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The independent domination number of maximal triangle-free graphs

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Abstract

A triangle-free graph is *maximal* if the addition of any edge produces a triangle. A set S of vertices in a graph G is called an *independent dominating set* if S is both an independent and a dominating set of G. The *independent domination number* $i(G)$ of G is the minimum cardinality of an independent dominating set of G. In this paper, we show that $i(G) \leq \delta(G) \leq \lfloor \frac{n}{2} \rfloor$ for maximal triangle-free graphs G of order n and minimum degree $\delta(G)$. We characterize the graphs attaining the latter bound. We also show that, given a positive integer $k \geq 2$ and any positive integer $n \geq \frac{5k}{2}$, there exists a non-bipartite maximal triangle-free graph G of order n with $i(G) = k$.

1 Introduction

Domination is a well studied subject in graph theory. A set S of vertices in a graph G is called a *dominating set* if every vertex not in S is adjacent to some vertex of S. A set S of vertices in a graph G is called an *independent dominating set* (**IDS**) if S is both an independent and a dominating set of G. The *independent domination number* $i(G)$ of G is the minimum cardinality of an IDS of G. For a comprehensive introduction to domination in graphs, the reader is directed to the books [8, 9].

All graphs in this paper are finite, simple, and undirected. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $x \in V(G)$, we write $N_G(v)$ for the set of vertices of $V(G)$ adjacent to v, $N_G[v]$ for $N_G(v) \cup \{v\}$, $d_G(v) = |N_G(v)|$ for the degree of v and $\delta(G)$ for the minimum degree of G. Other graph theoretic terminology not defined here can be found in [2].

A graph G is called *maximal triangle-free* (**MTF**) if G has no triangles but the addition of any edge produces a triangle. For instance, any complete bipartite graph is an MTF graph. However, not every MTF graph is complete bipartite. Consider, for example, the 5-cycle C_5. Barefoot et al. [1] showed that a triangle-free graph is maximal if and only if it is minimal of diameter 2. For $n \geq 5$, they showed that there is an n-node m-edge MTF graph if and only if it is either complete bipartite or $2n-5 \leq m \leq \lfloor (n-1)^2/4 \rfloor + 1$.

In this paper, we investigate the independent domination number of MTF graphs. It is not hard to show that, for an MTF graph G of order $n \geq 2$, $i(G) = 1$ if and only if $G \cong K_{1,n-1}$. Observe that $i(K_{s,t}) = \min\{s,t\}$ for all positive integers s and t. Thus, for a positive integer k and any integer $n \geq 2k$, there exists a bipartite MTF graph $K_{k,n-k}$ so that $i(K_{k,n-k}) = k$. A natural question which arises is whether there exist non-bipartite MTF graphs with the same property. We will answer this question in Section 3.

2 Upper bounds

In this section, we present an upper bound on the independent domination number of an MTF graph in terms of its minimum degree and then its order. We characterize the graphs attaining the latter bound.

The following result describes how an MTF graph of order $n+1$ can be constructed from an MTF graph of order n. However, not every MTF graph can be constructed in such way. For instance, Figure 1 demonstrates that H is MTF but $H - v$ is not MTF for all $v \in V(H)$.

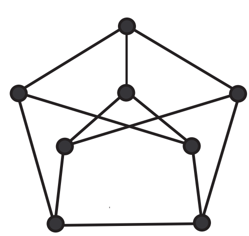

Figure 1: An MTF graph H satisfying the condition that $H - v$ is not MTF for each $v \in V(H)$

Lemma 1 *Let S be an IDS of an MTF graph G of order n. Let H be the graph obtained from G by adding a new vertex v adjacent to all vertices of S. Then H is an MTF graph of order $n+1$.*

Proof. Clearly, the graph H is triangle-free. Suppose that H is not an MTF graph. Thus, there exist two nonadjacent vertices $u, w \in V(H) - S$ such that $H + \{uw\}$ is triangle-free. As G is MTF, one of the vertices u and w, say w, must be v. As S is an IDS of G, u must be adjacent to some vertex x in S. Hence, the vertices v, u and x form a triangle in H, which is a contradiction. $\square$

