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A new infinite family of graceful generalised Petersen graphs, via “graceful collages” again

Andrea Vietri

Dipartimento Me. Mo. Mat.
Università La Sapienza
Roma
Italy

Abstract

By strengthening an edge-decomposition technique for gracefully labelling a generalised Petersen graph, we provide graceful labellings for a new infinite family of such graphs. The method seems flexible enough to provide graceful labellings for many other classes of graphs in the future.

1 Introduction

Among the numerous kinds of vertex labellings for graphs, *graceful* labellings owe their popularity to disparate reasons such as applicability in real-life contexts (see a pioneering work by Bloom and Golomb [3]), clear connections with graph decomposition theory (see e.g. [7, 11, 12]) and, we presume, an intrinsic appeal that has been enthralling combinatorialists for decades.

Definition 1.1. Let $G = (V, E)$ be a simple connected graph, $\lambda : V \to \{0, 1, 2, \ldots, |E|\}$ be an injective vertex labelling, and λ' be the induced labelling, on the edges, that assigns the value $|\lambda(u) - \lambda(v)|$ to the edge $\{u, v\}$. The labelling λ is termed *graceful* if λ' is injective (equivalently, if it is a bijection on $\{1, 2, \ldots, |E|\}$). A graph that admits a graceful labelling is termed *graceful* as well.

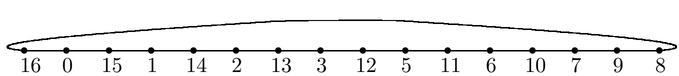

Figure 1: A graceful labelling of the 16-cycle

Without lingering on applicative aspects of graceful labellings, let us emphasise instead the relationship between these labellings and edge-decompositions of graphs into copies of assigned subgraphs. If we are given a graph $S = (V, E)$ and try to embed $2|E| + 1$ copies of S in the complete graph $K_{2|E|+1}$ with the proviso that any edge of the complete graph belongs to exactly one copy (we are in fact looking for a so-called $(K_{2|E|+1}, S)$-*design*, see e.g. [6]), then we get a quick solution to our problem whenever we can gracefully label S. For, by identifying V with the finite group $\mathbf{Z}_{2|E|+1}$, and regarding labels as integers (mod $2|E|+1$), the i-th copy is obtained by adding i to each vertex of S. It turns out that we have actually obtained a *cyclic* $(K_{2|E|+1}, S)$-design, because the further property we get for free is that copies are still changed into copies by any vertex automorphism φ_i of $K_{2|E|+1}$ defined, indeed, by $\varphi_i(v) = v + i$, with $0 \leq i \leq 2|E|$.

The already quoted paper by Rosa ([12]) was the starting point for the systematic search of classes of graphs that might be graceful. For example *trees* have been playing a star-role in this context, and they will still do at least until someone turns up with a proof that every tree is graceful, or with a stunning counterexample[1]. Complete graphs are graceful if and only if they have at most 4 vertices (see [9]). Many other graphs are by now without secrets as to their being graceful or not, whereas many others besides trees appear still difficult to understand.

Should a graceful labelling be hard to find, some relief might come from the discovery of a *relaxed* graceful labelling (see [13]), namely a still injective labelling λ which may however take larger values than $|E|$ (and the same with λ'). Asymptotical aspects of relaxed graceful labelling have been dealt with in [4], while in [2] these labellings have been related to *Golomb's rulers.*

A really satisfactory on-line survey, dealing with graceful labellings and many other kinds of labellings, is Gallian's ([7]), which counts several updates — the last in this very year.

Let us now come to the main object of study in the present paper, namely *generalised Petersen graphs.*

Definition 1.2. Let n, k be positive integers such that $n \geq 3$ and $1 \leq k \leq [(n-1)/2]$. The *generalised Petersen graph* $P_{n,k}$ is the graph whose vertex set is $\{a_i, b_i \colon 1 \leq i \leq n\}$ and whose edge set is $\{\{a_i, b_i\}, \{a_i, a_{i+1}\}, \{b_i, b_{i+k}\} : 1 \leq i \leq n\}$, where $a_{n+1} = a_1$ and $b_{n+c} = b_c$ for any positive $c \leq k$.

