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Dyck Paths and partial Bell polynomials

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Abstract

In the present paper, we consider two kinds of statistics "number of u-segments" and "number of internal u-segments" in Dyck paths. More precisely, using Lagrange inversion formula we present the generating function for the number of Dyck paths according to semilength and our new statistics by the partial Bell polynomials, namely,

$$\begin{aligned}\sum_{D\in\mathfrak{D}_n}\prod_{i\geq 1} t_i^{\alpha_i(D)} &= \sum_{i=1}^{n}\frac{1}{(n-i+1)!}\mathbf{B}_{n,i}\big(1!t_1, 2!t_2, \cdots\big),\\ \sum_{D\in\mathfrak{D}_n}\prod_{i\geq 1} t_i^{\beta_i(D)} &= \sum_{j=0}^{n}\sum_{i=0}^{j}\binom{n}{i}\frac{i!}{j!}\frac{n-j}{n}\mathbf{B}_{j,i}\big(1!t_1, 2!t_2, \cdots\big),\end{aligned}$$

where $\alpha_r(D)$ $\big(\beta_r(D)\big)$ is the number of u-segments (internal u-segments) of length r in a Dyck path D. Many important special cases are presented which lead to a lot of interesting results.

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1 Introduction

A *Dyck path* of length $2n$ is a lattice path from $(0,0)$ to $(2n,0)$ in the plane integer lattice $\mathbb{Z}\times\mathbb{Z}$ consisting of up-steps $(1,1)$ and down-steps $(1,-1)$, which never passes below the x-axis. Let $\mathfrak{D}_n$ denote the set of Dyck paths of length $2n$. Dyck paths are very well-known combinatorial objects that have been widely studied in the literature. Stanley [20] presents a lot of objects equivalent to Dyck paths of length $2n$, all of which are counted by *Catalan sequence* $\frac{1}{n+1}\binom{2n}{n}$. Many various statistics have been studied on the set of Dyck paths, such as *area* [9, 12, 24], *pyramid weight* [10], and number of *udu*'s [22]. Others [11, 14] are many that have carried out the important and earlier studies regarding statistics on Dyck paths.

Let D be any Dyck path. A *u-segment* of D is a maximum consecutive up steps in D and an *internal u-segment* of D is a u-segment between two down steps, i.e., all u-segments except for the first one are internal u-segments. Define $\alpha_k(D)$ and $\beta_k(D)$ to be the number of u-segments and internal u-segments of length k in D, respectively.

Recall that the potential polynomials [8] $\mathbf{P}_n^{(\lambda)}$ are defined for each complex number λ by

$$1+\sum_{n\geq 1}\mathbf{P}_n^{(\lambda)}\frac{x^n}{n!} \;=\; \Big\{1+\sum_{n\geq 1}f_n\frac{x^n}{n!}\Big\}^{\lambda},$$

which can be represented by Bell polynomials

$$\mathbf{P}_n^{(\lambda)}=\mathbf{P}_n^{(\lambda)}(f_1,f_2,f_3,\dots)=\sum_{1\leq k\leq n}\binom{\lambda}{k}k!\mathbf{B}_{n,k}(f_1,f_2,f_3,\dots), \tag{1.1}$$

where $\mathbf{B}_{n,r}\big(x_1,x_2,\cdots\big)$ is the partial Bell polynomial [1] on the variables $\{x_j\}_{j\geq 1}$, that is

$$\mathbf{B}_{m,r}\big(x_1,x_2,\cdots\big) \;=\; \sum_{\kappa_m(r)}\frac{m!}{r_1!r_2!\cdots r_m!}\left(\frac{x_1}{1!}\right)^{r_1}\left(\frac{x_2}{2!}\right)^{r_2}\cdots\left(\frac{x_m}{m!}\right)^{r_m},$$

where the summation $\kappa_m(r)$ is for all the nonnegative integer solutions of $r_1+r_2+\cdots+r_m=r$ and $r_1+2r_2+\cdots+mr_m=m$.

In this paper, we prove that

$$\sum_{D\in\mathfrak{D}_n}\prod_{i\geq 1}t_i^{\alpha_i(D)}=\sum_{i=1}^{n}\frac{1}{(n-i+1)!}\mathbf{B}_{n,i}\big(1!t_1,2!t_2,\cdots\big),$$

and

$$\sum_{D\in\mathfrak{D}_n}\prod_{i\geq 1}t_i^{\beta_i(D)}=\sum_{j=0}^{n}\sum_{i=0}^{j}\binom{n}{i}\frac{i!}{j!}\frac{n-j}{n}\mathbf{B}_{j,i}\big(1!t_1,2!t_2,\cdots\big).$$

As an application, we consider many special cases which lead to a few interesting results.

