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A note on independent domination in graphs of girth 6

Julie Haviland*

Exeter College
Hele Road, Exeter EX4 4JS
U.K.
`juliehaviland@exe-coll.ac.uk`

Abstract

Let G be a simple graph of order n, maximum degree Δ and minimum degree $\delta \geq 2$. The *independent domination number* $i(G)$ is defined to be the minimum cardinality among all maximal independent sets of vertices of G. The *girth* $g(G)$ is the minimum length of a cycle in G. We establish best possible upper and lower bounds, as functions of n, Δ and δ, for the independent domination number of graphs G with $g(G) = 6$.

1 Introduction

Let $G = (V, E)$ be a simple graph of order $|V| = n$, maximum degree Δ and minimum degree $\delta \geq 2$. An *independent set* is a set of pairwise non-adjacent vertices of G. A subset I of V is a *dominating set* if every vertex of $V - I$ has at least one neighbour in I. The *independent domination number* $i(G)$ is defined to be the minimum cardinality among all maximal independent sets of G. An independent set is maximal if and only if it is dominating, so $i(G)$ is also the minimum cardinality of an independent dominating set in G.

A number of previous papers on the parameter $i(G)$ have been focussed upon finding upper bounds, as functions of n and/or δ, for general and regular graphs [3]–[6], [9], [11]. The present author proved analogous results for triangle-free graphs in [7]. If the *girth* $g(G)$ is defined to be the minimum length of a cycle in G, clearly triangle-free graphs satisfy $g(G) \geq 4$, and indeed, all graphs from [7] which are either extremal or conjectured to be extremal have girth exactly 4. This observation prompted the research of [8], which featured sharp upper and lower bounds for $i(G)$ in graphs of girth 5, as functions of n, Δ and δ.

Motivated by these earlier investigations, the aim of this note is to provide best possible upper and lower bounds, as functions of n, Δ and δ, for the independent

* Honorary Research Fellow, School of Engineering, Computing and Mathematics, University of Exeter, North Park Road, Exeter EX4 4QF, U.K.

domination number in graphs of girth 6. Moreover, as we shall demonstrate, it is surprisingly straightforward to find minimum maximal independent sets in the corresponding extremal graphs. We conclude our study with a brief examination of why our results are unlikely to lead to obvious extensions for graphs of higher girth.

In what follows, we abbreviate $i(G)$ to i and $g(G)$ to g. The *open neighbourhood* in G of a vertex $v \in V$ will be denoted by $\Gamma(v) = \{u \in V : uv \in E\}$, and that of a set of vertices $X \subset V$ by $\Gamma(X) = \bigcup_{x\in X} \Gamma(x) \cap (V - X)$. For disjoint vertex sets $X, Y \subset V$, write $e(X, Y)$ for the number of edges with one endvertex in X and the other in Y.

2 Results

We start by determining the maximum value of δ for graphs of girth 6, which requires the following well-known result of Tutte [12].

Proposition 1 (Tutte [12]) *Any graph of order n, minimum degree $\delta \geq 3$ and girth $g \geq 3$ satisfies*

$$n \geq \begin{cases} 1 + \frac{\delta}{\delta-2}\left[(\delta-1)^{(g-1)/2} - 1\right] & \textit{if } g \textit{ is odd;} \\ \frac{2}{\delta-2}\left[(\delta-1)^{g/2} - 1\right] & \textit{if } g \textit{ is even.} \end{cases}$$

Corollary 2 *Any graph of order n, minimum degree $\delta \geq 2$ and girth $g = 6$ satisfies*

$$\delta \leq \begin{cases} \frac{1}{2}(n-2) & \textit{if } \delta = 2; \\ \frac{1}{2}[(2n-3)^{1/2} + 1] & \textit{if } \delta \geq 3. \end{cases}$$

Proof. If $\delta = 2$ then the proof of Proposition 1 (see for example [2], pp.68–9) is easily adapted for this case, as follows. Suppose $g = 6$. Choosing any two adjacent vertices x and y, there is no vertex z for which G contains x-z and y-z paths of length at most 2, since otherwise G has a cycle of length at most 5. Hence there are at least $\delta = 2$ vertices at distance j from $\{x, y\}$ for $j = 1, 2$. Therefore $n \geq 2 + 2\delta$, as claimed.

