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Further results on fine structures of three Latin squares*

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Abstract

Denote by $\mathrm{Fin}(v)$ the set of all integer pairs (t, s) for which there exist three Latin squares of order v on the same set having fine structure (t, s). The set $\mathrm{Fin}(v)$ with $v \geq 2$ and $v \neq 5, 6, 7, 8$ is determined in [2]. In the present article the set $\mathrm{Fin}(8)$ is finally determined. Some results on $\mathrm{Fin}(v)$ with $5 \leq v \leq 7$ are also updated.

1 Introduction

A *Latin square* of order v is a $v \times v$ array in which each cell contains a single element from a v-set S, such that each element occurs exactly once in each row and exactly once in each column.

Let $L_k = (a_{ij}^{(k)})_{v \times v}$, $k = 1, 2, \ldots, \lambda$ be Latin squares on symbol set S containing v elements. Define

$$\mathcal{C}_l = \{(a_{ij}^{(1)}, a_{ij}^{(2)}, \ldots, a_{ij}^{(\lambda)}) : \ |\{a_{ij}^{(1)}, a_{ij}^{(2)}, \ldots, a_{ij}^{(\lambda)}\}| = l, i, j = 1, 2, \ldots, v\}.$$

The *fine structure* of λ Latin squares $L_1, L_2, \ldots, L_\lambda$ is the vector $(c_1, c_2, \ldots, c_\lambda)$, where c_i is the number of elements in $\mathcal{C}_i$. Clearly, $c_1 + c_2 + \cdots + c_\lambda = v^2$. Let $\mathrm{Fin}[v, \lambda]$ be the set of λ-vectors $\mathbf{c} = (c_1, c_2, \ldots, c_\lambda)$ for which there exist λ Latin squares of order v with fine structure $\mathbf{c}$. When $\lambda = 3$, since any two of $\{c_1, c_2, c_3\}$ determine the third, we use a more convenient notation for the fine structure: (t, s) is said to be the fine structure of three Latin squares if $c_1 = v^2 - s$ and $c_2 = t$. Let $\mathrm{Fin}(v)$ denote the set of all integer pairs (t, s) for which there exist three Latin squares of order v with fine structure (t, s).

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A λ-*Latin trade of volume* s is a set of λ partial Latin squares of the same shape and order, containing the same filled cells, such that if cell (i, j) is filled, there exist at least two partial Latin squares with *different* entries at cell (i, j) among the λ partial Latin squares, and such that row i in each of the λ partial Latin squares contains, set-wise, the same symbols (for each relevant i), and column j, likewise, in each of the λ partial Latin squares, set-wise, the same symbols (for each relevant j).

Here λ-Latin trade can be regarded as a generalization of 3-way Latin trade, which comes from [1]. See [1] and the references therein for details.

Let $\{P_1, P_2, \ldots, P_\lambda\}$ be a λ-Latin trade of volume s. If cell (i, j) is filled, denote the entry in partial Latin square P_k, $k = 1, 2, \ldots, \lambda$, as $a_{ij}^{(k)}$. Define the number

$$c_l = |\{(a_{ij}^{(1)}, a_{ij}^{(2)}, \ldots, a_{ij}^{(\lambda)}) : \ |\{a_{ij}^{(1)}, a_{ij}^{(2)}, \ldots, a_{ij}^{(\lambda)}\}| = l, \text{ for any filled cell } (i, j)\}|$$

for $l = 1, 2, 3, \ldots, \lambda$ (note that $c_1 = 0$ by the definition of λ-Latin trade of volume s). Then $\{P_1, P_2, \ldots, P_\lambda\}$ is said to be a $(s, \lambda; [c_2, \ldots, c_\lambda])$-*Latin trade*. It is easy to see that $c_2 + c_3 + \ldots + c_\lambda = s$. When $\lambda = 3$, a $(s, 3; [c_2, c_3])$-Latin trade is briefly denoted as (c_2, s)-Latin trade, where c_2 is actually the number of filled cells in which the entry is the same only in two of the three partial Latin squares.

Let $\{P_1, P_2, P_3\}$ be a (t, s)-Latin trade. If cell (i, j) is filled, then cell (i, j) is called 2-*different* if there are two partial Latin squares with the same entry at cell (i, j) among the three partial Latin squares. A filled cell (i, j) is called 3-*different* if the entry at cell (i, j) is different in each of the three partial Latin squares.

In a (t, s)-Latin trade, if cell (i, j) is filled, $S(i, j)$ denotes the set of all entries at cell (i, j) among the three partial Latin squares. $R(i)$ denotes the set of all entries at row i among the three partial Latin squares. $C(j)$ denotes the set of all entries at column j among the three partial Latin squares.