2 The u-segment statistics

In this section we link Dyck paths to several combinatorial objects, such as partial Bell polynomials (see [1]), Riordan number r_n (see [3]), and the number of unlabeled plane tree on $n+1$ vertices in which every vertex has outdegree not greater than k (see [13, 5]). Define the ordinary generating functions for the number of Dyck paths D of length $2n$ according to the statistics $\alpha_1(D), \alpha_2(D), \ldots$, that is,

$$G(x;\mathbf{t}) = G(x;t_1,t_2,\ldots) = \sum_{n\geq 0} x^n \sum_{D\in\mathfrak{D}_n} \prod_{i\geq 1} t_i^{\alpha_i(D)}.$$

Proposition 2.1. *The ordinary generating function $G(x;\mathbf{t})$ is given by*

$$G(x;\mathbf{t}) = 1 + \sum_{j\geq 1} t_j x^j G(x;\mathbf{t})^j. \tag{2.1}$$

Proof. The ordinary generating function $G(x;\mathbf{t})$ can be written as $G(x;\mathbf{t}) = 1 + \sum_{j\geq 1} G_j(x;\mathbf{t})$, where $G_j(x;\mathbf{t})$ is the generating function for the number of Dyck paths D of length $2n$ starting with exactly j up steps according to the statistics $\alpha_1(D), \alpha_2(D), \ldots$. An equation for $G_j(x;\mathbf{t})$ is obtained from the first return decomposition of a Dyck path starting with a u-segment of length j: $D = u^j dD^{(j)} dD^{(j-1)} d \ldots D^{(2)} dD^{(1)}$, where $D^{(1)}, \ldots, D^{(j)}$ are Dyck paths; see Figure 1.

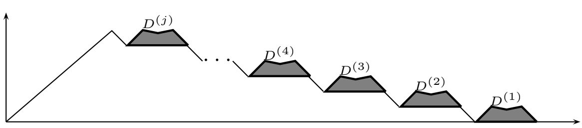

Figure 1: First return decomposition of a Dyck path starting with exactly j up steps.

Thus $G_j(x;\mathbf{t}) = t_j x^j G^j(x;\mathbf{t})$. Hence, the ordinary generating function $G(x;\mathbf{t})$ satisfies the following equation $G(x;\mathbf{t}) = 1 + \sum_{j\geq 1} G_j(x;\mathbf{t}) = 1 + \sum_{j\geq 1} t_j x^j G^j(x;\mathbf{t})$, as required. $\square$

Let $T(x) = \sum_{i\geq 0} t_i x^i$ be the ordinary generating function for the indeterminates $\{t_i\}_{i\geq 0}$ with $t_0 = 1$. Define $y(x,\mathbf{t}) = xG(x,\mathbf{t})$, or simply $y = xG(x,\mathbf{t})$. Then (2.1) reduces to $y = xT(y)$. Applying the Lagrange inversion formula [23] and the potential polynomials (1.1), we have

$$\begin{aligned}\sum_{D\in\mathfrak{D}_n} \prod_{i\geq 1} t_i^{\alpha_i(D)} &= [x^{n+1}]y = \frac{1}{n+1}[x^n]T(x)^{n+1} \\ &= \frac{1}{n+1}\sum_{i=1}^{n}\binom{n+1}{i}\frac{i!}{n!}\mathbf{B}_{n,i}\big(1!t_1, 2!t_2, \cdots\big), \\ &= \sum_{i=1}^{n}\frac{1}{(n-i+1)!}\mathbf{B}_{n,i}\big(1!t_1, 2!t_2, \cdots\big).\end{aligned}$$

Hence we obtain the first main results,

Theorem 2.2. *For any integer* $n \geq 1$,

$$\sum_{D\in\mathfrak{D}_n}\prod_{i\geq 1} t_i^{\alpha_i(D)} \quad = \quad \sum_{i=1}^{n}\frac{1}{(n-i+1)!}\mathbf{B}_{n,i}\big(1!t_1, 2!t_2, \cdots\big),$$

where $\mathbf{B}_{n,i}\big(x_1, x_2, \cdots\big)$ *is the partial Bell polynomial on the variables* $\{x_j\}_{j\geq 1}$.

Theorem 2.2 generates a lot of surprising results. Here we present some of these results; see Examples 2.3–2.11.

