AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **42** (2008), Pages 159–164

Decompositions of complete multigraphs into open trails

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Abstract

Balister [*Combin. Probab. Comput.* 12 (2003), 1–15] gave a necessary and sufficient condition for a complete multigraph ${}^rK_n$ to be arbitrarily decomposable into closed trails of prescribed lengths. In this article we solve the corresponding problem showing that the multigraphs ${}^rK_n$ are arbitrarily decomposable into open trails.

1 Introduction

Consider a graph G (without loops), whose number of edges we call the size of G and denote by $\|G\|$. Write $V(G)$ for the vertex set and $E(G)$ for the edge set of a graph G. Let ${}^rG$ be a graph obtained from G in such a way that each edge $xy \in E(G)$ occurs with multiplicity r. Notice that $V({}^rG) = V(G)$ and $\|{}^rG\| = r \cdot \|G\|$.

We say that a graph G is *Eulerian* if and only if there exists a closed trail which passes through every edge of G. Here and subsequently, a trail T of length n we identify with any sequence $(v_1, v_2, \ldots, v_{n+1})$ of vertices of T such that v_iv_{i+1} are distinct edges of T for $i = 1, 2, \ldots, n$. Notice that we do not require the v_i to be distinct. A trail T is closed if $v_1 = v_{n+1}$ and T is open if $v_1 \neq v_{n+1}$. However, a closed trail will be regarded as an Eulerian graph of size n.

A sequence of positive integers $\tau = (t_1, t_2, \ldots, t_p)$ is called *admissible for a graph* G if it adds up to $\|G\|$ and for each $i \in \{1, \ldots, p\}$ there exists an open trail of length t_i in G. Let $\tau = (t_1, t_2, \ldots, t_p)$ be an admissible sequence for G. If G is edge-disjointly decomposable into open trails $T_1, T_2, \ldots, T_p$ of lengths $t_1, t_2, \ldots, t_p$ respectively, then τ is called *realizable in* G and the sequence $(T_1, T_2, \ldots, T_p)$ is said to be *a* G*-realization of* τ or *a realization of* τ *in* G.

For edge-disjoint trails $T_1 = (v_1, \ldots, v_i)$ and $T_2 = (v_i, \ldots, v_n)$ let $T := T_1 \cup T_2$ denote a trail $(v_1, \ldots, v_i, \ldots, v_n)$. We say that the trail T is constructed by *gluing the trails* T_1 *and* T_2.

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This paper has been inspired by a paper of Balister [1] who considered graphs decomposable into closed trails. He proved that a complete graph K_n for n odd, and $K_n - I$, where I is a 1-factor in K_n for n even, is arbitrarily decomposable into closed trails. Among graphs arbitrarily decomposable into closed trails there are also complete bipartite graphs $K_{a,b}$ for a, b even (see Horňák and Woźniak [9]), complete tripartite graphs $K_{n,n,n}$ (see Billington and Cavenagh [3]), complete digraphs (see Balister [2]) and complete bipartite digraphs (see Cichacz [4]). This problem was also generalized to multigraphs; see [2]:

Theorem 1 ([2]) *Assume $n \geqslant 3$, $\sum_{i=1}^{p} t_i = r\binom{n}{2}$, and $t_i \geqslant 2$ for $i = 1, \ldots, p$. Then ${}^rK_n$ can be written as edge-disjoint union of closed trails of lengths $t_1, t_2, \ldots, t_p$ if and only if either*

a. r is even, or

b. r and n are both odd and $\sum_{t_i > 2} t_i \geqslant \binom{n}{2}$.

There is a natural question about a decomposition of graphs into edge-disjoint open trails. We proved the following theorem in [6]:

Theorem 2 ([6]) *The graph $K_{a,b}$ is arbitrarily decomposable into open trails if and only if one of the following conditions holds:*

1^0 $a = 1$ or

2^0 a and b are both even.