We define

$$\mathrm{Adm}(v) = (\{(t, s) : \ 0 \le t \le s \le v^2, s \notin [1, 11]\} \cup \mathcal{A}) \setminus \mathcal{E}$$

where $\mathcal{A} = \{(t, t) : \ t \in [4, 11] \setminus \{5\}\} \cup \{(6, 7), (6, 8), (0, 9), (6, 9), (8, 9), (8, 10), (9, 10), (6, 11), (8, 11), (9, 11), (10, 11)\}$, $\mathcal{E} = \{(1, 12), (2, 12), (5, 12), (0, 13), (1, 13), (2, 13), (3, 13), (5, 13), (0, 14), (1, 14), (1, 15), (1, 17)\}$, and $[a, b]$ denotes the set of integers x such that $a \le x \le b$.

The following results are from [2].

Lemma 1.1 $\mathrm{Fin}(v) \subseteq \mathrm{Adm}(v)$ *for any integer* $v \ge 2$.

Lemma 1.2 $\mathrm{Fin}(v) = \mathrm{Adm}(v)$ *for any integer* $v \ge 9$.

Lemma 1.3 $J_5 \setminus E_5 \subseteq \mathrm{Fin}(5) \subseteq J_5$ *where* $J_5 = \mathrm{Adm}(5) \setminus \{(6, 8), (0, 9), (6, 9), (6, 11), (0, 12), (3, 12), (4, 12), (0, 16), (1, 16), (0, 17)\}$ *and* $E_5 = \{(6, 12), (7, 12), (4, 13), (6, 13), (7, 13), (2, 14), (3, 14), (4, 14), (5, 14), (7, 14), (2, 15), (3, 15), (5, 15), (2, 16), (3, 16), (5, 16), (3, 17), (1, 18)\}$.

Lemma 1.4 $J_6 \setminus E_6 \subseteq \text{Fin}(6) \subseteq J_6$ *where* $J_6 = \text{Adm}(6) \setminus \{ (6,11), (0,12), (4,12), (0,15), (0,16), (1,16)\}$ *and* $E_6 = \{(6,12), (7,12), (4,13), (6,13), (7,13), (2,14), (3,14), (5,14), (6,14), (7,14), (2,15), (3,15), (4,15), (5,15), (7,15), (2,16), (3,16), (2,17), (3,17), (5,17), (1,18), (2,18), (3,18), (1,19), (1,20), (1,21) \}$.

Lemma 1.5 $J_7 \setminus E_7 \subseteq \text{Fin}(7) \subseteq J_7$ *where* $J_7 = \text{Adm}(7) \setminus (\{(0,s) : s \in [15,18]\} \cup \{(1,16)\})$ *and* $E_7 = \{(2,14), (3,14), (5,14), (6,14), (7,14), (2,15), (4,15), (2,16), (3,16), (2,17), (3,17), (6,17), (1,18), (2,18), (3,18), (4,18), (1,19), (2,19), (1,20), (2,20), (3,20), (1,21), (3,21), (1,22)\}$.

Lemma 1.6 $\text{Adm}(8) \setminus E_8 \subseteq \text{Fin}(8)$, *where* $E_8 = \{ (1,18), (2,18), (1,20), (1,21), (1,22), (2,23), (1,26), (1,29), (1,30), (1,33) \}$.

2 A non-existence example

The following lemmas are simple but very useful.

Lemma 2.1 [2]

(1) *Any (t,s)-Latin trade must contain at least two rows and at least two columns, with at least two entries in each row and each column.*

(2) *If row i (or column j) contains only two filled cells in a (t,s)-Latin trade, then each filled cell of row i (or column j) must be 2-different.*

(3) *If row i (or column j) contains only three filled cells, and two of them are 3-different in a (t,s)-Latin trade, then the third filled cell of row i (or column j) must also be 3-different.*

Lemma 2.2

(1) *If row i contains only four 3-different filled cells in a (t,s)-Latin trade, (i,a) and (i,b) are two filled cells, where $a \neq b$, then $R(i) = S(i,a) \cup S(i,b)$, $|(S(i,a) \cap S(i,b)| = 2$.*

(2) *If row i contains only three 3-different and one 2-different filled cells in a (t,s)-Latin trade, (i,a) is 2-different cell, (i,b) is 3-different cell, then $R(i) = S(i,a) \cup S(i,b)$.*

(3) *If row i contains only three 3-different and one 2-different filled cells in a (t,s)-Latin trade, then there exist only two 3-different filled cells (i,a) and (i,b) where $a \neq b$, such that $S(i,a) = S(i,b)$.*