Much work has been done on the independent domination number in graphs. For instance, Gimbel and Vestergaard [4], and Chartrand and Lesniak [2] discovered upper bounds on the independent domination number for general graphs and bipartite graphs in terms of their orders, respectively; while many others focussed on finding upper bounds as functions of minimum degree $\delta(G)$ and order n, including Favaron [3], Haviland [5, 6], Lam et al. [10], and Sun and Wang [11]. Recently, Haviland [7] studied the independent domination number of triangle-free graphs and proved the following result.

Theorem 2 *Any simple, triangle-free graph G of order n and minimum degree δ satisfies*

$$i(G) \leq \begin{cases} n + 2\delta - 2\sqrt{n\delta} & \text{if } 0 \leq \delta \leq 16n/121, \\ n + 3\delta - 2\sqrt{\delta(n+3\delta)} & \text{if } 16n/121 \leq \delta \leq n/6, \\ n/2 & \text{if } n/6 \leq \delta \leq n/4, \\ 3n/4 - \delta & \text{if } n/4 < \delta \leq n/3, \\ (2n-\delta)/4 & \text{if } n/3 < \delta < 2n/5. \end{cases}$$

In the following theorem, we present a much stronger result on the independent domination number $i(G)$ of MTF graphs.

Theorem 3 *Let G be an MTF graph of order $n \geq 2$ and minimum degree $\delta(G)$. For every vertex $v \in V(G)$, $N_G(v)$ is an IDS and so*

$$i(G) \leq \delta(G) \leq \lfloor \frac{n}{2} \rfloor.$$

Proof. It is not hard to check that the statement is true for $n = 2$ and 3, so we may assume that $n \geq 4$ in the following. We may also assume $G \not\cong K_{1,k}$ for some positive integer k, as otherwise the result follows. Since G is MTF, $N_G(v)$ is an independent set for every vertex $v \in V(G)$. Suppose that there exists a vertex $u \in V(G) - N_G[v]$ such that u is not adjacent to any vertex of $N_G(v)$. Then $G + uv$ is triangle-free, contradicting the fact that G is MTF. Thus, $N_G(v)$ is a dominating set, so $i(G) \leq \delta(G)$.

In order to complete the proof, suppose that $\delta(G) > \frac{n}{2}$. Thus,

$$\begin{aligned} 2|E(G)| &= \sum_{v \in V(G)} d_G(v) \\ &\geq n\delta(G) > \frac{n^2}{2}, \end{aligned}$$

which contradicts the well-known result of Turán [12] that every triangle-free graph of order n has at most $\lfloor \frac{n^2}{4} \rfloor$ edges. $\square$

Theorem 4 *Let G be an MTF graph of order $n \geq 6$. Then $i(G) = \lfloor \frac{n}{2} \rfloor$ if and only if $G \cong K_{\lfloor \frac{n}{2} \rfloor, n-\lfloor \frac{n}{2} \rfloor}$.*

Proof. Clearly $i(G) \leq \lfloor \frac{n}{2} \rfloor$ by Theorem 3. Since $i\left(K_{\lfloor \frac{n}{2} \rfloor, n-\lfloor \frac{n}{2} \rfloor}\right) = \lfloor \frac{n}{2} \rfloor$ is trivially true, it only remains to show that G is isomorphic to $K_{\lfloor \frac{n}{2} \rfloor, n-\lfloor \frac{n}{2} \rfloor}$ whenever $i(G) = \lfloor \frac{n}{2} \rfloor$. Suppose that $i(G) = \lfloor \frac{n}{2} \rfloor$. By Theorem 3, $i(G) = \delta(G) = \lfloor \frac{n}{2} \rfloor$. We consider the following two cases.

Case 1. $n = 2m$ for some positive integer m.

