i$ of $\frac{q+1}{2}$ lines from the opposite regulus R_i' can be chosen such that the lines of $\hat{R}_j$ are disjoint from the lines of $\hat{R}_i$ where $i \neq j$. The line-sets $\hat{R}_i$ will be referred to as *half-reguli.*

Let N be a half-regulus replaceable t-nest. The corresponding set, $\hat{N} = \{\hat{R}_1,\ldots,\hat{R}_t\}$ of half-reguli, is such that the lines of $\hat{R}_1,\ldots,\hat{R}_t$ are pairwise disjoint and cover the same points as the lines of $R_1,\ldots,R_t$. Thus $\hat{N}$ is a replacement set for N. It is known, [17], that for $t \leq q$, all t-nests which have a replacement set are half-regulus replaceable and $\hat{N}$, as described above, is the only possible replacement set.

We will need the following lemma which describes how a regulus of the spread $\mathcal{S}$ meets the lines of a nest.

Lemma 3.1 *A regulus R in the regular spread $\mathcal{S}$ meets a t-nest N of $\mathcal{S}$ in at most t lines of N or in $q+1$ lines of N.*

Proof If $R \in N$, then R meets the lines of N in $q+1$ lines.

So suppose that $R \notin N$. Let $N = \{R_1, R_2, \ldots, R_t\}$. We wish to compute the maximum number of lines in the intersection of R and N.

The process begins as follows, suppose N and R share at least one line l. Then l is covered by two reguli of N, say R_1 and R_2. The regulus R can share at most two lines with any regulus of N otherwise $R \in N$. Hence R_1 can contain at most one further line of R, similarly R_2 can contain at most one further line of R. Thus there are four cases:

Case 1: Suppose R_1 meets R in two lines l, m and R_2 meets R in two lines l, n $(m \neq n)$. Here there exists a regulus of N, say R_3, that covers m since the reguli of N doubly cover m. From here we have two possibilities:

Either, R_3 also covers n and then there are no more lines of R contained in R_1, R_2 or R_3. In that case we have doubly covered three lines in R. We have $t-3$ further reguli $R_4, \ldots R_t$ of N and need to consider whether these meet R. This will be called the maximal case, when the number of reguli of N used is the same as the number of lines of R used. Any case, where we have considered more reguli of N than lines of R, will not give us a maximal intersection between R and the lines of N. We may now start the process again, from the beginning of the proof, using the reguli $R_4, \ldots, R_t$.

Otherwise, there is a another regulus of N, say R_4, that covers n. From here, we have four possibilities,

Case 1a: R_3 and R_4 do not contain any more lines of the intersection of R and N.

This case is not maximal, as we have considered four reguli R_1, R_2, R_3, R_4 of N but only three lines l, m, n of R.

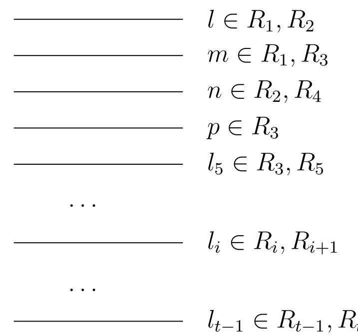
$l \in R_1, R_2$
$m \in R_1, R_3$
$n \in R_2, R_4$

Case 1b: R_3 and R_4 both share one further line $p \in R$.

Then we have no more reguli of N sharing lines of R with R_1, R_2, R_3 and R_4, hence we have $t-4$ reguli of N remaining and have considered four lines l, m, n, p of R. So we may start the entire process again, from the beginning of the proof, using the reguli $R_5, \ldots, R_t$.

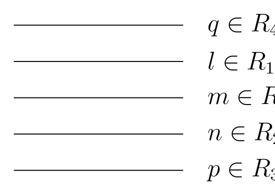
$l \in R_1, R_2$
$m \in R_1, R_3$
$n \in R_2, R_4$
$p \in R_3, R_4$

Case 1c: One of R_3 and R_4, say R_3 has another line p in common with R but $p \notin R_4$.