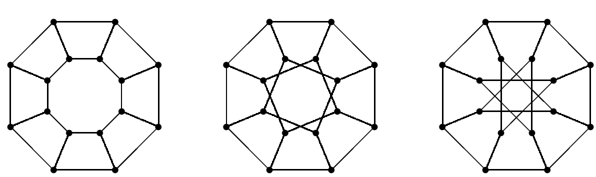

Figure 2: $P_{8,k}$, with $1 \leq k \leq 3$

In [8] every $P_{n,1}$ was shown to be graceful (see also [7]). In addition, a number of $P_{n,k}$'s with $k \geq 2$ and small values of n were shown to be graceful – this time

[1]While the general feeling is that the Graceful Tree Conjecture (*Ringel's conjecture*, or *Kotzig-Ringel's*, standing from the 60's, see [11]) is true, yet the author of the present paper has recently raised some little doubts about it; see [15].

with the precious aid of computers (see [10]). The first result that has "pierced the infinity" for k larger than 1 is the following.

Theorem 1.3 ([14]). *$P_{8t,3}$ is graceful for every positive integer t.*

The parametric construction of infinitely many graceful labellings in the proof of the above theorem is based on the splitting of any $P_{8t,3}$ into a $(12t)$-cycle (thus, containing half the number of all edges) and $4t$ 3-stars whose endpoints lie on the cycle. After gracefully labelling the cycle, and amplifying labels by two, one chooses suitable odd numbers for the centres of the stars, so as to generate all remaining (odd) differences. Clearly, this game can be played first of all because the length of the cycle allows for a graceful labelling (a classical result, see [12], characterises graceful cycles as those having length congruent to 0 or 3 (mod 4)). Such a constraint made it impossible, at the time of the proof, to apply the above method also to graphs of the form $P_{8t+4,3}$, though any of them was likewise decomposable into a (nongraceful) cycle and some 3-stars.

In the present paper we overcome the congruence obstacle using a "surgery" technique, by means of which two adjacent vertices of the $(12t+6)$-cycle of a given $P_{8t+4,3}$ are labelled by *odd* numbers, while the remaining vertices are assigned even numbers as in an ingenuous up-and-down labelling amplified by two (performed more or less in the same fashion as when gracefully labelling a cycle of admissible length). Further little adjustments are then required before launching the same, old method of odd labels for centres of 3-stars.

The Achilles heel of the above parametric construction is that it does not apply to exceedingly small graphs — this is a typical obstruction when dealing with such combinatorial problems. In the next section we shall in fact establish the following result.

Theorem 1.4. *$P_{8t+4,3}$ is graceful for every positive integer $t \geq 4$.*

Choosing suitable labels for some of the centres of the 3-stars when $1 \leq t \leq 3$ was indeed so hard to us that we gave up the search after a reasonable while. It might, however, very well be that a smarter labelling of the big cycle – such labellings could in fact be performed in many ways – yields the desired global labelling with little effort, in each of the three cases above (we leave it as an open question). At any rate, we have filled up the three holes using a computer algorithm by Del Fra ([1]), which has provided three graceful labellings for our purposes (see the last section).

It is worth remarking that no additional infinite class of generalised Petersen graphs are known to be graceful (see [7]). The method that we shall carefully describe in the next section is in our opinion likely to be applicable to many other classes of graphs, not necessarily of the form $P_{n,k}$. However it would be desirable, first of all, to find graceful labellings for infinitely many graphs of the form $P_{2,k}$ (none of which, unfortunately, seems to contain a suitable cycle).

Our extreme confidence moved us to give the present technique a proper name. Let us then call it *graceful collage*.

2 The construction

The whole section is devoted to proving the main theorem.

Proof of Theorem 1.4. Starting from a given generalised Petersen graph of the form $P_{8t+4,3}$, and using the above notation for vertices, let us consider the cycle (b_3 a_3 a_2 a_1 b_1 b_4 b_7 a_7 a_6 a_5 b_5 b_8 b_{11} a_{11} ... a_{8t-1} a_{8t-2} a_{8t-3} b_{8t-3} b_{8t} b_{8t+3} a_{8t+3} a_{8t+2} a_{8t+1} b_{8t+1} b_{8t+4}) — many of the present definitions are in keeping with [14]. The reader can check with few difficulties that the deletion of this cycle leaves $P_{8t+4,3}$ with a disjoint union of 3-stars that can be collected in two subfamilies: stars of class 1, having b_{4i-2} connected to b_{4i-5}, a_{4i-2}, b_{4i+1} for $1 \le i \le 2t+1$, and stars of class 2, having a_{4i} connected to a_{4i-1}, b_{4i}, a_{4i+1} for $1 \le i \le 2t+1$.

