AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **42** (2008), Pages 299–300

Note: A short proof of a theorem of Tutte

S. Akbari

Department of Mathematical Sciences
Sharif University of Technology
Tehran
Iran
s_ akbari@sharif.edu

A. Mahmoodi

Department of Mathematics
Tarbiat Modares University
Tehran
Iran
amahmoodi@modares.ac.ir

Abstract

Let G be a graph. A spanning subgraph of G is called a $\{1,2\}$-*factor* if each of its components is a regular graph of degree one or two. In this paper we provide a short proof of a theorem of Tutte which says that a graph G has a $\{1,2\}$-*factor* if and only if $i(G \setminus S) \leq |S|$ for any $S \subseteq V(G)$, where $i(G \setminus S)$ denotes the number of isolated vertices of $G \setminus S$.

Let G be a graph with vertex set $V(G)$. For each $S \subseteq V(G)$, $\Gamma_G(S)$ and $i(G \setminus S)$ denote the set of all neighbors of S in G and the number of isolated vertices of $G \setminus S$, respectively. A subgraph F of G is called a *factor* of G if $V(F) = V(G)$. A factor F is called a k-*factor* if each of its components is a regular graph of degree k. A factor F is a $\{1,2\}$-*factor* if each of its components is a regular graph of degree one or two. If F is a $\{1,2\}$-factor, then we denote the subgraph of F containing all single edges and all cycles by F^1 and F^2, respectively. In (7.1) of [2], Tutte gave a necessary and sufficient condition for a finite or locally finite graph G to have a $\{1,2\}$-factor. In this note we give a simple proof of this result in the finite case.

Theorem 1. *A graph G has a $\{1,2\}$-factor if and only if $i(G \setminus S) \leq |S|$ for all $S \subseteq V(G)$.*

Proof. First suppose that F is a $\{1,2\}$-factor of G. Let $S \subseteq V(G)$. If $S \subseteq V(F^1)$, then $i(F^1 \setminus S) \leq |S|$. If C is a cycle and $S \subseteq V(C)$, then $i(C \setminus S) \leq |S|$. To see this suppose that $i(C \setminus S) = |S| + t$, where $t \geq 1$. There are two edges between S and every component of $C \setminus S$. So, there are $2|S| + 2t$ edges between S and the components of $C \setminus S$. Hence, by the pigeonhole principle, there exists a vertex in S with degree at least 3, a contradiction. Thus if $S \subseteq V(F^2)$, then we have $i(F^2 \setminus S) \leq |S|$. Therefore if $S = S_1 \cup S_2$, where $S_1 \subseteq V(F^1)$ and $S_2 \subseteq V(F^2)$, then $i(G \setminus S) \leq |S_1| + |S_2| = |S|$.

For the reverse implication, let $V(G) = \{v_1, \ldots, v_n\}$ and construct an auxiliary bipartite graph H with two parts $V'(G) = \{v'_1, \ldots, v'_n\}$ and $V''(G) = \{v''_1, \ldots, v''_n\}$ and join v'_i to v''_j and v'_j to v''_i if and only if the edge $v_i v_j$ is in G. We claim that every 1-factor of H gives a $\{1,2\}$-factor for G. Suppose that H has a 1-factor and both the edges $v'_i v''_j$ and $v''_i v'_j$ appear in this 1-factor. These two edges are corresponding to the edge $v_i v_j$ in G. Now, consider those edges $v'_i v''_j$ in the 1-factor of H which does not contain the edge $v''_i v'_j$. It is not hard to see that these edges form a disjoint union of cycles for G, and the claim is proved.

Now, by Hall's Theorem [1, Theorem 5.2], it suffices to show that $|\Gamma_H(S')| \geq |S'|$ for every $S' \subseteq V'(G)$. Suppose that there exists a subset $S' = \{v'_{i_1}, \ldots, v'_{i_r}\} \subseteq V'(G)$ such that $|\Gamma_H(S')| < |S'|$. Let $S = \{v_{i_1}, \ldots, v_{i_r}\}$ and $S_1 \subseteq S$ be the set of all isolated vertices in the induced subgraph on S. We have

$$\begin{aligned} i(G \setminus \Gamma_G(S_1)) \geq |S_1| &= |S| - |S \cap \Gamma_G(S)| \\ &> |\Gamma_H(S')| - |S \cap \Gamma_G(S)| \\ &= |\Gamma_G(S)| - |S \cap \Gamma_G(S)|. \end{aligned}$$

On the other hand, we have $\Gamma_G(S_1) \subseteq \Gamma_G(S) \setminus (S \cap \Gamma_G(S))$. Now, we obtain $i(G \setminus \Gamma_G(S_1)) > |\Gamma_G(S_1)|$, a contradiction. □

Acknowledgement

The first author is indebted to the Research Council of Sharif University of Technology for support.

References

[1] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, North-Holland, 1976.

[2] W.T. Tutte, The 1-factors of oriented graphs, *Proc. Amer. Math. Soc.* 4 (1953), 922–931.

(Received 14 Jan 2008)