Example 2.3

If $T(x) = 1 + q(e^x - 1) = (1-q) + qe^x$, then $t_i = q/i!$ for all $i \geq 1$. Note that $\mathbf{B}_{n,i}\big(q, q, q, \cdots\big) = S(n,i)q^i$, where $S(n,i)$ are the second kind of Stirling numbers [8, Page 135]. Thus Theorem 2.2 leads to

$$\begin{aligned}\sum_{D\in\mathfrak{D}_n}\frac{n!q^{\sum_{i\geq 1}\alpha_i(D)}}{\prod_{i\geq 1}(i!)^{\alpha_i(D)}} &= \frac{1}{n+1}\sum_{i=1}^{n}\binom{n+1}{i} i!S(n,i)q^i = \sum_{i=1}^{n}\frac{n!}{i!}S(n,n-i+1)q^{n-i+1}\\ &= \frac{1}{n+1}\sum_{i=0}^{n}\binom{n+1}{i} i^n q^i(1-q)^{n-i+1},\end{aligned}$$

which reduces to $(n+1)^{n-1}$ when $q = 1$. We note that it is well known that $(n+1)^{n-1}$ counts the set of labeled trees on $n+1$ vertices or the set of rooted labeled trees with a fixed root on $n+1$ vertices. Specially the number of rooted labeled trees with a special root on $n+1$ vertices with i leaves is $\frac{n!}{i!}S(n, n-i+1)$ [6].

Example 2.4

Let $T(x) = 1 + qxe^x$. This case implies $t_i = q/(i-1)!$ for all $i \geq 1$. Note that $\mathbf{B}_{n,i}\big(q, 2q, 3q, \cdots\big) = \binom{n}{i} i^{n-i}q^i$, which are called the idempotent numbers [8, Page 135] when $q = 1$. Then Theorem 2.2 leads to

$$\sum_{D\in\mathfrak{D}_n}\prod_{i\geq 1}\left\{\frac{q}{(i-1)!}\right\}^{\alpha_i(D)} \quad = \quad \sum_{i=1}^{n}\binom{n}{i}\frac{i^{n-i}q^i}{(n-i+1)!},$$

for any integer $n \geq 1$.

Example 2.5

Let $T(x) = (e^x - 1)/x$. This case implies $t_i = 1/(i+1)!$ for all $i \geq 1$. Note that $\left(\frac{e^x-1}{x}\right)^k/k! = \sum_{n\geq 0} S(n+k, k)x^n/(n+k)!$. Then Theorem 2.2 gives

$$\begin{aligned}\sum_{D\in\mathfrak{D}_n}\prod_{i\geq 1}\frac{1}{\big((i+1)!\big)^{\alpha_i(D)}} &= \sum_{i=1}^{n}\frac{1}{(n-i+1)!}\mathbf{B}_{n,i}\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right)\\ &= \frac{n!}{(2n+1)!}S(2n+1, n+1),\end{aligned}$$

for any integer $n \geq 1$.

Example 2.6

Here we give a relation between our Dyck paths and the first kind of Stirling numbers [8, Page 135]. More precisely, the case $T(x) = \frac{1}{x}\ln\frac{1}{1-x}$ implies $t_i = 1/(i+1)$ for all $i \geq 1$. Note that $\left(\frac{1}{x}\ln\frac{1}{1-x}\right)^k/k! = \sum_{n\geq 0}|s(n+k,k)|x^n/(n+k)!$, where $s(n,k)$ are the first kind of Stirling numbers. Then Theorem 2.2 produces

$$\begin{aligned}\sum_{D\in\mathfrak{D}_n}\prod_{i\geq 1}\frac{1}{(i+1)^{\alpha_i(D)}} &= \sum_{i=1}^{n}\frac{1}{(n-i+1)!}\mathbf{B}_{n,i}\left(\frac{1!}{2},\frac{2!}{3},\frac{3!}{4},\cdots\right)\\ &= \frac{n!}{(2n+1)!}|s(2n+1,n+1)|,\end{aligned}$$

for any integer $n \geq 1$.

Example 2.7

Another relation between our Dyck paths and first kind of Stirling numbers can be state as follows. The case $T(x) = 1 + q\ln\frac{1}{1-x}$ implies $t_i = q/i$ for all $i \geq 1$. Note that $\mathbf{B}_{n,i}\big(0!q, 1!q, 2!q, \cdots\big) = |s(n,i)|q^i$ [8, pp.135]. Then Theorem 2.2 gives

$$\sum_{D\in\mathfrak{D}_n}\prod_{i\geq 1}\left(\frac{q}{i}\right)^{\alpha_i(D)} = \sum_{i=1}^{n}\frac{|s(n,i)|q^i}{(n-i+1)!},$$

for any integer $n \geq 1$.