In this case the line p needs to be doubly covered by reguli of N, so let p also be a line of R_5, say. If R_5 shares no further line with R, then we have considered five reguli to cover four lines of intersection, which gives us a non-maximal case. So assume R_5 shares a further line with R, say l_5. We can continue this case, each time covering the new line l_i shared between some R_i of N and R with a new regulus R_{i+1} of N until we consider all the lines of R or all the reguli of N. In the former case, we have $q+1$ lines of intersection. In the latter case, when $i = t-1$, we have only one more regulus R_t of N left and R_t must doubly cover the extra line shared between R_{t-1} and R. The regulus R_t can not share any more lines with R, otherwise we would not be able to doubly cover this extra line, hence we have considered t reguli but only have $t-1$ lines of intersection. Again, this is a non-maximal case.

Case 1d: R_3 has a line p in common with R and R_4 has a line q in common with R, but $p \notin R_4$ and $q \notin R_3$.



In this case, we can repeat the process as in Case 1 with two new reguli R_5 and R_6. It is possible to continue with this case considering R_i and R_{i+1} each having distinct lines in common with R. We finish when either we run out of reguli in N or we reach a point when there is only one remaining regulus of N. In the former case we have considered more reguli of N than lines of R, hence we have a non-maximal case. In the latter case, the final regulus R_t of N must complete the double covering of the last two reguli R_{t-2} and R_{t-1} as in Case 1b.

Case 2: R_1 and R_2 both meet R in the same two lines l, m.
Here R_1 and R_2 can contain no more lines of R. So we have two lines R and a further $t-2$ reguli of N left to consider. Since we have considered the same number of reguli of N as lines of R, we can start the process again, from the beginning of the proof, with the reguli $R_3, \dots, R_t$.

Case 3: R_1 and R_2 only meet R in the same unique line l.
Here we have considered one line of R and have $t-2$ reguli of N left to consider. This case is not maximal, since we have considered more reguli of N than lines of R.

Case 4: Only one of R_1 and R_2 meets R in a second line m.
We are again in a non-maximal case.

We repeat the above process, assuming R and N share additional lines in common until we have considered all the reguli of N or exhausted all the lines of R. At this point we reach the maximum number of shared lines between R and N. So, from above, the maximal case is that we use one reguli of N for each line of intersection between R and N. So R contains at most t lines from the reguli of N.

If $t > q + 1$, then we can only have a maximum of $q + 1$ lines shared between R and the reguli of N. □

4 Unitals after t-nest replacement

We wish to see what happens to the known unitals in $\mathcal{P}(\mathcal{S}) \cong PG(2, q^2)$ after we perform a t-nest replacement. Given a unital $\overline{U}$ in $\mathcal{P}(\mathcal{S})$, does the corresponding affine point set in the new plane $\mathcal{P}(\mathcal{S}')$ complete to a unital or not?

We work in the Bruck-Bose correspondence of $PG(2, q^2)$ in $PG(4, q)$, where q is odd. So $PG(2, q^2) \cong \mathcal{P}(\mathcal{S})$ where $\mathcal{S}$ is a regular spread of Σ_∞. Let l_∞ be the line at infinity in $\mathcal{P}(\mathcal{S})$. Suppose the spread $\mathcal{S}$ contains a half-regulus replaceable t-nest N. When we replace N with $\hat{N}$ we form the new spread $\mathcal{S}' = (\mathcal{S} \backslash N) \cup \hat{N}$ which corresponds to the plane $\mathcal{P}(\mathcal{S}')$. Let l'_∞ be the line at infinity in $\mathcal{P}(\mathcal{S}')$. We denote the reguli of N by $R_1, \ldots, R_t$ and the corresponding half-reguli of $\hat{N}$ by $\hat{R}_1, \ldots, \hat{R}_t$.

Denote our unital in $\mathcal{P}(\mathcal{S})$ by $\overline{U}$ and the corresponding point-set in $PG(4, q)$ by $\overline{\mathcal{U}}$. Denote the affine point-set of $\overline{U}$ in $\mathcal{P}(\mathcal{S})$ by U and let $\mathcal{U}$ be the corresponding affine point-set in the Bruck-Bose representation of $PG(2, q^2)$ in $PG(4, q)$. Let U' represent the affine point set in $\mathcal{P}(\mathcal{S}')$ that corresponds to U.

We first consider those unitals that correspond to a non-singular quadric in the Bruck-Bose representation of $PG(2, q^2)$ in $PG(4, q)$. We then consider those unitals that correspond to an ovoidal cone in the Bruck-Bose representation in $PG(4, q)$.