Arbitrarily decomposing into open trails for complete graphs is solved by Cichacz, Woźniak and Egawa [5]:

Theorem 3 ([5]) *If n is odd, then a complete graph K_n is arbitrarily decomposable into open trails.*

For oriented graphs the similar problem was considered by Meszka and Skupień ([10]). They showed that complete multidigraphs are arbitrarily decomposable into nonhamiltonian paths. However we proved that ([7]):

Theorem 4 ([7]) *If $\sum_{i=1}^{p} t_i = \|\overleftrightarrow{K}_n\|$ and $t_i \geqslant 1$ for $i = 1, \ldots, p$, then $\overleftrightarrow{K}_n$ can be decomposed as arc-disjoint unions of directed open trails of lengths $t_1, t_2, \ldots, t_p$, except in the case when $n = 3$ and $t_i = 2$ for all $i = 1, \ldots, p$.*

In this paper we prove a necessary and sufficient condition for any complete multigraph ${}^rK_n$ to be decomposable into open trails.

2 Main results

Our main theorem is the following:

Theorem 5 *A complete multigraph* ^rK_n *is arbitrarily decomposable into open trails if and only if one of the following conditions holds:*

1^0 r is even

2^0 n is odd

3^0 $n = 2$ or $n = 4$.

Proof. **Necessity.** We show that if $n \geqslant 6$ is even then there exists an admissible sequence τ for ^rK_n with odd r such that there is no realization for τ in ^rK_n. It easy to check, that in ^rK_n there exists an open trail of length $\frac{n-2}{2}$. Moreover, if we remove $\frac{n-2}{2}$ independent edges from G, then by Euler's theorem we obtain an open trail of length $\|G\| - \frac{n-2}{2}$. It implies that the sequence $\tau = (\frac{n-2}{2}, \|G\| - \frac{n-2}{2})$ is admissible for ^rK_n, but it is not realizable (because if T_1 denotes an open trail of length $\frac{n-2}{2}$ in ^rK_n, then in $^rK_n - T_1$ there is at least four vertices of odd degree).

Sufficiency. Let $\tau = (t_1, t_2, \ldots, t_p)$ be an admissible sequence for ^rK_n. Note that $p > 1$. Let $s_i := t_i + \cdots + t_p$ for $i \in \{1, \ldots, p\}$. We divide the proof into two parts:
Case 1. Let us consider a complete multigraph ^rK_n with even r. Observe that an open trail in $^rK_2$ is of odd lengths. It obviously follows that $^rK_2$ is arbitrarily decomposable into open trails. From now on assume that $n \geqslant 3$. We argue by induction on r. The basic idea of the proof is to consider ^rK_n as an edge-disjoint union of multigraphs ^2K_n and $^{r-2}K_n$.

Forgetting the orientation of the edges, by Theorem 4 any multigraph ^2K_n different than $^2K_3$ is arbitrarily decomposable into open trails. It is also easy to see that there exists a realization of $\tau = (2, 2, 2)$ in $^2K_3$. Hence $^2K_3$ is arbitrarily decomposable.

Let $r \geq 4$. A multigraph ^rK_n is an edge-disjoint union of an Eulerian multigraph ^2K_n and a multigraph $^{r-2}K_n$ of sizes $n(n-1)$ and $(r-2)\frac{n(n-1)}{2}$, respectively. Suppose first that one of terms of τ is greater than $n(n-1)$. Possibly permuting our sequence without loss of generality we can assume that $t_p > n(n-1)$. Define $t'_p = t_p - n(n-1)$ and consider a sequence $\tau' = (t_1, \ldots, t_{p-1}, t'_p)$. By induction there exists a realization $T_1, \ldots, T_{p-1}, T'_p$ of the sequence τ' in $^{r-2}K_n$. Let $T_p := T'_p \cup {}^2K_n$. Then we obtain a realization of a sequence τ in ^rK_n. Assume now that $t_i \leq n(n-1)$ for each $i \in \{1, \ldots, p\}$. Observe that then there exists $i_0 \in \{1, \ldots, p-1\}$ such that $s_{i_0+1} \leq (r-2)\frac{n(n-1)}{2}$ and $s_{i_0} > (r-2)\frac{n(n-1)}{2}$. We consider the following cases:
Case 1.1: Let $s_{i_0+1} < (r-2)\frac{n(n-1)}{2}$ and $s_{i_0} > (r-2)\frac{n(n-1)}{2}$. Let $\tau_1 = (t'_{i_0}, \ldots, t_{p-1}, t_p)$ and $\tau_2 = (t_1, \ldots, t''_{i_0})$, where $t'_{i_0} := (r-2)\frac{n(n-1)}{2} - s_{i_0+1} > 0$ and $t''_{i_0} := t_{i_0} - t'_{i_0} > 0$. By induction we can find realizations of the sequences τ_1 and τ_2 in $^{r-2}K_n$ and ^2K_n, respectively. Moreover, because $n \geqslant 3$ we can glue the trails T'_{i_0} and T''_{i_0} this way that we obtain an open trail T_{i_0} of length t_{i_0} and hence we get a ^rK_n-realization of τ.