Otherwise $\delta \geq 3$, so taking $g = 6$ in the even case of Proposition 1, we obtain $n \geq 2(\delta^2 - \delta + 1)$. Solving this quadratic inequality for δ yields the stated upper bound. □

Note that if $\delta = 2$ and $n = g = 6$ then $\delta = (n-2)/2 = [(2n-3)^{1/2} + 1]/2$. The proofs of Proposition 1 and Corollary 2 imply that any extremal graph is regular of degree δ; such graphs are called Moore graphs of degree $\delta = [(2n-3)^{1/2} + 1]/2$ and girth $g = 6$. When $\delta = 2$ and $g = 6$, the only Moore graph is the hexagon C_6. In addition, Singleton [10] proved that, for every prime number p and natural number s, there is a bipartite Moore graph of degree $\delta = 1 + p^s$ and girth $g = 6$ for each finite projective geometry of dimension 2 with δ points on a line. In the case $\delta = 3$ this graph is unique and known as the Heawood graph (see for example Bollobás [1], p.106).

We now derive best possible lower and upper bounds for the independent domination number of graphs of girth 6. Observe that all such graphs have the following three properties:
(†) G is triangle-free, so no pair of adjacent vertices have a common neighbour.
(‡) G is C_4-free, so every pair of non-adjacent vertices have at most one common neighbour.
($*$) G is C_5-free.
We exploit these conditions heavily in our proofs. Henceforth we assume that G is an extremal graph containing a minimum maximal independent set I.

Theorem 3 *Any graph of order n, minimum degree $\delta \geq 2$ and girth 6 satisfies $i \geq 2\delta - 2$.*

Proof. First we claim that $V - I$ is not an independent set. For suppose otherwise; since $\delta \geq 2$ then every vertex of $V - I$ has at least two neighbours in I. Choose any $z \in V - I$ and form the sets $Z = \Gamma(z) \cap I$ and $R = \{v \in V - I : \Gamma(v) \cap I \subseteq Z\}$. By the maximality of I, any vertex of $R - \{z\}$ has a common neighbour in Z with z. Thus by (†), R is an independent set. Therefore $R \cup (I - Z)$ is maximal independent for G, so $|R| + (i - |Z|) \geq i$, implying $|R| \geq |Z| \geq 2$ and $R - \{z\} \neq \emptyset$. However, by (‡), every vertex of $R - \{z\}$ has precisely one neighbour in Z, and hence in I. This contradiction proves our claim, and so we deduce that $G[V - I]$ contains at least one edge, xy say.

Let $X_1 = \Gamma(x) \cap I$ and $Y_1 = \Gamma(y) \cap I$, where $X_1 \cap Y_1 = \emptyset$ by (†). Now let $X_2 = \Gamma(x) \cap (V - I - \{y\})$ and $Y_2 = \Gamma(y) \cap (V - I - \{x\})$. Again by (†), $X_2 \cap Y_2 = \emptyset$ and both X_2 and Y_2 are independent sets. Moreover, we have the obvious lower bounds

$$|X_2| \geq \delta - 1 - |X_1| \quad \text{and} \quad |Y_2| \geq \delta - 1 - |Y_1|. \tag{1}$$

Define the sets $X_3 = \Gamma(X_2) \cap I$ and $Y_3 = \Gamma(Y_2) \cap I$. By (†), $X_3 \cap X_1 = \emptyset$ and $Y_3 \cap Y_1 = \emptyset$. As X_2 (resp. Y_2) is an independent set with all vertices therein adjacent to x (resp. y), then by (‡) all vertices of X_2 (resp. Y_2) have disjoint neighbourhoods in X_3 (resp. Y_3), so

$$|X_3| = e(X_2, X_3) \geq |X_2| \quad \text{and} \quad |Y_3| = e(Y_2, Y_3) \geq |Y_2|. \tag{2}$$

Furthermore $X_3 \cap Y_3 = \emptyset$, for any $z \in X_3 \cap Y_3$ must have neighbours $x' \in X_2$ and $y' \in Y_2$ and so $zx'xyy'z$ is a cycle in G, which contradicts ($*$).