(4) *If row i contains only five 3-different filled cells (i,a_j) $(j = 1,2,3,4,5)$ in a (t,s)-Latin trade with $S(i,a_1) = S(i,a_2)$, then $|S(i,a_3) \cap S(i,a_4) \cap S(i,a_5)| = 2$.*

(5) *If row* i *contains only four filled cells* (i, a_j) $(j = 1, 2, 3, 4)$ *in a* (t, s)*-Latin trade with* $S(i, a_1) = S(i, a_2)$ *and* $|S(i, a_1)| = 3$*, then* $|S(i, a_3)| = 2$ *or* $|S(i, a_4)| = 2$*.*

Proof The conclusions follow by an exhaustive hand search. □

Lemma 2.3

(1) *If row* i *contains only* n *filled cells* (i, a_j) $(j = 1, 2, \ldots, n)$ *in a* (t, s)*-Latin trade,* (i, a_k) *is a filled cell, then* $R(i) = \bigcup_{j=1, j \neq k}^{n} S(i, a_j)$*.*

(2) *If row* i *contains only* n *filled cells* (i, a_j) $(j = 1, 2, \ldots, n)$ *in a* (t, s)*-Latin trade,* (i, a_k) *and* $(i, a_{k'})$ *are two* 3*-different cells where* $k \neq k'$*, then*

$$R(i) = \bigcup_{j=1, j \neq k, k'}^{n} S(i, a_j).$$

Proof (1). For any $x \in R(i)$, if $x \notin S(i, a_k)$, then $x \in \bigcup_{j=1, j \neq k}^{n} S(i, a_j)$. If $x \in S(i, a_k)$, then x appears exactly 3 times in row i among the three partial Latin squares of the Latin trade. We have $x \in \bigcup_{j=1, j \neq k}^{n} S(i, a_j)$, that is to say $R(i) \subseteq \bigcup_{j=1, j \neq k}^{n} S(i, a_j)$. Clearly, $\bigcup_{j=1, j \neq k}^{n} S(i, a_j) \subseteq R(i)$. The conclusion then follows.

(2). For any $x \in R(i)$, if $x \notin S(i, a_k)$ and $x \notin S(i, a_{k'})$, then

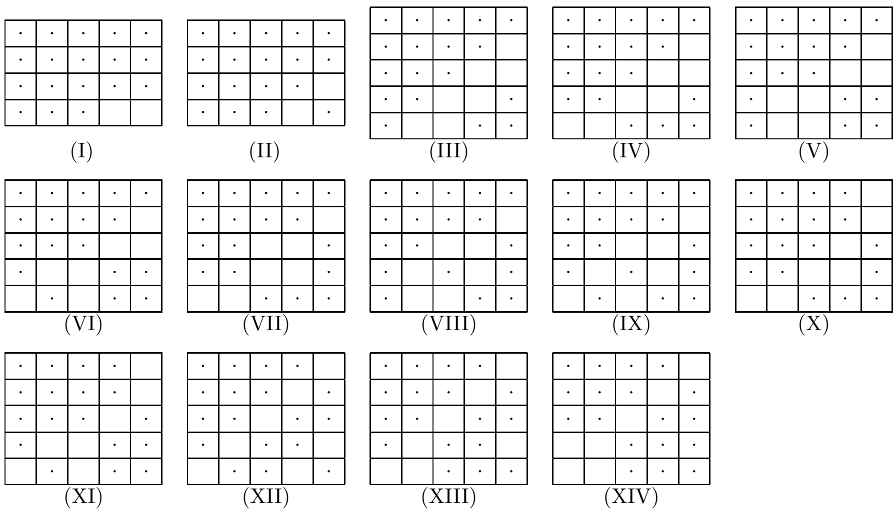
$$x \in \bigcup_{j=1, j \neq k, k'}^{n} S(i, a_j).$$

If $x \in S(i, a_k)$ or $x \in S(i, a_{k'})$, then x appears exactly 3 times in row i among the three partial Latin squares of the Latin trade. Since cell (i, a_k) and cell $(i, a_{k'})$ are 3-different, we have $x \in \bigcup_{j=1, j \neq k, k'}^{n} S(i, a_j)$, that is to say $R(i) \subseteq \bigcup_{j=1, j \neq k, k'}^{n} S(i, a_j)$. Clearly, $\bigcup_{j=1, j \neq k, k'}^{n} S(i, a_j) \subseteq R(i)$. The assertion thus follows. □

Lemmas 2.2–2.3 are also true if row i is replaced by column i and $R(i)$ is replaced by $C(i)$ correspondingly.

Lemma 2.4 $(1, 18) \notin \mathrm{Fin}(8)$.