Example 2.8

Let $T(x) = (1+x)^\lambda$. This case implies $t_i = \binom{\lambda}{i}$ for all $i \geq 1$, where λ is an indeterminant. Then Theorem 2.2 leads to

$$\sum_{D\in\mathfrak{D}_n}\prod_{i\geq 1}\binom{\lambda}{i}^{\alpha_i(D)} = \frac{1}{n+1}\binom{(n+1)\lambda}{n},$$

for any integer $n, k \geq 1$. Specially, replacing λ by $-\lambda$, we have

$$\sum_{D\in\mathfrak{D}_n}\prod_{i\geq 1}\binom{\lambda+i-1}{i}^{\alpha_i(D)} = \frac{1}{n+1}\binom{(n+1)\lambda+n-1}{n}.$$

Example 2.9

Let $T(x) = 1 + x + x^2 + \cdots + x^k$. This case implies $t_i = 1$ for $1 \le i \le k$ and $t_i = 0$ for all $i \ge k+1$. Then Theorem 2.2 gives

$$\sum_{D\in\mathfrak{D}_n}\prod_{i\ge 1}^{k} 1^{\alpha_i(D)}\prod_{i\ge k+1} 0^{\alpha_i(D)} = \frac{1}{n+1}\sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i}\binom{2n-(k+1)i}{n},$$

which generates the following result (by convention $0^0 = 1$). The number of Dyck paths D of length $2n$ with no u-segments of length greater than k (i.e. $\alpha_i(D) = 0$ for $i > k$) is given by

$$U_{n,k} = \frac{1}{n+1}\sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i}\binom{2n-(k+1)i}{n}.$$

We note that the number $U_{n,k}$ also counts the unlabeled plane tree on $n+1$ vertices in which every vertex has outdegree not greater than k. Klarner [13] first considers this problem, which is solved by Chen [6]. When $k = 2$, Callan [4] gives a bijection between the set of Dyck paths D of length $2n$ with no u-segments of length greater than 2 and the set of Motzkin paths of length n.

Example 2.10

Let $T(x) = \dfrac{1}{1-x} + (q-1)x^k$. This case implies $t_i = 1$ for all $i \ge 1$ except for $i = k$ and $t_k = q$. Then Theorem 2.2 yields

$$\sum_{D\in\mathfrak{D}_n} q^{\alpha_k(D)} = \frac{1}{n+1}\sum_{j=0}^{n}\binom{n+1}{j}\binom{2n-(k+1)j}{n-j}(q-1)^j, \tag{2.2}$$

which produces the following result. The number of Dyck paths D of length $2n$ with exactly m u-segments of length k (namely, $\alpha_k(D) = m$) is given by

$$\frac{1}{n+1}\sum_{j=m}^{n}(-1)^{j+m}\binom{n+1}{j}\binom{2n-(k+1)j}{n-j}\binom{j}{m},$$

which is the n-th Riordan number r_n [3] when $q = 0$ and $k = 1$. This result for $k = 1, 2, 3$, and $m = 0$ gives the following table.

$k\backslash n$	0	1	2	3	4	5	6	7	8	9	10	11	12	Reference
1	1	0	1	1	3	6	15	36	91	232	603	1585	4213	[18, Seq. A005043]
2	1	1	1	2	6	17	46	128	372	1109	3349	10221	31527	[18, Seq. A102403]
3	1	1	2	4	10	27	79	240	750	2387	7711	25214	83315	[18, Seq. A114507]

Table 1: The number of Dyck paths D of length $2n$ such that $\alpha_k(D) = 0$ for $k = 1, 2, 3$.

Example 2.11

Let $T(x) = 1 + \frac{qx^k}{1-x}$. This case implies $t_i = q$ for all $i \geq k$ and $t_i = 0$ for $1 \leq i \leq k-1$. Then Theorem 2.2 gives (here assumed that 0^0 is 1)

$$\sum_{D \in \mathfrak{D}_n} \prod_{i \geq 1}^{k-1} 0^{\alpha_i(D)} \prod_{i \geq k} q^{\alpha_i(D)} = \frac{1}{n+1} \sum_{j=0}^{n} \binom{n+1}{j} \binom{n-(k-1)j-1}{j-1} q^j,$$

which yields the n-th Riordan number r_n [3] when $q = 1$ and $k = 2$. Then the number of Dyck paths D of length $2n$ such that each u-segment has length not less than k is given by

$$\frac{1}{n+1} \sum_{j=0}^{n} \binom{n+1}{j} \binom{n-(k-1)j-1}{j-1}.$$

Specially, the number of Dyck paths D of length $2n$ such that each u-segment has length not less than 2 is given by the n-th Riordan number r_n. More precisely, the number of Dyck paths D of length $2n$ with exactly j u-segments such that each u-segment has length not less than k is given by $\frac{1}{n+1}\binom{n+1}{j}\binom{n-(k-1)j-1}{j-1}$, which is the Narayana number when $k = 1$.