In this case, $i(G) = \delta(G) = m \geq 3$. Let $S = \{u_1, u_2, \ldots, u_m\}$ be an IDS of G. Let $T = V(G) - S = \{v_1, v_2, \ldots, v_m\}$. As S is an IDS of G and $\delta(G) = m$, u_i must be adjacent to all vertices v_j $(1 \leq j \leq m)$ for each $1 \leq i \leq m$. It turns out that v_i must be adjacent to all vertices u_j $(1 \leq j \leq m)$ for every $1 \leq i \leq m$. Moreover, v_i and v_j are not adjacent whenever $i \neq j$, since G is MTF. Hence, G is isomorphic to the complete bipartite graph $K_{m,m}$.

Case 2. $n = 2m + 1$ for some positive integer m.

We have that $i(G) = \delta(G) = m \geq 3$. Similarly to Case 1, we let $S = \{u_1, u_2, \ldots, u_m\}$ be an IDS of G and let $T = V(G) - S = \{v_1, v_2, \ldots, v_m, v_{m+1}\}$.

As $\delta(G) = m$, $d_G(u_i) \geq m$ for every $1 \leq i \leq m$. Suppose that $d_G(u_i) = m$ for each $1 \leq i \leq m$. Without loss of generality, we assume that u_1 is adjacent to $v_1, \ldots, v_m$, and v_{m+1} is adjacent to u_2. By Theorem 3, we know that $v_1, \ldots, v_m$ form an independent set of G, and so does $N_G(u_2)$. Hence, v_{m+1} is adjacent to $u_2, \ldots, u_m$ and exactly one of $v_1, \ldots, v_m$, say v_m. So v_m can not be adjacent to any of $u_2, \ldots, u_m$, as otherwise a triangle is created. Hence, $d_G(v_m) = 2$, which contradicts the assumption $\delta(G) \geq 3$. Therefore, there exists some integer $k \in \{1, \ldots, m\}$ such that $d_G(u_k) = m + 1$. As $S = \{u_1, u_2, \ldots, u_m\}$ is an IDS of G, u_k must be adjacent to all vertices $v_1, \ldots, v_m, v_{m+1}$. By Theorem 3, $v_1, \ldots, v_m, v_{m+1}$ form an independent set of G. Hence, G is a bipartite graph with partite sets of order m and $m + 1$. As G is MTF, it must be isomorphic to $K_{m,m+1}$. □

3 More on the independent domination number

In this section, we mainly answer a question posed in the introduction. Specifically, we prove the following:

Theorem 5 *Let $k \geq 2$ be an integer. For any positive integer $n \geq \frac{5k}{2}$, there exists a nonbipartite MTF graph G of order n with $i(G) = k$.*

Remark 1 *The condition $n \geq \frac{5k}{2}$ in Theorem 5 cannot be weakened any further, since there are no non-bipartite MTF graphs G of order 4, and by Theorem 4, there are no non-bipartite MTF graphs of order 7 and 9 with the independent domination number 3 and 4, respectively.*

To prove Theorem 5, we need the following lemmas.

Lemma 6 *Let S be an IDS of an MTF graph G with $i(G) = |S|$. Let H be the graph obtained from G by adding a new vertex v and all possible edges from v to S. Then H is an MTF graph with $i(H) = i(G)$.*

Proof. By Lemma 1, we know that H is MTF. It is clear from the construction of H that S is an IDS in H. Hence, $i(H) \leq |S| = i(G)$.

We now prove $i(H) \geq i(G)$ by contradiction. Suppose that $i(H) < i(G)$. Let S' be an IDS of H such that $|S'| = i(H) < i(G)$.

Claim 1. $v \in S'$ and $S \cap S' = \emptyset$.

Suppose that $v \notin S'$. Then S' is an IDS of G with $|S'| < i(G)$, which is impossible. By hypothesis, S' is an IDS of H. Hence, $S \cap S' = \emptyset$.