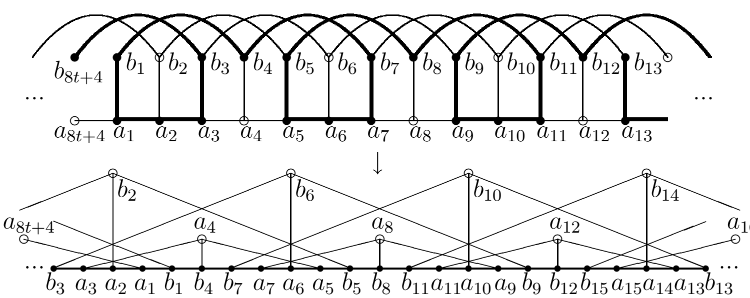

Figure 3: Decomposing $P_{8t+4,3}$ into a $(12t+6)$-cycle and $4t+2$ 3-stars

We now endow the above cycle with the following labelling (the two odd numbers in bold refer to the modified edge, as hinted in the Introduction, while the just preceding increase by two — in bold as well — is the standard trick used for gracefully labelling cycles).

$$(\overset{b_3}{24t+12}, \overset{a_3}{0}, \overset{a_2}{24t+10}, \overset{a_1}{2}, ..., \overset{b_{4t-3}}{18t+14}, \overset{b_{4t}}{6t-2}, \overset{b_{4t+3}}{18t+12}, \overset{a_{4t+3}}{6t+\mathbf{2}}, \overset{a_{4t+2}}{\mathbf{18t}+\mathbf{11}},$$

$$\overset{a_{4t+1}}{\mathbf{6t}+\mathbf{3}}, \overset{b_{4t+1}}{18t+8}, \overset{b_{4t+4}}{6t+6}, \overset{b_{4t+7}}{18t+6}, \overset{a_{4t+7}}{6t+8}, ..., \overset{a_{8t+2}}{12t+10}, \overset{a_{8t+1}}{12t+4}, \overset{b_{8t+1}}{12t+8}, \overset{b_{8t+4}}{12t+6}).$$

More formally, for any $i \equiv 1, 2, 3 \pmod 4$ and $i \notin [4t+1, 4t+3]$, the vertex a_i is labelled respectively by $2 + 3(i-1)/2 + \varepsilon$, $24t + 10 - 3(i-2)/2$, $3(i-3)/2 + \varepsilon$, where $\varepsilon = 2$ if $i \ge 4t+5$ and $\varepsilon = 0$ otherwise; we do not write again the labels for the three remaining a_i's, which show well in the above sequence. Instead, for any $i \equiv 0, 1, 3 \pmod 4$, the vertex b_i is labelled respectively by $-2 + 3i/2 + \varepsilon$, $24t + 8 - 3(i-1)/2$, $24t + 12 - 3(i-3)/2$, where $\varepsilon = 2$ if $i \ge 4t+4$ and $\varepsilon = 0$ otherwise.

The differences obtained along the cycle are therefore all the even numbers from 2 to $24t+12$ except $12t+4$ and $12t+12$, plus the two odd differences $12t+5$ and $12t+9$. The missing even numbers are now obtained by labelling the star centres a_{4t} and b_{4t+2} respectively by $18t+15$ and $6t+7$, which as a consequence yields the

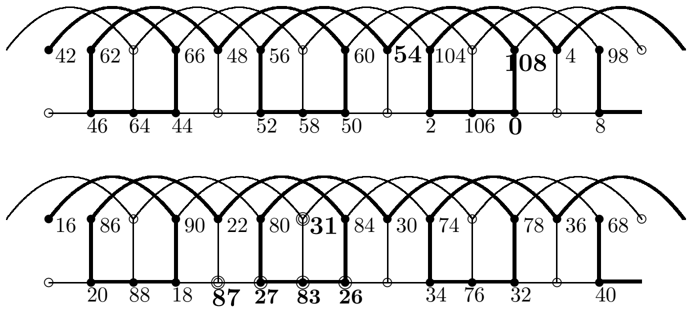

Figure 4: Near the cycle extremes and the modified trait, when $t = 4$

further differences $12t - 5$, $12t + 11$, $12t + 17$, $12t + 21$ (see the two distinguished circles out of the cycle, in the lower side of figure 4).