Non-singular-Buekenhout unitals

Let $\overline{U}$ be a non-singular-Buekenhout unital in $PG(2, q^2)$. That is, $\overline{U}$ is secant to l_∞ and corresponds to a non-singular quadric $\overline{\mathcal{U}}$ in the Bruck-Bose representation in $PG(4, q)$ such that $\overline{\mathcal{U}}$ meets Σ_∞ in a regulus of the spread $\mathcal{S}$.

Let R be the regulus of S that is contained in $\overline{\mathcal{U}}$. We need to consider three different cases, depending on how the lines of R meet the lines of the reguli of N. The possibilities are: R is a regulus of N, R shares lines with the lines of N but is not a regulus of N or R shares no lines with the lines of N.

Lemma 4.1 *If $\overline{\mathcal{U}}$ meets $\mathcal{S}$ in a regulus of N, then the point set U' in $\mathcal{P}(\mathcal{S}')$ can not be completed to a unital in $\mathcal{P}(\mathcal{S}')$.*

Proof Suppose $\overline{\mathcal{U}}$ meets $\mathcal{S}$ in the regulus R_t of N. Consider, $\hat{N} = \{\hat{R}_1, \ldots, \hat{R}_t\}$, the replacement set of opposite half-reguli to $N = \{R_1, \ldots, R_t\}$. The quadric $\overline{\mathcal{U}}$ contains the $\frac{q+1}{2}$ lines of $\hat{R}_t$. Every line of a half-regulus $\hat{R}_i$, where R_i shares a line with R_t, intersects $\overline{\mathcal{U}}$ in either one or two points.

Let m be a line of $\hat{R}_t$. Let P be a point of $\mathcal{U}$. Now, the plane $\langle P, m\rangle$ meets $\overline{\mathcal{U}}$ in the $q+1$ points of m and the point P. A plane in $PG(4,q)$ intersects a quadric in $1, q+1$ or $2q+1$ points. Hence $\langle P, m\rangle$ meets $\overline{\mathcal{U}}$ in $2q+1$ points. So $\langle P, m\rangle$ meets $\mathcal{U}$ in q points, since we remove the points of m when looking at the affine part of $\overline{\mathcal{U}}$.

Suppose we replace the nest N with $\hat{N}$ to form the new spread $\mathcal{S}'$. The plane $\langle P, m\rangle$ corresponds to a line m_P in $\mathcal{P}(\mathcal{S}')$ and the line m_P is a q-secant of U'. Now, every line of $\mathcal{P}(\mathcal{S}')$ intersects a unital in 1 or $q+1$ points. Hence, to complete U' to a unital of $\mathcal{P}(\mathcal{S}')$, we need to add the point of infinity $m_P \cap l'_\infty$ to U'. This point corresponds to the line l in the spread. Hence, for each line m, there is a corresponding point of l'_∞ that we need to add to U'. There are $\frac{q+1}{2}$ lines in $\hat{R}_t$, so we need to add $\frac{q+1}{2}$ points of l'_∞ to U'.

Now, suppose R_t shares lines with k other reguli of N, say $\{R_1, \ldots, R_k\}$. Let P be a point of $\mathcal{U}$. Let l be a line of some $\hat{R}_i$, $i \in 1, \ldots, k$, so l is a line in an opposite half-regulus to a regulus of N that shares lines with R_t. Denote the particular half-regulus to which l belongs by $\hat{R}_l$. The line l intersects $\overline{\mathcal{U}}$ in one or two points according to whether $\hat{R}_l$ intersects one or two lines of R_t. Thus the plane $\langle P, l\rangle$ meets $\overline{\mathcal{U}}$ in at least two points, P and $l \cap \overline{\mathcal{U}}$.

We note again that, in $PG(4,q)$, a plane meets a quadric in $1, q+1$ or $2q+1$ points. Here, $\langle P, l\rangle$ contains at least two points of $\overline{\mathcal{U}}$, hence $\langle P, l\rangle$ meets $\overline{\mathcal{U}}$ in either $q+1$ or $2q+1$ points.