Case 1.2: Let $s_{i_0+1} = (r-2)\frac{n(n-1)}{2}$. Suppose first that $t_1 = \cdots = t_p = n(n-1)$. Below we show that a multigraph ${}^rK_n$ with even r is decomposable into $\frac{r}{2}$ edge-disjoint open trails of lengths $n(n-1)$.

Let us consider first a multigraph ${}^rK_3$ with the set of vertices $V({}^rK_3) = (x_1, x_2, x_3)$. For $r = 4$ we define two edge-disjoint open trails of length six: $T_1 = (x_1, x_2, x_3, x_1, x_3, x_1, x_3)$ and $T_2 = (x_1, x_2, x_1, x_2, x_3, x_2, x_3)$ in ${}^4K_3$. Assume that $r = 6$. Let

$$T_i = (x_i, x_{(i+1) \bmod 3}, x_{(i+2) \bmod 3}, x_{(i+1) \bmod 3}, x_{(i+2) \bmod 3}, x_i, x_{(i+2) \bmod 3})$$

for $i \in \{1, 2, 3\}$. Observe that (T_1, T_2, T_3) is ${}^6K_3$-realization of $\tau = (6, 6, 6)$.

Let us assume now that $r \geq 8$. Notice that for each even $r \geq 8$ there exist integers α, β such that $r = \alpha \cdot 4 + \beta \cdot 6$. So, we can consider a multigraph ${}^rK_3$ as an edge-disjoint union of α multigraphs ${}^4K_3$ and β multigraphs ${}^6K_3$. Hence, ${}^rK_3$ is decomposable into open trails of length six.

Consider now a multigraph ${}^rK_n$ for $n \geqslant 4$. Let us introduce $t_1' = 1$, $t_2' = n(n-1) - 1$, $t_1'' = n(n-1) - 1$, $t_2'' = 1$ and let us consider the sequences $\tau_1 = (t_1', t_2')$, and $\tau_2 = (t_1'', t_2'', t_3, \ldots, t_p)$. There exists a ${}^2K_n$-realization (T_1', T_2') of the sequence τ_1 and by induction a ${}^{r-2}K_n$-realization $(T_1'', T_2'', T_3, \ldots, T_p)$ of the sequence τ_2. Denote the consecutive vertices which create T_1' by $(v_1, v_{n(n-1)})$, the vertices which create T_2' by $(v_1, \ldots, v_{n(n-1)})$, T_1'' by $(w_1, \ldots, w_{n(n-1)})$ and T_2'' by (a_1, a_2) (obviously, all of vertices $v_1, \ldots, v_{n(n-1)}, w_1, \ldots, w_{n(n-1)}, a_1, a_2$ are in the set of vertices of ${}^rK_n$).

Assume first that $w_1 = a_1$. Because $n > 3$ we can assume that $v_1 = w_1 = a_1$ and $v_{n(n-1)} \neq w_{n(n-1)}$ and $v_{n(n-1)} \neq a_2$ (analogously for $w_{n(n-1)} = a_2$). For $w_1 \neq a_1$ we can assume that $v_1 = w_1$, $v_{n(n-1)} = a_1$, $v_1 \neq a_2$ and $v_{n(n-1)} \neq w_{n(n-1)}$. It implies that T_1 constructed by gluing the trails T_1', T_1'' and T_2 constructed by gluing T_2', T_2'' are open trails of length $n(n-1)$ and we obtain a ${}^rK_n$-realization of τ.