It can be easily seen (with the possible help of the whole Figure 4) that apart from two exceptions, and leaving aside the two stars whose centres have already been labelled, the endvertices of each class-1 star are assigned three labels of the form $\langle x, x-8, x-16\rangle$, while the labels for each class-2 star are, apart from one exception, of the form $\langle x, x+4, x+8\rangle$ (moving in the same direction along the cycle). In details, denoting by $[x]$ and $[[x]]$ the above forms, such triples of labels are the following:

$$\{\,[6x]\colon 2t+4 \le x \le 4t+2\,,\ x \ne 3t+3\}\,,$$
$$\{\,[[6x]]\colon 0 \le x \le t-2\}\,,\ \{\,[[6x+2]]\colon t \le x \le 2t-1\}\,,$$
$$\langle 2, 12t+2, 12t+6\rangle\,,\ \langle 12t+10, 12t+18, 24t+8\rangle\,,\ \langle 12t+12, 24t+2, 24t+10\rangle\,.$$

The choice of odd labels, for all these triples except the two already labelled, will split into two cases depending on the parity of t. In both cases, every arrow $\mapsto$ in the scheme indicates an assignment. The labels used at each time are specified, unless there is a unique label. The resulting differences are in any case displayed. Three groups of assignments (starting with "IF") must be skipped only if $t = 4$ and $t = 5$ respectively. The symbol $[x, y]_z$ denotes all integers from x to y that are congruent to x (mod z). Finally, notice that the two modifications subject to $t \ge 8$ in the first case prevent repeating the very last label, $12t - 25$ (something similar occurs, though less explicitly, also in the second case).

<u>Case 1</u>: *t even.*

$[18t+30+12x] \mapsto 6t-23-12x\colon 0 \le x \le t/2-2 \implies$
diff. $[12t+37, 24t+5]_8$, labels $[1, 6t-23]_{12}$

$[18t+36+12x] \mapsto 6t-21-12x\colon 0 \le x \le t/2-2 \implies$
diff. $[12t+41, 24t+9]_8$, labels $[3, 6t-21]_{12}$

$[18t+24] \mapsto 6t-7 \implies$ diff. $\{12t+15, 12t+23, 12t+31\}$

IF $t \geq 6$: $[12t+42+12x] \mapsto 12t-15-12x\colon 0 \leq x \leq t/2-3$, $x \neq 1$,
$[12t+54] \mapsto 12t-31$ (**if** $t \geq 8$) $\implies$
diff. $[41, 12t-15]_8 \setminus \{65, 73, 81\} \cup \{69, 77, 85\}$,
labels $[6t+21, 12t-15]_{12} \setminus \{12t-27\} \cup \{12t-31\}$
(**do not** perform the "$(\setminus, \cup)$-replacements", **when** $t = 6$)

IF $t \geq 6$: $[12t+48+12x] \mapsto 12t-13-12x\colon 0 \leq x \leq t/2-3$, $x \neq 1$,
$[12t+60] \mapsto 12t-21$ (**if** $t \geq 8$) $\implies$
diff. $[45, 12t-11]_8 \setminus \{69, 77, 85\} \cup \{65, 73, 81\}$,
labels $[6t+23, 12t-13]_{12} \setminus \{12t-25\} \cup \{12t-21\}$
(**do not** perform the "$(\setminus, \cup)$-replacements", **when** $t = 6$)

$[12t+36] \mapsto 12t-3 \implies$ diff. $\{23, 31, 39\}$

$[12t+30] \mapsto 12t-5 \implies$ diff. $\{19, 27, 35\}$

$[12t+24] \mapsto 12t-9 \implies$ diff. $\{17, 25, 33\}$

$\langle 12t+10, 12t+18, 24t+8 \rangle \mapsto 12t-11 \implies$ diff. $\{21, 29, 12t+19\}$

$\langle 12t+12, 24t+2, 24t+10 \rangle \mapsto 12t-25 \implies$ diff. $\{37, 12t+27, 12t+35\}$

. . . .