Claim 2. Each vertex of $V(G)\backslash S$ is adjacent to some vertex of $S'\backslash\{v\}$.

Note that v is only adjacent to all vertices of S in G. By a hypothesis that S' is an IDS of H, and Claim 1, the result follows.

Claim 3. There exists a unique $u \in S$ such that u is adjacent to none of $S'\backslash\{v\}$.

Suppose that there exists no vertex $u \in S$ such that u is adjacent to none of $S'\backslash\{v\}$. Hence, by Claim 2, we know that $S'\backslash\{v\}$ is an IDS of G with $|S'\backslash\{v\}| < i(G) - 1$, a contradiction. The uniqueness follows from the fact that G is MTF.

By Claims 1, 2 and 3, $(S'\backslash\{v\}) \cup \{u\}$ is an IDS of G with size $|S'| < i(G)$. This final contradiction completes the proof. □

In [1], Barefoot et al. define a family of non-bipartite MTF graphs which generalizes the 5-cycle C_5. Let p, q, r, s and t be positive integers. The graph $C_5[p, q, r, s, t]$ on $p+q+r+s+t$ vertices has the vertex set V partitioned into five subsets P, Q, R, S, T with p, q, r, s, t vertices, respectively. Each subset is an independent set and the only edges are between all pairs of vertices in P and Q, Q and R, R and S, etc. (see Figure 2). Obviously, $C_5[1, 1, 1, 1, 1]$ is the 5-cycle. Applying Lemma 1 $p + q + r + s + t - 5$ times, one can show that $C_5[p, q, r, s, t]$ is triangle-free and maximal for all positive integers p, q, r, s and t.

Lemma 7 *Let p, q, r, s and t be positive integers. The graphs $C_5[p, q, r, s, t]$ satisfy $i\left(C_5[p, q, r, s, t]\right) = \min\{p + s, p + r, q + s, q + t, r + t\}$.*

Proof. Let $P = \{p_1, ..., p_p\}, Q = \{q_1, ..., q_q\}, R = \{r_1, ..., r_r\}$, $S = \{s_1, ..., s_s\}$ and $T = \{t_1, ..., t_t\}$. By the construction of $C_5[p, q, r, s, t]$, we can see that $P \cup S$, $P \cup R$, $Q \cup S$, $Q \cup T$ and $R \cup T$ are IDSs of $C_5[p, q, r, s, t]$ with size $p+s$, $p+r$, $q+s$, $q+t$ and $r+t$, respectively. To prove that $i\left(C_5[p, q, r, s, t]\right) = \min\{p+s, p+r, q+s, q+t, r+t\}$, it suffices to show that there are no IDSs other than the above five in $C_5[p, q, r, s, t]$. Let W be an IDS of $C_5[p, q, r, s, t]$. We consider the following two cases.

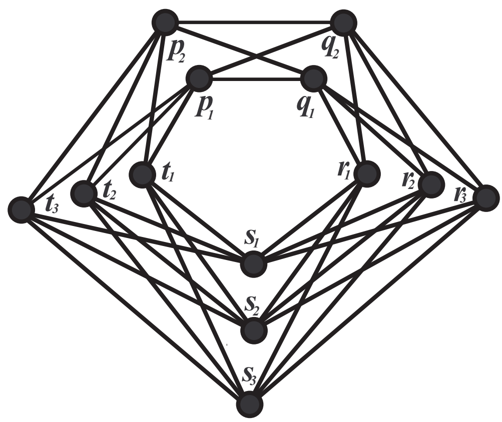

Figure 2: The graph $C_5[2, 2, 3, 3, 3]$

Case 1. $p_p \in W$.

As W is an IDS of $C_5[p, q, r, s, t]$, then by the construction, $v \notin W$ for each $v \in Q \cup T$. Hence, $P \subseteq W$.

Subcase 1.1. $s_1 \in W$.