Proof By (1) and (2) of Lemma 2.1, every non-empty row (or column) has at least 3 entries in a $(1, 18)$-Latin trade. If the number of non-empty rows (or columns) in a $(1, 18)$-Latin trade is more than 6, we have $s > 18$. It is impossible. If the number of non-empty rows (or columns) in a $(1, 18)$-Latin trade is 6, then every row (or column) has exactly 3 filled cells. It is also impossible by (3) of Lemma 2.1. Hence, the only possible partitions of 18 for the number of entries per row (or per column) are (1) 5, 5, 5, 3; (2) 5, 5, 4, 4; (3) 5, 4, 3, 3, 3; (4) 4, 4, 4, 3, 3. The possible skeletons for a $(1, 18)$-Latin trade are of the following, or of their transposes:

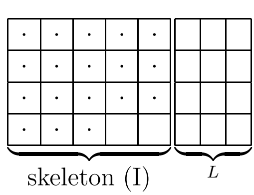


Case (I): By (3) of Lemma 2.1, without loss of generality, we may assume that cell $(1,1)$ is 2-different. Then all cells except cell $(1,1)$ in the skeleton (I) are 3-different. By (3) of Lemma 2.2, without loss of generality, we may assume that $S(2,1)=S(3,1)$, or $S(2,1)=S(4,1)$.

If $S(2,1)=S(3,1)$, by (2) of Lemma 2.3, $C(4)$, $C(5)\subseteq R(2)$. Since $S(3,1)$, $S(3,4)$, $S(3,5)\subseteq R(2)$, it then follows $R(3)\subseteq R(2)$ by (2) of Lemma 2.3. Evidently, $S(3,2)$, $S(3,3)\subseteq R(2)$. Similarly by (2) of Lemma 2.3, we have $C(2)$, $C(3)\subseteq R(2)$. By (2) of Lemma 2.3, $R(4)\subseteq R(2)$. Finally, according to (1) of Lemma 2.3, $C(1)\subseteq R(2)$.

If $S(2,1)=S(4,1)$, by (2) of Lemma 2.3, $R(4)$, $C(4)$, $C(5)\subseteq R(2)$. Since $S(2,2)$, $S(2,3)$, $S(4,2)$, $S(4,3)\subseteq R(2)$, it then follows $C(2)$, $C(3)\subseteq R(2)$ by (2) of Lemma 2.3. By (1) of Lemma 2.3, $R(1)$, $R(3)\subseteq R(2)$.

Hence, in both two cases, for any filled cell (a,b) in the skeleton (I), we have $S(a,b)\subseteq R(2)$, so the skeleton (I) can only be filled with 5 symbols. Suppose that the skeleton (I) could be embedded in a Latin square of order 8 on symbol set X. Let L be a 4×3 array as below.



Let Y be the set of symbols in the skeleton (I). Clearly, $|Y|=5$ and $|X|=8$. In L, the first three rows contain every symbol in $X\setminus Y$ respectively. The last row contains at least one symbol in $X\setminus Y$. So L contains at least 10 symbols in $X\setminus Y$.

However, noticing any symbol in $X \setminus Y$ appears at most 3 times in L, L contains at most 9 symbols in $X \setminus Y$. That is impossible.

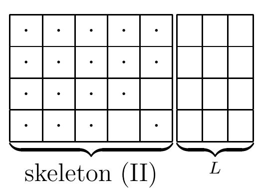
Case (II): By (3) of Lemma 2.1, without loss of generality, we may assume that cell $(1,1)$ or cell $(3,1)$ is 2-different.

When cell $(1,1)$ is 2-different, then all cells except cell $(1,1)$ in the skeleton (II) are 3-different. Without loss of generality, we may assume that $S(2,1) = S(3,1)$ or $S(3,1) = S(4,1)$ by (3) of Lemma 2.2.

If $S(2,1) = S(3,1)$, by (2) of Lemma 2.3, $C(4)$, $C(5) \subseteq R(2)$. Since $S(3,1)$, $S(3,4) \subseteq R(2)$, it then follows $R(3) \subseteq R(2)$ by (2) of Lemma 2.3. Evidently, $S(3,2)$, $S(3,3) \subseteq R(2)$. Similarly by (2) of Lemma 2.3, $C(2)$, $C(3) \subseteq R(2)$. Finally, by (1) of Lemma 2.3, $R(1)$, $R(4) \subseteq R(2)$.

If $S(3,1) = S(4,1)$, let $A = R(3) \cup S(4,5)$. By (1) of Lemma 2.2, we have $|S(4,1) \cap S(4,5)| = 2$, and hence $|A| \leq 5$. Similar arguments as the subcase of $S(2,1) = S(3,1)$ show that for any filled cell (a,b) in the skeleton, we have $S(a,b) \subseteq A$.