$[[6x]] \mapsto 24t+11-6x\colon 0 \leq x \leq t-3 \implies$
diff. $[12t+39, 24t+11]_4$, labels $[18t+29, 24t+11]_6$

$[[6t-12]] \mapsto 18t+21 \implies$ diff. $\{12t+25, 12t+29, 12t+33\}$

$[[6t+2]] \mapsto 18t+3 \implies$ diff. $\{12t-7, 12t-3, 12t+1\}$

IF $t \geq 6$: $[[6t+8+6x]] \mapsto 18t-1-6x\colon 0 \leq x \leq t-5 \implies$
diff. $[43, 12t-9]_4$, labels $[12t+29, 18t-1]_6$

$[[12t-16]] \mapsto 12t-1 \implies$ diff. $\{7, 11, 15\}$

$[[12t-10]] \mapsto 12t-7 \implies$ diff. $\{1, 3, 5\}$

$[[12t-4]] \mapsto 24t+3 \implies$ diff. $\{12t-1, 12t+3, 12t+7\}$

$\langle 2, 12t+2, 12t+6 \rangle \mapsto 12t+15 \implies$ diff. $\{9, 13, 12t+13\}$

The reader can easily check that this scheme yields, as differences, all the odd numbers from 1 to $24t+11$ with the exception of the six previously obtained, while no repetition of label occurs (consider also the labels $6t+7$ and $8t+15$, already used) and all the 3-star centres to label are taken into account.

We now proceed with the second half of the construction.

Case 2: *t odd.*

$[18t+36+12x] \mapsto 6t-29-12x\colon 0 \le x \le (t-5)/2 \implies$
diff. $[12t+49, 24t+5]_8$, labels $[1, 6t-29]_{12}$

$[18t+42+12x] \mapsto 6t-27-12x\colon 0 \le x \le (t-5)/2 \implies$
diff. $[12t+53, 24t+9]_8$, labels $[3, 6t-27]_{12}$

$[18t+30] \mapsto 6t-15 \implies$ diff. $\{12t+29, 12t+37, 12t+45\}$

$[18t+24] \mapsto 6t-17 \implies$ diff. $\{12t+25, 12t+33, 12t+41\}$

$[18t+12] \mapsto 6t-19 \implies$ diff. $\{12t+15, 12t+23, 12t+31\}$

IF $t \ge 7$: $[12t+54+12x] \mapsto 12t-39-12x\colon 0 \le x \le (t-9)/2$,
diff. $[77, 12t-15]_8$, labels $[6t+15, 12t-39]_{12}$
IF $t \ge 7$: $[12t+60+12x] \mapsto 12t-37-12x\colon 0 \le x \le (t-9)/2$,
diff. $[83, 12t-11]_8$, labels $[6t+17, 12t-37]_{12}$

$[12t+48] \mapsto 12t-21 \implies$ diff. $\{53, 61, 69\}$

$[12t+42] \mapsto 12t-31 \implies$ diff. $\{57, 65, 73\}$

$[12t+36] \mapsto 12t-15 \implies$ diff. $\{35, 43, 51\}$

$[12t+30] \mapsto 12t-17 \implies$ diff. $\{31, 39, 47\}$

$[12t+24] \mapsto 12t-9 \implies$ diff. $\{17, 25, 33\}$

$\langle 12t+10, 12t+18, 24t+8\rangle \mapsto 12t-11 \implies$ diff. $\{21, 29, 12t+19\}$

$\langle 12t+12, 24t+2, 24t+10\rangle \mapsto 12t-25 \implies$ diff. $\{37, 12t+27, 12t+35\}$

. . . .

$[[6x]] \mapsto 24t+11-6x\colon 0 \le x \le t-4 \implies$
diff. $[12t+51, 24t+11]_4$, labels $[18t+35, 24t+11]_6$

$[[6t-18]] \mapsto 18t+29 \implies$ diff. $\{12t+39, 12t+43, 12t+47\}$

$[[6t-12]] \mapsto 18t-5 \implies$ diff. $\{12t-1, 12t+3, 12t+7\}$

$[[6t+2]] \mapsto 18t+3 \implies$ diff. $\{12t-7, 12t-3, 12t+1\}$

IF $t \ge 7$: $[[6t+8+6x]] \mapsto 18t-1-6x\colon 0 \le x \le t-6 \implies$
diff. $[55, 12t-9]_4$, labels $[12t+35, 18t-1]_6$

$[[12t-22]] \mapsto 12t+27 \implies$ diff. $\{41, 45, 49\}$

$[[12t-16]] \mapsto 12t-1 \implies$ diff. $\{7,11,15\}$

$[[12t-10]] \mapsto 12t-7 \implies$ diff. $\{1,3,5\}$

$[[12t-4]] \mapsto 12t+23 \implies$ diff. $\{19,23,27\}$

$\langle 2, 12t+2, 12t+6\rangle \mapsto 12t+15 \implies$ diff. $\{9,13,12t+13\}$

Also in this case, the routine checks are left to the reader.