So suppose $|\langle P, l\rangle \cap \overline{\mathcal{U}}| = 2q+1$. Then $\langle P, l\rangle$ meets $\mathcal{U}$ in $2q$ or $2q-1$ points depending on whether l intersects R_t in one or two points. If we replace the nest N, then $\langle P, l\rangle$ corresponds to a line l_P of $\mathcal{P}(\mathcal{S}')$ which is either a $(2q)$- or $(2q-1)$-secant of U'. This means U' cannot be completed to a unital in $\mathcal{P}(\mathcal{S}')$ as a unital must have every line of $\mathcal{P}(\mathcal{S}')$ intersecting it in 1 or $q+1$ points.

Now, suppose $|\langle P, l\rangle \cap \overline{\mathcal{U}}| = q+1$ and l intersects R_t in two points. We then have that $\langle P, l\rangle$ meets $\mathcal{U}$ in $q-1$ points. If we replace the nest N, the plane $\langle P, l\rangle$ corresponds to a line l_P of $\mathcal{P}(\mathcal{S}')$ that is a $(q-1)$-secant of U'. Adding the point $l_p \cap l'_\infty$ to U' does not give us the required $(q+1)$-secant. Hence U' can not be completed to a unital in $\mathcal{P}(\mathcal{S}')$.

Suppose then that $|\langle P, l\rangle \cap \overline{\mathcal{U}}| = q+1$ and l intersects R_t in one point. We also assume that this is the case for every $l \in R_i, i = 1, \ldots, k$, since we have considered the other possible cases above. That is, every regulus of N that contains a line of R_t, contains exactly one line of R_t. So there are $q+1$ reguli of N, say $R_1, \ldots, R_{q+1}$ that share one line with R_t. The line l belongs to one of the corresponding half-reguli $\hat{R}_1, \ldots, \hat{R}_{q+1}$.

Now, the plane $\langle P, l\rangle$ meets $\mathcal{U}$ in q points. If we replace the nest N then this plane corresponds to a line l_P in $\mathcal{P}(\mathcal{S}')$ which is a q-secant of U'. For U' to be completed to a unital in $\mathcal{P}(\mathcal{S}')$, the line l_P needs to be completed to a $(q+1)$-secant. To do this, we must add the point of infinity $l_p \cap l'_\infty$ to U' and this point corresponds to the line l in the spread. This will be the case for every such $l \in \hat{R}_1, \dots, \hat{R}_{q+1}$. Now, there are $q+1$ half-reguli $\hat{R}_1, \dots, \hat{R}_{q+1}$ whose lines intersect lines of R_t. Each of these half-reguli contains $\frac{q+1}{2}$ lines. Hence we must add $(q+1)\frac{q+1}{2}$ points of l'_∞ to U' to try to complete it to a unital.

From above, we know we have already added at least $\frac{q+1}{2}$ points of l'_∞, hence we must add a total of $\frac{q+1}{2} + (q+1)\frac{q+1}{2}$ points. This means l'_∞ is a $(\frac{q+1}{2} + (q+1)\frac{q+1}{2})$-secant. Since $\frac{q+1}{2} + (q+1)\frac{q+1}{2}$ is greater than $q+1$, the line l'_∞ is not a tangent or a $(q+1)$-secant to U'. Hence, the point-set U' can not be completed to a unital of $\mathcal{P}(\mathcal{S})$. □

Lemma 4.2 *If $\overline{\mathcal{U}}$ meets $\mathcal{S}$ in a regulus R that is not a regulus of N but R contains at least one line of N, then the point set U' can not be completed to a unital after nest replacement.*

Proof We know from Lemma 3.1 that, if R contains at least one line of N, then R contains at most t lines of N or $q+1$ lines of N.

Suppose R contains exactly one line n_1 from the reguli of N. Let R_1 and R_2 be the two reguli of N that doubly cover n_1. Let $\hat{R}_1$ and $\hat{R}_2$ be the corresponding opposite half-reguli in $\hat{N}$ to R_1 and R_2. Now, n_1 intersects $\frac{q+1}{2}$ lines of $\hat{R}_1$ and $\frac{q+1}{2}$ lines of $\hat{R}_2$. So $\overline{\mathcal{U}}$ meets the new spread $\mathcal{S}'$ in $2.\frac{q+1}{2}$ lines plus the q lines of $R\backslash n_1$, which are not in N. This gives a total of $2q+1$ lines of $\mathcal{S}'$ that $\overline{\mathcal{U}}$ meets. Using a similar argument to the proof of Lemma 4.1, we show that we have to add too many points of l'_∞ to U' and hence U' can not be completed to a unital in $\mathcal{P}(\mathcal{S}')$.