Hence, in both subcases when $(1,1)$ is 2-different, the skeleton (II) can only be filled with 5 symbols. When cell $(3,1)$ is 2-different, similarly the skeleton (II) can only be filled with 5 symbols. Suppose that the skeleton (II) could be embedded in a Latin square of order 8 on symbol set X. Let L be a 4×3 array as below.



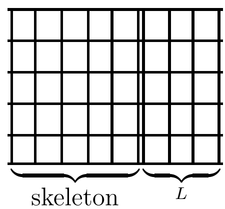
Let Y be the set of symbols in skeleton (II). Clearly, $|Y| = 5$ and $|X| = 8$. In L, the first two rows contain every symbol in $X \setminus Y$ respectively. The last two rows contain at least 2 symbols in $X \setminus Y$. So L contains at least 10 symbols in $X \setminus Y$. However, noting that any symbol in $X \setminus Y$ appears at most 3 times in L, L contains at most 9 symbols in A. That is impossible.

Case (III): By (2) of Lemma 2.3, we have $C(3) = R(3)$, and $C(4) = R(4) = C(5) = R(5)$. Clearly, $|C(3)| = 3$ and $|C(4)| = 3$. So, $C(3)$, $R(3)$, $C(4)$, $R(4)$, $C(5)$, $R(5) \subseteq R(2)$, then by (1) of Lemma 2.3, $C(1)$, $C(2) \subseteq R(2)$. For any filled cell (a,b) in the skeleton (III), we have $S(a,b) \subseteq R(2)$. It is impossible.

For Cases (IV) and (VI)-(IX), similar arguments as Case (III) show that all these skeletons do not exist.

Case (V): According to (2) of Lemma 2.3, it then follows $R(4)$, $R(5) \subseteq C(5)$ and $C(2)$, $C(3) \subseteq R(3)$. By (5) of Lemma 2.2, cell $(2,4)$ is 2-different. Since $S(1,2) = S(1,3)$, we have $|S(1,1) \cap S(1,4) \cap S(1,5)| = 2$ by (4) of Lemma 2.2. In a similar way, we have $|S(1,1) \cap S(2,1) \cap S(3,1)| = 2$. It is easy to see that there exists an element a with $a \in S(1,1) \cap S(1,4) \cap S(1,5) \cap S(2,1) \cap S(3,1)$. Evidently, $S(3,1) = S(1,2) = S(1,3)$. Hence, a appears 5 times in row 1, which is impossible.

For Cases (X)-(XIV), similar arguments as Case (II) show that the skeletons can only be filled with 4 symbols. Suppose that one of the skeletons (X)-(XIV) could be embedded in a Latin square of order 8 on symbol set X. Let L be a 5×3 array as below.



Let Y be the set of symbols in the skeleton. Clearly, $|Y| = 4$ and $|X| = 8$. In L, the first three rows contain at least 3 symbols in $X \setminus Y$ respectively. The last two rows contain 2 symbols in $X \setminus Y$ respectively. So L contains at least 13 symbols in $X \setminus Y$. However, noting that any symbol in $X \setminus Y$ appears at most 3 times in L, L contains at most 12 symbols in $X \setminus Y$. That is impossible. □

Remark: Based on the arguments of Lemma 2.4, the possible skeletons for a $(1,18)$-Latin trade are of Case (I), Case (II), and Cases (X)-(XIV), or of their transposes. Similarly, we have that $(1,18) \notin \mathrm{Fin}(v)$ for $v = 6,7$.

3 New examples

A partial Latin square P is called $(m \times n)$-*completable* if the empty cells of P can be filled to form an $m \times n$ Latin rectangle. Clearly, we should note that if one of the three partial Latin squares in a (t,s)-Latin trade is $(m \times n)$-completable to an $m \times n$ Latin rectangle then all of them are. Thus, we use the term $(m \times n)$-*completable* (t,s)-*Latin trade*, if any one of the partial Latin squares is $(m \times n)$-completable in a (t,s)-Latin trade.

In Appendix, the three partial Latin squares are superimposed, the position of each of the three entries in each filled cell determines to which partial Latin square that entry belongs. In each example of Appendix, the three Latin rectangles can be filled to form Latin squares. From Appendix, we have the following lemmas.

Lemma 3.1 $(6,13)$, $(5,16) \in \mathrm{Fin}(5)$.

Lemma 3.2 $(6,14)$, $(7,14)$, $(3,17)$, $(5,17)$, $(1,20)$, $(1,21) \in \mathrm{Fin}(6)$.