Example 2.12

Let $T(x) = 1 + \frac{qx^k}{1-x^k} = \frac{1+(q-1)x^k}{1-x^k}$. This case implies $t_{jk} = q$ for all $j \geq 1$ and $t_i = 0$ for otherwise. Then Theorem 2.2 generates

$$\sum_{D \in \mathfrak{D}_{kn}, \alpha_i(D)=0 \, if \, i \neq 0 \, mod \; k} q^{\sum_{j \geq 1} \alpha_{jk}(D)} = \frac{1}{kn+1} \sum_{m=1}^{n} \binom{kn+1}{m} \binom{n-1}{n-m} q^m \tag{2.3}$$

$$= \frac{1}{kn+1} \sum_{m=0}^{n} \binom{kn+1}{m} \binom{(k+1)n-m}{n-m} (q-1)^m. \tag{2.4}$$

Identity (2.3) produces the following result: The number of Dyck paths D of length $2kn$ such that the length of any u-segment is a multiplicity of k (namely, $q = 1$) is given by $\frac{1}{kn+1}\binom{(k+1)n}{n}$ (which is obtained by Vandermonde convolution). More precisely, the number of Dyck paths D of length $2kn$ with exactly m u-segments such that the length of any u-segment is a multiplicity of k is given by $\frac{1}{kn+1}\binom{kn+1}{m}\binom{n-1}{m-1} = \frac{1}{n}\binom{kn}{m-1}\binom{n}{m}$, which is the Narayana number for $k = 1$.
When $k = 1$, (2.3) and (2.4) produce many interesting results: A new expression for Narayana polynomials is given by

$$\mathfrak{N}_n(q) = \sum_{m=1}^{n} \frac{1}{n} \binom{n}{m-1} \binom{n}{m} q^m = \frac{1}{n+1} \sum_{m=0}^{n} \binom{n+1}{m} \binom{2n-m}{n} (q-1)^m. \tag{2.5}$$

Define the associated Narayana polynomials as $\mathcal{N}_n(q) = q^n \mathfrak{N}_n(\frac{1}{q})$; then we have

$$\begin{aligned}\mathcal{N}_n(q) &= \sum_{m=1}^{n} \frac{1}{n}\binom{n}{m-1}\binom{n}{m} q^{n-m} \\ &= \frac{1}{n+1}\sum_{m=0}^{n}\binom{n+1}{m}\binom{2n-m}{n}(1-q)^m q^{n-m}. \end{aligned} \tag{2.6}$$

Note that several authors have investigated the polynomials $\mathcal{N}_n(q)$. For examples, see Rogers [16], Rogers and Shapiro [17], Sulanke [21], Bonin, Shapiro and Simion [2], Coker [7].

If setting $q = -1$ in (2.6) and using the identity proved in [2, 22],

$$\sum_{m=1}^{n} \frac{1}{n}\binom{n}{m-1}\binom{n}{m}(-1)^{n-m} = \begin{cases} 0 & \text{if } n = 2r, \\ \frac{(-1)^r}{r+1}\binom{2r}{r} & \text{if } n = 2r+1, \end{cases}$$

then we get an expression for Catalan numbers

$$\frac{1}{n+1}\binom{2n}{n} = \sum_{m=0}^{2n+1} \frac{(-1)^{n-m+1}}{n+1}\binom{2n+2}{m}\binom{4n-m+2}{2n+1} 2^{m-1}.$$

Example 2.13

Let $T(x) = f^k(x)$. Define $T(x) = f^k(x)$, where $f(x)$ is the generating function for the m-ary plane trees, which satisfies the relation $f(x) = 1 + xf^m(x)$. Another form of Lagrange formula [8, Page 149] generates

$$[x^n]\Phi(y) = \frac{1}{n}[x^{n-1}]\Phi'(x)T(x)^n, \tag{2.7}$$

for $n \geq 1$, where $\Phi(x)$ is any formal power series and $\Phi'(x)$ is its derivative on the variable x. Thus, by (2.7), we can deduce $t_i = \frac{k}{(m-1)i+k}\binom{mi+k-1}{i}$. Then Theorem 2.2 gives

$$\begin{aligned} &\sum_{D\in\mathfrak{D}_n}\prod_{i\geq 1}\Big(\frac{k}{(m-1)i+k}\binom{mi+k-1}{i}\Big)^{\alpha_i(D)} \\ &= \sum_{i=1}^{n}\frac{1}{(n-i+1)!}\mathbf{B}_{n,i}\Big(\frac{1!k}{m-1+k}\binom{m+k-1}{1}, \frac{2!k}{2(m-1)+k}\binom{2m+k-1}{2}, \cdots\Big) \\ &= \frac{k}{(m+k-1)n+k}\binom{(m+k)n+k-1}{n}. \end{aligned}$$

for any integer $n, k, m \geq 1$.