Assume then that one of terms of τ is less then $n(n-1)$. Without loss of generality we can assume that $t_1 < n(n-1)$. Since $s_{i_0+1} = (r-2)\frac{n(n-1)}{2}$ and $t_i \leq n(n-1)$ for each i, it implies that in τ there is another element, let us say t_2, less than $n(n-1)$. The sequences $\tau_1 = (t_1, \ldots, t_{i_0})$ and $\tau_2 = (t_{i_0+1}, \ldots, t_p)$ are realizable in ${}^2K_n$ and ${}^{r-2}K_n$ respectively, except in the case $r = 4$ and $t_p = n(n-1)$. Notice that $p > 2$ for $t_p = n(n-1)$ and considering a sequence $\tau' := (t_1', \ldots, t_p')$ such that $t_i' := t_i$ for each $i \in \{1, \ldots, p\} \setminus \{2, p\}$, $t_2' := t_p$ and $t_p' := t_2$ we obtain Case 1.1.

Case 2. Let us consider a complete multigraph ${}^rK_n$ with odd r. Let us assume that n is odd or $n = 2$ or $n = 4$. This part of the proof is analogous to the proof of Case 1. Applying the same arguments as above one may check that ${}^rK_2$ is arbitrarily decomposable into open trails, so we can assume that $n \geqslant 3$. Observe that K_n is arbitrarily decomposable into open trails for n odd by Theorem 3. It is easy to check that K_2 and K_4 are also arbitrarily decomposable into open trails. Assume that $r \geq 3$. We consider ${}^rK_n$ as an edge-disjoint union of arbitrarily decomposable multigraphs K_n and ${}^{r-1}K_n$, of sizes $\frac{n(n-1)}{2}$ and $(r-1)\frac{n(n-1)}{2}$, respectively. Assume that $t_i < \frac{n(n-1)}{2}$ for each $i \in \{1, \ldots, p\}$. Then there exists $i_0 \in \{2, \ldots, p-1\}$ such that $s_{i_0+1} \leq (r-1)\frac{n(n-1)}{2}$ and $s_{i_0} > (r-1)\frac{n(n-1)}{2}$. For $s_{i_0+1} < (r-1)\frac{n(n-1)}{2}$ let $t_{i_0}' := (r-1)\frac{n(n-1)}{2} - s_{i_0+1}$, $t_0'' := t_{i_0} - t_{i_0}'$. Let $\tau_1 := (t_1, \ldots, t_{i_0-1}, t_{i_0}'')$ and $\tau_2 := (t_{i_0}', t_{i_0+1}, \ldots, t_p)$. Then the sequence τ_1 is realizable in K_n and the sequence τ_2

is realizable in ${}^{(r-1)}K_n$ by Case 1. Let T_{i_0} be an open trail of length t_{i_0} constructed by gluing T'_{i_0} and T''_{i_0}. Hence τ is realizable in ${}^rK_n$. If $s_{i_0+1} = (r-1)\frac{n(n-1)}{2}$, then the sequence $(t_{i_0+1}, \ldots, t_p)$ is realizable in ${}^{(r-1)}K_n$ and the sequence $(t_1, \ldots, t_{i_0})$ is realizable in K_n, because $p-2 \geq i_0 \geq 2$.

Suppose then that there exists an element of τ greater or equal than $\frac{n(n-1)}{2}$. Notice that without loss of generality we can assume that $t_1 \leq \cdots \leq t_p$ so let $t_p \geq \frac{n(n-1)}{2}$. Let us consider the following cases:

Case 2.1 Assume that $n = 4$. So $t_p \geq 6$. Let $V({}^rK_4) = \{x, y, z, u\}$. Let us introduce $t'_{p-1} = 1$, $t''_{p-1} = t_p - 1$, $t'_p = 5$, $t''_p = t_p - 5$. By Case 1 we can find a realization $(T_1, \ldots, T_{p-2}, T''_{p-1}, T''_p)$ in ${}^{r-1}K_4$ of the sequence $(t_1, \ldots, t_{p-2}, t''_{p-1}, t''_p)$ and a realization (T'_{p-1}, T'_p) of the sequence (t'_{p-1}, t'_p) in K_4 such that $T'_{p-1} = (x, y)$, $T'_p = (u, x, z, u, y, z)$. Let $T''_{p-1} = (w_1, \ldots, w_{t''_{p-1}})$ and $T''_p = (v_1, \ldots, v_{t''_p})$. If $w_1 = v_1$ and $w_{t''_{p-1}} = v_{t''_p}$, then we can assume that $w_1 = v_1 = y$ and $w_{t''_{p-1}} = v_{t''_p} = u$. If $w_1 \neq v_1$ and $w_{t''_{p-1}} \neq v_{t''_p}$, then let $w_1 = y$, $w_{t''_{p-1}} = z$, $v_1 = x$ and $v_{t''_p} = u$. Assume now that $w_1 = v_1$ and $w_{t''_{p-1}} \neq v_{t''_p}$, let $w_1 = v_1 = y$ and $w_{t''_{p-1}} = z$, $v_{t''_p} = u$. Denote $T_{p-1} := T'_{p-1} \cup T''_{p-1}$, $T_p := T'_p \cup T''_p$. Hence we get a ${}^rK_4$-realization of τ.

Case 2.2 Let $n \geqslant 3$ be odd. If $t_p > \frac{n(n-1)}{2}$ then let $t'_p = t_p - \frac{n(n-1)}{2}$. By Case 1 there exists a realization $(T_1, \ldots, T_{p-1}, T'_p)$ of the sequence $\tau' = (t_1, \ldots, t_{p-1}, t'_p)$ in ${}^{r-1}K_n$. Let $T_p = T'_p \cup K_n$ and we obtain a ${}^rK_n$-realization of τ. Assume then that $t_p = \frac{n(n-1)}{2}$. Using similar method as in Case 1 one may check that τ is ${}^rK_n$-realizable except the case $t_1 = \cdots = t_p = \frac{n(n-1)}{2}$. For $n = 3$ the sequence $\tau = (3, 3, 3)$ is obviously realizable in ${}^3K_3$, see Figure 1.

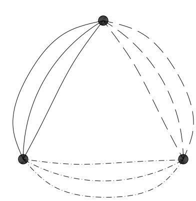

Figure 1: A realization of $\tau = (3, 3, 3)$ in ${}^3K_3$.

For $r \geqslant 5$ we consider ${}^rK_3$ as an edge-disjoint union of ${}^3K_3$ and ${}^{r-3}K_3$ and we obtain a ${}^rK_3$-realization of τ by Case 1. Assume that $n \geqslant 5$. Let us introduce $t'_1 = 1$, $t'_2 = \frac{n(n-1)}{2} - 1$, $t''_1 = \frac{n(n-1)}{2} - 1$, $t''_2 = 1$. The sequences $\tau_1 = (t'_1, t'_2)$, and $\tau_2 = (t''_1, t''_2, t_3, \ldots, t_p)$ are realizable in K_n and ${}^{r-1}K_n$, respectively. Let $T'_1 = (v_1, v_{\frac{n(n-1)}{2}})$, $T'_2 = (v_1, \ldots, v_{\frac{n(n-1)}{2}})$ and $T''_1 = (w_1, \ldots, w_{\frac{n(n-1)}{2}})$, $T''_2 = (a_1, a_2)$. If $w_1 = a_1$, then we can assume (because $n > 3$) that $v_1 = w_1 = a_1$, $v_{\frac{n(n-1)}{2}} \neq w_{\frac{n(n-1)}{2}}$ and $v_{\frac{n(n-1)}{2}} \neq a_2$. If $w_1 \neq a_1$ we can assume that $v_1 = w_1$, $v_{\frac{n(n-1)}{2}} = a_1$, $v_1 \neq a_2$ and $v_{\frac{n(n-1)}{2}} \neq w_{\frac{n(n-1)}{2}}$. Let $T_1 := T'_1 \cup T''_1$, $T_2 := T'_2 \cup T''_2$. We obtain a realization of τ in ${}^rK_n$ and the proof is finished. $\square$

Acknowledgement

The work of the first author was partially supported by MNiSW grant no. N201 036 31/3064.

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(Received 14 Aug 2007; revised 29 Oct 2007)