Lemma 3.3 $(5,14)$, $(6,14)$, $(7,14)$, $(2,18)$, $(2,19)$, $(3,20)$, $(1,21)$, $(3,21)$, $(1,22) \in \mathrm{Fin}(7)$.

Lemma 3.4 $(2,18)$, $(1,20)$, $(1,21)$, $(1,22)$, $(2,23)$, $(1,26)$, $(1,29)$, $(1,30)$, $(1,33) \in \mathrm{Fin}(8)$.

Using Lemmas 3.1–3.3 and the Remark, we update Lemmas 1.3–1.5 as follows.

Lemma 3.5 $J_5 \setminus E_5 \subseteq \mathrm{Fin}(5) \subseteq J_5$ *where* $J_5 = \mathrm{Adm}(5) \setminus \{(6,8),\ (0,9),\ (6,9),\ (6,11),\ (0,12),\ (3,12),\ (4,12),\ (0,16),\ (1,16),\ (0,17)\}$ *and* $E_5 = \{\ (6,12),\ (7,12),\ (4,13),\ (7,13),\ (2,14),\ (3,14),\ (4,14),\ (5,14),\ (7,14),\ (2,15),\ (3,15),\ (5,15),\ (2,16),\ (3,16),\ (3,17),\ (1,18)\}$.

Lemma 3.6 $J_6 \setminus E_6 \subseteq \mathrm{Fin}(6) \subseteq J_6$ *where* $J_6 = \mathrm{Adm}(6) \setminus \{\ (6,11),\ (0,12),\ (4,12),\ (0,15),\ (0,16),\ (1,16),\ (1,18)\}$ *and* $E_6 = \{(6,12),\ (7,12),\ (4,13),\ (6,13),\ (7,13),\ (2,14),\ (3,14),\ (5,14),\ (2,15),\ (3,15),\ (4,15),\ (5,15),\ (7,15),\ (2,16),\ (3,16),\ (2,17),\ (2,18),\ (3,18),\ (1,19)\}$.

Lemma 3.7 $J_7 \setminus E_7 \subseteq \mathrm{Fin}(7) \subseteq J_7$ *where* $J_7 = \mathrm{Adm}(7) \setminus (\{(0,s) : s \in [15,18]\} \cup \{(1,16),\ (1,18)\})$ *and* $E_7 = \{(2,14),\ (3,14),\ (2,15),\ (4,15),\ (2,16),\ (3,16),\ (2,17),\ (3,17),\ (6,17),\ (3,18),\ (4,18),\ (1,19),\ (1,20),\ (2,20)\}$.

Theorem 3.8 $\mathrm{Adm}(8) \setminus \{(1,18)\} = \mathrm{Fin}(8)$.

Proof By Lemmas 1.6 and 3.4, $\mathrm{Adm}(8) \setminus \{(1,18)\} \subseteq \mathrm{Fin}(8)$. By Lemma 1.1 and 2.4, it then follows that $\mathrm{Fin}(8) \subseteq \mathrm{Adm}(8) \setminus \{(1,18)\}$. The conclusion then follows immediately. $\square$

Appendix

$(6,13) \in \mathrm{Fin}(5)$

1	0 2 2	3 0 3	4	2 3 0
2	3 3 0	4	0 1 1	1 0 3
3	4	1 1 0	2 0 2	0 2 1
4	2 0 3	0 3 1	1 2 0	3 1 2

$(5,16) \in \mathrm{Fin}(5)$

1	0 2 3	3 0 4	4 4 2	2 3 0
2	3 4 0	4 3 1	0 1 4	1 0 3
3	4 0 4	1 4 0	2 2 1	0 1 2
4	2 3 2	0 1 3	1 0 0	3 2 1

$(6,14) \in \mathrm{Fin}(6)$

0 1 2	1 2 0	2 0 3	4	3 3 1	5
3 2 0	2 0 1	0 3 2	5	1 1 3	4
1 0 1	0 1 2	4	2 2 0	5	3
2 3 3	5	3 2 0	0 0 2	4	1

$(7,14) \in \mathrm{Fin}(6)$

0 0 1	1 2 0	2 1 2	4	5	3
3 1 2	2 0 1	1 2 3	5	0 3 0	4
1 2 3	0 1 2	3 3 1	2 0 0	4	5
2 3 0	4	5	0 2 2	3 0 3	1

$(3,17) \in \mathrm{Fin}(6)$

0 1 2	2 4 0	1 3 1	3 0 4	4 2 3	5
1 0 3	0 1 1	3 4 4	4 3 0	5	2
2 3 1	1 2 4	4 1 3	5	3 4 2	0
3 2 0	4 0 2	5	0 4 3	2 3 4	1