Counting vertices of I and then applying the pairs of inequalities (2) and (1), we obtain

$$\begin{aligned} i &\geq |X_1| + |X_3| + |Y_1| + |Y_3| \\ &\geq |X_1| + |X_2| + |Y_1| + |Y_2| \\ &\geq 2\delta - 2, \end{aligned}$$

as claimed. □

Theorem 4 *Any graph of order n, maximum degree Δ, minimum degree $\delta \geq 2$ and girth 6 satisfies*

$$i \leq n - \Delta\delta + \Delta - \delta^2 + 3\delta - 4,$$

where $\Delta \leq (n - \delta^2 + 2\delta - 2)/\delta$.

Proof. Choose any vertex x of degree Δ and any vertex y such that $xy \in E$. By (†), x and y have disjoint neighbourhoods in G which are independent sets. Thus the open neighbourhood of $\{x, y\}$ is a (not necessarily maximal) independent set $X \cup Y$ of order at least $\Delta + \delta - 2$. As all vertices of X (resp. Y) have x (resp. y) as a common neighbour, then by (‡) all vertices of X (resp. Y) have disjoint neighbourhoods in $V - \{x, y\}$. Additionally, $\Gamma(X) \cap \Gamma(Y) = \emptyset$, because any $z \in \Gamma(X) \cap \Gamma(Y)$ necessarily has neighbours $x' \in X$ and $y' \in Y$ and so G contains a cycle $zx'xyy'z$, thereby contradicting (∗). We conclude that

$$\begin{aligned} i &\leq |V| - |\Gamma(X \cup Y)| \\ &= n - |\{x, y\}| - |\Gamma(X) - \{x\}| - |\Gamma(Y) - \{y\}| \\ &\leq n - 2 - (\Delta - 1)(\delta - 1) - (\delta - 1)^2 \\ &= n - \Delta\delta + \Delta - \delta^2 + 3\delta - 4, \end{aligned}$$

as stated. In order to verify the upper bound for Δ, it is easily seen that

$$\Delta + \delta - 2 \leq |X \cup Y| \leq |V| - |\Gamma(X \cup Y)| \leq n - \Delta\delta + \Delta - \delta^2 + 3\delta - 4,$$

which rearranges to yield the required result. □

From Theorems 3 and 4, we have $2\delta - 2 \leq i \leq n - \Delta\delta + \Delta - \delta^2 + 3\delta - 4$. When $\delta = \Delta = [(2n - 3)^{1/2} + 1]/2$, this chain of inequalities becomes an equality. Therefore Theorems 3 and 4 are best possible, with their respective bounds attained by the Moore graphs of girth 6. Furthermore, our final result reveals these extremal graphs to have an interesting property.

Theorem 5 *The Moore graphs of girth 6 have independent domination number $2\delta - 2$ and the open neighbourhood of every pair of adjacent vertices is a minimum maximal independent set.*

Proof. Let G be a Moore graph of girth 6 and choose any $xy \in E$. Arguing similarly to the proof of Theorem 4, the open neighbourhood of $\{x, y\}$ is an independent set of order $2\delta - 2$ with $2 + 2(\delta - 1)^2 = 2\delta^2 - 4\delta + 4 = n - 2\delta + 2$ neighbours, and hence is maximal. By Theorem 3, it must be a minimum maximal independent set. □

3 Concluding Remarks

The theorems proved here parallel directly those of [8] for the case $g = 5$, with Moore graphs attaining both upper and lower bounds for i whilst admitting a simple means of finding minimum maximal independent sets. Consequently, it is natural to seek an obvious generalisation of these results. However, as discussed by Bollobás in [1], pp.106–7, the only feasible larger values of g for which there can exist Moore graphs are 8 (Singleton gave examples of such graphs in [10]) and 12, so it seems unlikely that a neat characterisation is possible. Nevertheless, it remains an interesting open problem to investigate the independent domination number of graphs G with $g(G) > 6$.

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