If $s_1 \in W$ then $v \notin W$ for each $v \in R$. Thus, $v \in W$ for each $v \in S$. Therefore, $W = P \cup S$.

Subcase 1.2. $s_1 \notin W$.

If $s_1 \notin W$ then $r_j \in W$ for some $1 \leq j \leq r$. Thus, $v \notin W$ for each $v \in S$. This implies $R \subseteq W$, so $W = P \cup R$.

Case 2. $p_p \notin W$.

In this case, either $t_i \in W$ for some $1 \leq i \leq t$ or $q_j \in W$ for some $1 \leq j \leq q$.

Subcase 2.1. $t_i \in W$ for some $1 \leq i \leq t$.

Then $P \cup S \nsubseteq W$, so $T \subseteq W$. By a similar argument to that of Case 1, we can derive that either $W = R \cup T$ or $W = Q \cup T$.

Subcase 2.2. $q_j \in W$ for some $1 \leq j \leq q$.

Then $P \cup R \nsubseteq W$, so $Q \subseteq W$. By a similar argument to that of Case 1, we can derive that either $W = Q \cup S$ or $W = Q \cup T$. □

Remark 2 *For the graphs $C_5[p, q, r, s, t]$ of order $n = p + q + r + s + t$, if $i(C_5[p, q, r, s, t]) = k$, then $n \geq \frac{5k}{2}$.*

Proof. By Lemma 7, we have that

$$\min\{p+s, p+r, q+s, q+t, r+t\} = k.$$

Therefore, adding the five sums in the brackets, we obtain $2(p+q+r+s+t) \geq 5k$. □

Remark 3 *Let $k \geq 2$ be a positive integer and let $n = \lceil \frac{5k}{2} \rceil$. Then there exist positive integers p, q, r, s and t such that $n = p+q+r+s+t$ and $i\,(C_5[p,q,r,s,t]) = k$.*

Proof. If $k = 2l$ for some positive integer l, then choose $p = q = r = s = t = l$. Otherwise $k = 2l+1$ for some positive integer l, so $n = 5l+3$, and we choose $p = q = t = l+1$ and $s = t = l$. □

Proof of Theorem 5. By Remark 3, there exists a nonbipartite MTF graph $C_5[p,q,r,s,t]$ of order $\lceil \frac{5k}{2} \rceil$ with $i\,(C_5[p,q,r,s,t]) = k$. The result now follows by Lemma 6 and induction on the order of the graph. □

Theorem 8 *Let $k \geq 3$ be an integer. For any positive integers m and n with $n \geq 3k$ and $kn - k^2 - 1 \leq m \leq (k+1)n - k^2 - 3k - 1$, there is a nonbipartite MTF graph G of order n and size m such that $i(G) = k$.*

Proof. It suffices to construct a nonbipartite MTF graph G of order n and size m for each $kn - k^2 - 1 \leq m \leq (k+1)n - k^2 - 3k - 1$. Indeed, a nonbipartite MTF graph $C_5[1,2,r,s,t]$ can be constructed as follows. Let $f(x) = x(n-x-5)$. As $kn - k^2 - 1 \leq m \leq (k+1)n - k^2 - 3k - 1$, it is easily verified that

$$f(k-1) \geq m - 2n + 5, \tag{1}$$

and

$$f(k-2) \leq m - 2n - k + 7. \tag{2}$$

Let $s = k-1$ and $t = f(k-1) - m + 2n + k - 7$. By (1), we have $t \geq k-2$. Similarly, from the definition of t and (2), we deduce that $t \leq f(k-1) - f(k-2) = n - 2k - 2$. Therefore, it follows that $r = n - s - t - 3 \geq n - (k-1) - (n-2k-2) - 3 = k$.

Finally, from the definition of s, the bounds for r and t derived above, and Lemma 7, we have that

$$\begin{aligned} i(C_5[p,q,r,s,t]) &= \min\{1+s, 1+r, 2+s, 2+t, r+t\} \\ &= k. \end{aligned}$$

□

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