Remark 2.14

As a remark more interesting cases should be considered further such as $T(x)$ is the generating function of the elementary symmetric function, or complete symmetric function, or Schur symmetric function, or q-factorials, or the generating function of some special sequences such as Fibonacci, Bell sequences and so on (for definitions see [20, 23]).

3 The internal u-segment statistics

Recall that an *internal u-segment* of Dyck path D is a u-segment between two d steps, i.e., all u-segments except for the first one are internal u-segments. A Dyck path D is said to be *k-partial* (resp. k-complete) if the length of any internal u-segment (resp. u-segment) is a multiplicity of k. Define $\beta_k(D)$ to be the number internal u-segments in Dyck path D. In this section, we study the ordinary generating functions for the number of Dyck paths D of length $2n$ according to the statistics $\beta_1(D), \beta_2(D), \ldots$, that is,

$$F(x;\mathbf{t}) = F(x;t_1,t_2,\ldots) = 1 + \sum_{n\geq 1} x^n \sum_{D\in\mathcal{D}_n} \prod_{i\geq 1} t_i^{\beta_i(D)},$$

which can be represented in terms of the generating function $G(x;\mathbf{t})$ as follows.

Proposition 3.1. *The ordinary generating function $F(x;\mathbf{t})$ is given by*

$$\frac{1}{1-xG(x;\mathbf{t})} = \frac{1}{1-x-x\sum_{j\geq 1} t_j x^j G^j(x;\mathbf{t})}.$$

Proof. An equation for the ordinary generating function $F(x;\mathbf{t})$ is obtained from the first return decomposition of a Dyck paths: $D = uD'dD''$, where D' and D'' are Dyck paths. Therefore, the ordinary generating function $F(x;\mathbf{t})$ satisfies $F(x;\mathbf{t}) = 1 + xF(x;\mathbf{t})G(x;\mathbf{t})$. Hence, using Proposition 2.1 we get the desired result. □

Applying the Lagrange inversion formula (2.7) in which $\Phi(x) = \frac{1}{1-x}$ and the potential polynomials (1.1), we have

$$\begin{aligned}\sum_{D\in\mathfrak{D}_n} \prod_{i\geq 1} t_i^{\beta_i(D)} &= \sum_{j=0}^{n} [x^n] y^{n-j} = \sum_{j=0}^{n} \frac{n-j}{n} [x^j] T(x)^n \\ &= \sum_{j=0}^{n} \sum_{i=0}^{j} \binom{n}{i} \frac{i!}{j!} \frac{n-j}{n} \mathbf{B}_{j,i}(1!t_1, 2!t_2, \cdots).\end{aligned}$$

Hence, we obtain our second main result,

Theorem 3.2. *For any integer $n \geq 1$,*

$$\sum_{D\in\mathfrak{D}_n} \prod_{i\geq 1} t_i^{\beta_i(D)} = \sum_{j=0}^{n} \sum_{i=0}^{j} \binom{n}{i} \frac{i!}{j!} \frac{n-j}{n} \mathbf{B}_{j,i}(1!t_1, 2!t_2, \cdots).$$

Theorem 3.2 generates a lot of surprising results. Here we present some of these results; see Examples 3.3–3.10.

Example 3.3

Let $T(x) = 1 + q(e^x - 1) = (1-q) + qe^x$. This case implies $t_i = q/i!$ for all $i \geq 1$. Then Theorem 3.2 gives

$$\sum_{D\in\mathfrak{D}_n}\prod_{i\geq 1}\left\{\frac{q}{i!}\right\}^{\beta_i(D)} = \sum_{j=0}^{n}\sum_{i=0}^{j}\frac{n-j}{n}\binom{n}{i}\frac{i!}{j!}S(j,i)q^i$$
$$= \sum_{j=0}^{n}\sum_{i=0}^{n}\frac{n-j}{n}\binom{n}{i}\frac{i^j}{j!}(1-q)^{n-i}q^i$$

which leads to $\frac{n^n}{n!}$ when $q = 1$ for any integer $n \geq 1$.