Suppose $\overline{\mathcal{U}}$ contains exactly r lines of N, where $1 < r \leq q+1$. Then by a similar argument to that above, we show that $\overline{\mathcal{U}}$ meets at least $\lceil \frac{r}{2} \rceil (q+1)$ lines of $\hat{N}$ and $(q+1) - r$ lines of $\mathcal{S}\backslash N$. Hence, by an argument similar to the proof of Lemma 4.1, we have that U' can not be completed to a unital in $\mathcal{P}(\mathcal{S}')$. □

Lemma 4.3 *If $\overline{\mathcal{U}}$ meets $\mathcal{S}$ in a regulus R and R contains no lines of N, then U' can be completed to a unital in $\mathcal{P}(\mathcal{S}')$.*

Proof Let P_N denote the set of points in $\mathcal{P}(\mathcal{S})$ that correspond to the lines of N. The lines of R, in the spread $\mathcal{S}$, correspond to the points $\overline{U} \cap l_\infty$, in $\mathcal{P}(\mathcal{S})$. Since R contains no lines of N, in $\mathcal{P}(\mathcal{S})$ the points $\overline{U} \cap l_\infty$ are disjoint from the points P_N. If we replace the t-nest N, then the $q+1$ points of $\overline{U} \cap l_\infty$ remain unchanged as $q+1$ points of l'_∞ in $\mathcal{P}(\mathcal{S}')$. In $\mathcal{P}(\mathcal{S}')$ we denote U' together with these $q+1$ points as the set $\overline{U'}$. We denote the corresponding set in $PG(4,q)$, under the Bruck-Bose representation, as $\overline{\mathcal{U}'}$.

We want to show that $\overline{U'}$ is a unital of $\mathcal{P}(\mathcal{S}')$. So we want to show that every line of $\mathcal{P}(\mathcal{S}')$ meets $\overline{U'}$ in either 1 or $q+1$ points. The line l'_∞ meets $\overline{U'}$ in $q+1$ points since l_∞ meets $\overline{U}$ in $q+1$ points.

Let l be a line of $\mathcal{P}(\mathcal{S}')$ that meets l'_∞ in the point P. The point P corresponds to a line p of $\mathcal{S}$. If p is not a line of $\hat{N}$, then the points of l lie on a line of $\mathcal{P}(\mathcal{S})$ and so l contains 1 or $q+1$ points of $\overline{U}$. Hence l contains 1 or $q+1$ points of $\overline{U'}$.

Suppose P does correspond to some line p of $\hat{N}$, then $P \notin \overline{U'}$ since $\overline{U} \cap P_N$ is empty. Now, using the Bruck-Bose correspondence, let α be the plane of $PG(4,q)$ corresponding to the line l, so $\alpha \cap \Sigma_\infty = p$. Now α meets $\overline{\mathcal{U}'}$ in either a point, a line, a conic or two lines.

Suppose $\alpha \cap \overline{\mathcal{U}'}$ contains a line l_α. Since α is a plane through p, l_α meets p in a point. This implies $p \cap \overline{\mathcal{U}'}$ is not empty. In $\mathcal{P}(\mathcal{S}')$, this corresponds to P being a point of $\overline{U'}$, which is a contradiction. Hence α meets $\overline{\mathcal{U}'}$ in either a point or a conic and thus, in $\mathcal{P}(\mathcal{S}')$, l meets $\overline{U'}$ in 1 or $q+1$ points. Hence $\overline{U'}$ is a unital of $\mathcal{P}(\mathcal{S}')$. □

Buekenhout-Metz unitals

Let $\overline{U}$ be an ovoidal-Buekenhout-Metz unital of $PG(2,q^2)$. That is, $\overline{U}$ is tangent to l_∞ and corresponds to an ovoidal cone $\overline{\mathcal{U}}$ in the Bruck-Bose representation in $PG(4,q)$ such that $\overline{\mathcal{U}}$ meets Σ_∞ in exactly one line p_∞ of the spread $\mathcal{S}$.

There are two possibilities for p_∞. One case is that the line p_∞ is one of the lines of N, so p_∞ is contained in two reguli of N. The second case is that p_∞ is disjoint from the lines of N.