$(5,17) \in \mathrm{Fin}(6)$

0 1 2	1 2 3	3 0 4	4 3 1	2 4 0	5
1 0 0	4 3 1	0 4 3	3 1 4	5	2
2 2 3	3 4 2	4 3 0	5	0 0 4	1
3 3 1	2 1 4	5	1 4 3	4 2 2	0

$(1,20) \in \mathrm{Fin}(6)$

0 0 1	2 4 3	1 2 4	4 3 0	3 1 2	5
1 2 3	0 3 2	2 4 1	3 0 4	5	4 1 0
2 3 0	3 0 4	4 1 2	5	1 2 3	0 4 1
3 1 2	4 2 0	5	0 4 3	2 3 1	1 0 4

$(1,21) \in \mathrm{Fin}(6)$

0 1 2	1 3 5	2 5 0	3 0 3	5 2 1	4
1 0 3	3 1 2	0 2 1	4	2 3 0	5
2 3 0	5 2 3	4	0 5 2	3 0 5	1
3 2 1	4	5 1 2	2 3 5	1 5 3	0
4	2 5 1	1 0 5	5 2 0	0 1 2	3

$(5,14) \in \mathrm{Fin}(7)$

0 1 2	1 2 3	2 3 1	3 4 4	4 0 0	5	6
1 2 3	2 3 1	3 1 2	0	6	4	5
2 3 1	3 1 2	1 2 3	6	5	0	4
3 0 0	5	6	4 3 3	0 4 4	1	2

$(6,14) \in \mathrm{Fin}(7)$

0 0 1	1 2 0	3 1 2	2 3 3	6	4	5
1 2 0	0 1 2	2 3 1	4	3 0 3	5	6
3 1 2	2 0 1	1 2 3	5	0 3 0	6	4
2 3 3	4	6	3 2 2	5	0	1

$(7,14) \in \mathrm{Fin}(7)$

0 0 1	1 2 0	2 1 2	6	5	4	3
3 1 2	2 0 1	1 2 3	5	0 3 0	6	4
1 2 3	0 1 2	3 3 1	2 0 0	4	5	6
2 3 0	4	6	0 2 2	3 0 3	1	5

$(2,18) \in \mathrm{Fin}(7)$

0 1 2	1 0 0	2 3 4	4 2 3	3 4 1	5	6
2 0 0	0 3 4	4 2 3	3 4 2	5	6	1
1 3 4	3 4 1	0	6	4 1 3	2	5
3 4 1	4 1 3	6	5	1 3 4	0	2
4 2 3	5	3 4 2	2 3 4	6	1	0

$(2,19) \in \mathrm{Fin}(7)$

0 1 2	1 2 0	2 0 3	3 4 1	4 3 4	5	6
2 4 4	0 3 2	3 2 0	4 0 3	6	1	5
1 2 0	2 1 3	0 3 2	5	3 0 1	6	4
4 0 1	6	5	0 1 4	1 4 0	2	3
5	3 0 1	6	1 3 0	0 1 3	4	2

$(3,20) \in \mathrm{Fin}(7)$

0 0 1	1 4 0	2 5 4	4 3 5	3 1 2	5 2 3	6
1 2 3	4 1 5	5 4 2	3 5 4	2 3 1	6	0
2 3 0	0 0 4	4 2 5	5 4 3	6	3 5 2	1
3 1 2	5 5 1	6	0	1 2 3	2 3 5	4

$(1, 21) \in \mathrm{Fin}(7)$

0 0 1	1 4 5	5 1 4	3 5 2	2 3 0	4 2 3	6
1 2 3	4 1 0	0 4 1	6	3 0 2	2 3 4	5
2 3 0	0 5 4	4 0 5	5 2 3	6	3 4 2	1
3 1 2	5 0 1	1 5 0	2 3 5	0 2 3	6	4

$(3, 21) \in \mathrm{Fin}(7)$

0 0 1	2 2 4	1 4 3	3 1 0	4 3 2	5	6
1 2 3	4 3 2	3 1 4	5	6	2 4 1	0
2 3 0	3 4 3	5	6	0 2 4	4 0 2	1
3 1 2	5	6	0 3 1	2 0 3	1 2 0	4
5	6	4 3 1	1 0 3	3 4 0	0 1 4	2

$(1, 22) \in \mathrm{Fin}(7)$

0 1 2	1 0 3	2 4 0	3 2 6	6 3 4	4 6 1	5
1 2 0	0 3 1	4 0 2	2 6 3	3 4 6	5	6 1 4
2 0 3	3 1 0	0 2 4	6 3 2	5	1 4 6	4 6 1
3 3 1	2	5	0	4 6 3	6 1 4	1 4 6