Example 3.4

Let $T(x) = (e^x - 1)/x$. Again, we can state another relation between Dyck paths and second kind of Stirling number. The case $T(x) = (e^x - 1)/x$ implies $t_i = 1/(i+1)!$ for all $i \geq 1$. Note that $\left(\frac{e^x-1}{x}\right)^k/k! = \sum_{n\geq 0} S(n+k,k)x^n/(n+k)!$. Then Theorem 3.2 obtains

$$\sum_{D\in\mathfrak{D}_n}\prod_{i\geq 1}\frac{1}{\big((i+1)!\big)^{\beta_i(D)}} = \sum_{j=0}^{n-1}\frac{n-j}{n}\frac{n!}{(n+j)!}S(n+j,n),$$

for any integer $n \geq 1$.

Example 3.5

Let $T(x) = \frac{1}{x}\ln\frac{1}{1-x}$. This case implies $t_i = 1/(i+1)$ for all $i \geq 1$. Note that $\left(\frac{1}{x}\ln\frac{1}{1-x}\right)^k/k! = \sum_{n\geq 0}|s(n+k,k)|x^n/(n+k)!$. Then Theorem 3.2 gives

$$\sum_{D\in\mathfrak{D}_n}\prod_{i\geq 1}\frac{1}{(i+1)^{\beta_i(D)}} = \sum_{j=0}^{n-1}\frac{n-j}{n}\frac{n!}{(n+j)!}|s(n+j,n)|,$$

for any integer $n \geq 1$.

Example 3.6

Let $T(x) = (1+x)^\lambda$, so $t_i = \binom{\lambda}{i}$ for all $i \geq 1$, where λ is an indeterminant. Then Theorem 3.2 gives

$$\sum_{D\in\mathfrak{D}_n}\prod_{i\geq 1}\binom{\lambda}{i}^{\beta_i(D)} = \sum_{j=0}^{n-1}\frac{n-j}{n}\binom{n\lambda}{j},$$

for any integer $n \geq 1$.

Example 3.7

Let $T(x) = 1/(1-x)^k$, so $t_i = \binom{k+i-1}{i}$ for all $i \geq 1$. Then Theorem 3.2 obtains

$$\sum_{D\in\mathfrak{D}_n}\prod_{i\geq 1}\binom{k+i-1}{i}^{\beta_i(D)} = \sum_{j=0}^{n-1}\frac{n-j}{n}\binom{nk+j-1}{j} = \frac{1}{kn+1}\binom{(k+1)n}{n},$$

for any integer $n, k \geq 1$.

Example 3.8

Let $T(x) = 1/(1-x^k)$, so $t_{jk} = 1$ for all $j \geq 1$ and $t_i = 0$ for otherwise. Then Theorem 3.2 obtains

$$\sum_{D\in\mathfrak{D}_n,\beta_i(D)=0\ if\ i\neq 0\ mod\ k} 1^{\sum_{j\geq 1}\beta_{jk}(D)} = \sum_{j=0}^{[n/k]}\frac{n-jk}{n}\binom{n+j-1}{j} = \frac{\ell+1}{n+1}\binom{n+p}{n},$$

where $n = pk+\ell$ for $0 \leq \ell \leq k-1$, which generates the following result. The number of k-partial Dyck paths D of length $2n$ is given by

$$S_{n,\ell}^{(k)} = \frac{\ell+1}{n+1}\binom{n+p}{n}, \tag{3.1}$$

where $n = pk+\ell$ for $0 \leq \ell \leq k-1$.

Example 3.9

Let $T(x) = 1 + \frac{qx^k}{1-x^k}$, so $t_{jk} = q$ for all $j \geq 1$ and $t_i = 0$ for otherwise. Then Theorem 3.2 obtains

$$\begin{aligned}\sum_{D\in\mathfrak{D}_n,\beta_i(D)=0\ if\ i\neq 0\ mod\ k} q^{\sum_{j\geq 1}\beta_{jk}(D)} &= \sum_{m=0}^{[n/k]} q^m\binom{n}{m}\sum_{j=m}^{[n/k]}\frac{n-kj}{n}\binom{j-1}{j-m} \\ &= \sum_{m=0}^{p} q^m\frac{n+m\ell-km}{n(m+1)}\binom{n}{m}\binom{p}{m},\end{aligned}$$

which gives rise to the following result. The number of k-partial Dyck paths D of length $2n$ with exactly m internal u-segments is given by

$$\frac{n+m\ell-km}{n(m+1)}\binom{n}{m}\binom{p}{m}, \tag{3.2}$$

where $n = pk+\ell$ for $0 \leq \ell \leq k-1$.