Lemma 4.4 *If $\overline{\mathcal{U}}$ meets $\mathcal{S}$ in one line p_∞ and p_∞ is a line of N, then U' can not be completed to a unital of $\mathcal{P}(\mathcal{S}')$.*

Proof As p_∞ is a line of N, it is contained in two reguli of N, say R_1 and R_2. Let $\hat{R}_1$ and $\hat{R}_2$ be the opposite half-reguli in $\hat{N}$ corresponding to R_1 and R_2 respectively. Let l be a line of either $\hat{R}_1$ or $\hat{R}_2$. The line l meets p_∞ in one point. Let P be a point of $\mathcal{U}$. In $PG(4,q)$ a plane meets an ovoidal cone in $1, q+1$ or $2q+1$ points. Here, the plane $\langle P, l\rangle$ meets $\overline{\mathcal{U}}$ in at least two points, hence it meets $\overline{\mathcal{U}}$ in $q+1$ or $2q+1$ points.

If $\langle P, l\rangle$ meets $\overline{\mathcal{U}}$ in $2q+1$ points, then $\langle P, l\rangle$ meets $\mathcal{U}$ in $2q$ points. If we replace the nest N, then the plane $\langle P, l\rangle$ corresponds to a line l_p of $\mathcal{P}(\mathcal{S}')$ which is a $(2q)$-secant to U'. So U' can not be completed to a unital in $\mathcal{P}(\mathcal{S}')$ as every line of $\mathcal{P}(\mathcal{S})$ must intersect it in 1 or $q+1$ points.

Now, suppose $\langle P, l\rangle$ meets $\overline{\mathcal{U}}$ in $q+1$ points, then $\langle P, l\rangle$ meets $\mathcal{U}$ in q points. If we replace the nest N, then $\langle P, l\rangle$ corresponds to a line l_P that is a q-secant of U'. Each line of $\mathcal{P}(\mathcal{S}')$ intersects a unital in either 1 or $q+1$ points. Hence, if we wish to complete U' to a unital of $\mathcal{P}(\mathcal{S}')$, we need to add the point $l_p \cap l'_\infty$ to U'. This points corresponds to the line l of the spread $\mathcal{S}$.

There are $\frac{q+1}{2}$ lines in both $\hat{R}_1$ and $\hat{R}_2$, so there are $q+1$ different choices for l. Hence we need to add $q+1$ points of l'_∞ to U'. The affine set U' contains q^3 points plus the extra $q+1$ points from l'_∞ which gives us q^3+q+1 points. A unital has only q^3+1 points, hence we can not complete U' to a unital in $\mathcal{P}(\mathcal{S}')$. □

Lemma 4.5 *If $\overline{\mathcal{U}}$ meets $\mathcal{S}$ in one line p_∞ and p_∞ is not a line of N, then U' can be completed to a unital in $\mathcal{P}(\mathcal{S}')$.*

Proof This proof is similar to the proof given in Lemma 4.3. □

In summary, we have shown the following:

Theorem 4.6 *Let $\overline{U}$ be a unital of $PG(2,q^2)$, where q is odd. Let $\mathcal{S}$ be a regular spread corresponding to $PG(2,q^2) \cong \mathcal{P}(\mathcal{S})$ via the Bruck-Bose representation. Let N be a half-regulus replaceable t-nest in $\mathcal{S}$. Let P_N be the point set on l_∞ in $\mathcal{P}(\mathcal{S})$ corresponding to the lines of N. Suppose we replace the t-nest N by $\hat{N}$ to form the new spread $\mathcal{S}' = (\mathcal{S}\backslash N) \cup \hat{N}$. Let U' be the points of $\mathcal{P}(\mathcal{S}')$ corresponding to the affine points of $\overline{U}$. Then we have,*