$(2, 18) \in \mathrm{Fin}(8)$

0 1 2	1 0 0	2 3 4	4 2 3	3 4 1	5	6	7
2 0 0	0 3 4	4 2 3	3 4 2	7	6	5	1
1 3 4	3 4 1	5	6	4 1 3	7	0	2
3 4 1	4 1 3	7	5	1 3 4	0	2	6
4 2 3	6	3 4 2	2 3 4	0	1	7	5

$(1, 20) \in \mathrm{Fin}(8)$

0 1 2	1 4 0	3 0 1	2 3 4	4 2 3	5	6	7
1 3 4	4 0 1	0 1 3	6	7	3 4 0	2	5
3 4 1	0 1 4	1 3 0	7	6	4 0 3	5	2
2 0 0	5	7	4 2 3	3 4 2	0 3 4	1	6
4 2 3	6	5	3 4 2	2 3 4	7	0	1

$(1, 21) \in \mathrm{Fin}(8)$

0 1 2	2 0 3	3 4 0	1 2 4	4 3 1	5	7	6
1 0 3	3 4 0	0 3 4	6	7	4 1 1	2	5
2 3 0	0 2 4	4 0 3	7	5	3 4 2	6	1
3 2 1	6	7	4 1 2	1 4 3	2 3 4	5	0
5	4 3 2	6	2 4 1	3 1 4	1 2 3	0	7

$(1, 22) \in \mathrm{Fin}(8)$

0 1 2	1 2 3	2 5 6	3 0 5	6 3 0	5 6 1	7	4
1 2 0	2 3 1	5 6 2	0 5 3	3 0 6	4	6 1 5	7
2 3 3	3 1 2	6 2 5	4	7	1 5 6	5 6 1	0
3 0 1	4	7	5 3 0	0 6 3	6 1 5	1 5 6	2

$(2, 23) \in \mathrm{Fin}(8)$

0 1 2	3 0 1	2 4 3	4 2 0	1 3 4	5	6	7
1 0 3	4 1 0	3 2 4	7	5	2 4 1	0 3 2	6
2 3 1	6	7	0 4 2	4 1 3	1 2 4	3 0 0	5
3 2 0	0 4 3	4 3 2	2 0 4	7	6	5	1
7	1 3 4	6	5	3 4 1	4 1 2	2 2 3	0

$(1, 26) \in \mathrm{Fin}(8)$

0 1 2	1 0 3	2 4 5	5 3 1	3 5 4	4 2 0	7	6
1 0 3	0 4 1	5 1 4	3 5 0	4 3 5	7	6	2
2 3 0	3 1 4	4 2 1	1 0 3	6	5	0 4 2	7
3 2 1	4 3 0	6	7	1 1 3	0 4 2	2 0 4	5
6	7	1 5 2	0 1 5	5 4 1	2 0 4	4 2 0	3

$(1, 29) \in \mathrm{Fin}(8)$

0 1 2	1 0 3	2 3 0	3 4 1	5 6 7	7 5 4	4 2 6	6 7 5
1 0 3	0 1 2	3 2 1	4 3 5	6 7 0	5 4 7	2 6 4	7 5 6
2 3 0	3 2 1	0 1 3	1 5 4	7 0 6	4 7 5	6 4 2	5 6 7
3 2 1	2 3 0	1 0 2	5 1 3	0 5 5	6	7	4

$(1, 30) \in \mathrm{Fin}(8)$

0 1 2	1 0 3	2 3 0	3 4 1	5 6 7	7 5 4	4 2 6	6 7 5
1 0 3	0 1 2	3 2 1	4 3 5	6 7 0	5 4 7	2 6 4	7 5 6
2 3 0	3 2 1	0 1 3	1 5 4	7 0 5	4 7 6	6 4 2	5 6 7
3 2 1	2 3 0	1 0 2	5 1 3	0 5 6	6 6 5	7	4

$(1, 33) \in \mathrm{Fin}(8)$

3	2	1 4 5	0 5 4	7	6	5 0 1	4 1 0
4	5 6 6	6 5 1	7	0	2 1 3	3 2 5	1 3 2
5 6 7	4 7 5	7 1 6	6 4 0	1 3 2	0 2 1	2 5 3	3 0 4
6 7 5	7 5 4	4 6 7	5 0 6	3 2 1	1 3 0	0 1 2	2 4 3
7 5 6	6 4 7	5 7 4	4 6 5	2 1 3	3 0 2	1 3 0	0 2 1

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