Remark 3.11

Results (3.1) and (3.2) give the following identity

$$\frac{\ell+1}{n+1}\binom{n+p}{n} = \sum_{m=0}^{n} \frac{n+m\ell-km}{n(m+1)}\binom{n}{m}\binom{p}{m},$$

where $n = pk+\ell$ for $0 \le \ell \le k-1$. This identity can be proved by Vondermonde convolution, maybe it is interesting to prove it by some other powerful methods such as WZ method [15] and Riordan arrays [19].

Example 3.12

Define $T(x) = f^k(x)$, where $f(x)$ is the generating function for the complete m-ary trees, which satisfies the relation $f(x) = 1 + xf^m(x)$. By Lagrange inversion formula (2.7), we can deduce $t_i = \frac{k}{(m-1)i+k}\binom{mi+k-1}{i}$. Then Theorem 3.2 gives

$$\sum_{D\in\mathfrak{D}_n}\prod_{i\ge1}\left(\frac{k}{(m-1)i+k}\binom{mi+k-1}{i}\right)^{\beta_i(D)} = \sum_{j=0}^{n}\frac{k(n-j)}{kn+mj}\binom{kn+mj}{j},$$

for any integer $n \ge 1, k, m \ge 0$.

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References

[1] E. T. Bell, Partition Polynomials, *Annals Math.* **29** (1927), 38–46.

[2] J. Bonin, L. Shapiro and R. Simion, Some q-analogues of the Schröder numbers arising from combinatorial statistics on lattice paths, *J. Statist. Plann. Inference* **34** (1993), 35–55.

[3] F. R. Bernhart, Catalan, Motzkin and Riordan numbers, *Discrete Math.* **204** (1999), 73–112.

[4] D. Callan, Two Bijections for Dyck Path Parameters, preprint, `http://www.stat.wisc.edu/~callan/papers/motzkin_dyck/`.

[5] W.Y.C. Chen, A general bijective algorithm for increasing trees, *Systems Sci. Math. Sci.* **12** (1999), 193–203.

[6] W.Y.C. Chen, A general bijective algorithm for trees, *Proc. Natl. Acad. Sci. USA* **87** (1990), 9635–9639.

[7] C. Coker, Enumerating a class of lattice paths, *Dicrete Math.* **271** (2003), 13–28.

[8] L. Comtet, *Advanced Combinatorics*, D.Reidel, Dordrecht-Holland, 1970.

[9] M. Delest and X.G. Viennot, Algebraic languages and polyonimoes enumeration, *Theoret. Comp. Sci.* **34** (1984), 169–206.

[10] A. Denise and R. Simion, Two combinatorial statistics Dyck paths, *Discrete Math.* **187** no. 1–3 (1998), 71–96.

[11] E. Deutsch, Dyck path enumration, *Discrete Math.* **204** (1999), 167–202.

[12] J.M. Fédou, *Grammairs et q-énumération de polyominos*, Ph.D. Thesis, Université de Bordeaux I, 1989.

[13] D.A. Klarner, Correspondences between plane trees and binary sequences, *J. Combin. Theory Ser. A.* **9** (1970), 401–411.

[14] D. Merlini, R. Sprugnoli and M.C. Verri, Some statistics on Dyck paths, *J. Statist. Plann. Inference* **101** (2002), 211–227.

[15] M. Petkovsek, H.S. Wilf and D. Zeilberger, A=B, A.K. Peters, Wellesley, MA, 1996.

[16] D.G. Rogers, Rhyming schemes: crossings and coverings, *Discrete Math.* **33** (1981), 67–77.

[17] D.G. Rogers, L.W. Shapiro, Deques, trees and lattice paths, in: *Lect. Notes Math.* **884** (1981), 293–303.

[18] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, `http://www.research.att.com/~njas/sequences`.

[19] R. Sprugnoli, Riordan array proofs of identities in Gould's book, `http://www.dsi.unifi.it/~resp`, 2007.

[20] R. Stanley, *Enumerative Combinatorics*, vol. 1, Wadsworth and Brooks/Cole, Pacific Grove, CA, 1986, second printing, CUP, Cambridge, 1996.

[21] R. Sulanke, Counting lattice paths by Narayana polynomials, *Electron. J. Combin.* **2** (2000), #R40.

[22] Y. Sun, The statistic "number of *udu*'s" in Dyck paths, *Discrete Math.* **287** no. 1–3 (2004), 177–186.

[23] H. Wilf, *Generatingfunctionology*, Academic Press, New York, 1990.

[24] W. Woan, Area of Catalan paths, *Discrete Math.* **226** (2001), 439–444.

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