1. *If $\overline{U}$ is a non-singular-Buekenhout unital of $PG(2,q^2)$, then*
   - *(a) if $1 \leq |\overline{U} \cap P_N| \leq q+1$, then the point-set U' can not be completed to a unital in $\mathcal{P}(\mathcal{S}')$.*
   - *(b) if $|\overline{U} \cap P_N| = 0$ then the point-set U' can be completed to a unital in $\mathcal{P}(\mathcal{S}')$.*
2. *If $\overline{U}$ is an ovoidal-Buekenhout-Metz unital of $PG(2,q^2)$, then*
   - *(a) if $\overline{U} \cap l_\infty \in P_N$, then the point-set U' can not be completed to a unital in $\mathcal{P}(\mathcal{S}')$.*
   - *(b) if $\overline{U} \cap l_\infty \notin P_N$ then the point-set U' can be completed to a unital in $\mathcal{P}(\mathcal{S}')$.*

5 Conclusions

We have shown that some unitals of $\mathcal{P}(\mathcal{S}) \cong PG(2,q^2)$ give rise to unitals of $\mathcal{P}(\mathcal{S}')$. We now consider the known unitals of $\mathcal{P}(\mathcal{S}')$ and discuss whether they are inherited from unitals of $\mathcal{P}(\mathcal{S})$.

Using the Bruck-Bose representation, in $PG(4,q)$, through any line l in the spread $\mathcal{S}$, we can form ovoidal cones that meet Σ_∞ in the line l. In the new spread $\mathcal{S}' = (\mathcal{S}\backslash N) \cup \hat{N}$, any line of $\hat{N}$ will give rise to ovoidal cones in $PG(4,q)$ that do not meet $\mathcal{S}$ in a line of the spread. These cones correspond to ovoidal-Buekenhout-Metz unitals in the corresponding plane $\mathcal{P}(\mathcal{S}')$ that are not inherited from ovoidal-Buekenhout-Metz unitals in $\mathcal{P}(\mathcal{S})$.

The case for non-singular-Buekenhout unitals is more complex. We know that for a regulus R in the spread $\mathcal{S}'$ there are non-singular quadrics of $PG(4,q)$ that intersect Σ_∞ in the lines of R. We also know that such quadrics correspond to non-singular-Buekenhout unitals in $\mathcal{P}(\mathcal{S}')$, secant to l'_∞, via the Bruck-Bose correspondence. So to examine non-singular-Buekenhout unitals in $\mathcal{P}(\mathcal{S}')$ we need to examine the reguli in the new spread $\mathcal{S}'$.

In [17] there is the following result. Let R be a regulus contained in the spread $\mathcal{S}' = (\mathcal{S}\backslash N) \cup \hat{N}$. Then for $q \geq 5$, R is not completely contained in $\hat{N}$ whenever $t \leq q$. Further, R contains no lines of $\hat{N}$ whenever $t \leq q-2$.

For the case $t \leq q-2$, there are no reguli of $\mathcal{S}'$ that share lines with $\hat{N}$, hence there are no reguli in $\mathcal{S}'$ that are not in $\mathcal{S}\backslash N$. We know from Theorem 4.6 that all non-singular-Buekenhout unitals in $\mathcal{P}(\mathcal{S})$ that correspond to quadrics that meet Σ_∞ in a regulus of the line-set $\mathcal{S}\backslash N$ are inherited in $\mathcal{P}(\mathcal{S}')$. This means there are no non-singular-Buekenhout unitals in $\mathcal{P}(\mathcal{S}')$ that are not inherited from unitals in $\mathcal{P}(\mathcal{S}) \cong PG(2,q^2)$, since there are no reguli in $\mathcal{S}'$ that are not contained in $\mathcal{S}\backslash N$.

When $t \geq q-1$, there can exist a regulus R in $\mathcal{S}'$ that is not a regulus of $\mathcal{S}\backslash N$. In [12], [17] there are examples of t-nests that, when replaced, give rise to such spreads. We know from Theorem 4.6 that all non-singular-Buekenhout unitals in $\mathcal{P}(\mathcal{S})$ that correspond to quadrics that meet Σ_∞ in a regulus and that share lines with N can not be completed to unitals in $\mathcal{P}(\mathcal{S}')$. Hence there do exist spreads $\mathcal{S}'$ that correspond to planes $\mathcal{P}(\mathcal{S}')$ which contain non-singular Buekenhout unitals, not inherited from unitals of $PG(2,q^2)$. It is interesting to note that, in [4], it is shown that for the Hall plane $\mathcal{H}(q^2)$ of order q^2, all non-singular-Buekenhout unitals of $\mathcal{H}(q^2)$ are inherited from non-singular-Buekenhout unitals in $PG(2,q^2)$.

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