□

3 Some final remarks

As anticipated in the Introduction, the three cases not covered by Theorem 1.4 were managed by Del Fra with the aid of a computer. For each case, the "exterior cycle" was labelled once for all (in a fashion that will remind the reader of the graceful labelling of a cycle), then a suitable labelling for the "interior" was successfully found by an algorithm. The relevant results can be therefore recorded as follows.

Corollary 3.1. *There exist graceful labellings for* $P_{12,3}$ *,* $P_{20,3}$ *, and* $P_{28,3}$ *.*

Proof. For each $n \in \{12, 20, 28\}$ we simply exhibit the two ordered sequences of labels assigned to $a_1, a_2, ..., a_n$ and to $b_1, b_2, ..., b_n$ respectively, leaving the easy — though a little tedious — routine calculations to the reader.

$(36, 0, 35, 1, 34, 2, 33, 3, 32, 4, 17, 10)$,
$(11, 18, 12, 23, 15, 26, 13, 20, 5, 19, 16, 21)$.

. . . .

$(0, 60, 1, 59, 2, 58, 3, 57, 4, 56, 5, 54, 6, 53, 7, 52, 8, 51, 9, 50)$,
$(21, 23, 37, 19, 32, 24, 25, 28, 31, 17, 38, 30, 29, 22, 33, 14, 36, 16, 34, 18)$.

. . . .

$(0, 84, 1, 83, 2, 82, 3, 81, 4, 80, 5, 79, 6, 78, 7, 76, 8, 75, 9, 74, 10, 73, 11, 72, 12, 71, 13, 70)$,
$(29, 38, 35, 27, 51, 31, 33, 44, 54, 25, 49, 34, 37, 39, 43, 22, 50, 40, 41, 36, 58, 20, 52, 32, 45, 28, 60, 18)$.

□

The above corollary, together with Theorem 1.4, Theorem 1.3, and the gracefulness theorem for the prism $P_{4,1}$ (see [8]), settle in the affirmative the gracefulness problem when $k = 3$ and n is a multiple of 4:

Corollary 3.2. $P_{4s,3}$ *is graceful for every positive integer* s*.*

We end by spending the last corollary to obtain the expected edge decompositions of complete graphs into generalised Petersen graphs (the standard proof of the relevant claim has already been outlined in the Introduction).

Corollary 3.3. *There exists a cyclic* $(K_{24s+1}, P_{4s,3})$*-design for any positive integer* s*.*

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References

[1] A. Del Fra, personal communication.

[2] B. Beavers, *Golomb Rulers and Graceful Graphs*, Math 7999 Communicating Mathematics,
`http://www.math.lsu.edu/~beavers/Golomb_Rulers_and_Graceful_Graphs.pdf`

[3] G.S. Bloom and S.W. Golomb, *Applications of Numbered Undirected Graphs*, Proc. IEEE **65(4)** (1977), 562–570.

[4] B. Bollobás and O. Pikhurko, *Integer sets with prescribed pairwise differences being distinct*, Europ. J. Comb. **26(5)** (2005), 607–616.

[5] V. Bryant, *Aspects of Combinatorics, A Wide-ranging Introduction*, Cambridge University Press, 1993.

[6] C.J. Colbourn and J.H. Dinitz (eds.), *The CRC handbook of combinatorial designs*, Boca Raton, FL, CRC Press, 1996.

[7] J.A. Gallian, *A Dynamic Survey of Graph Labeling*, Electr. J. Comb. **DS6** (2002).

[8] J.A. Gallian and R.W. Frucht, Labeling prisms, *Ars Combin.* **26** (1988), 69–82.

[9] S.W. Golomb, *How to number a graph*, in: R.C. Read, *Graph Theory and Computing*, Academic Press, New York, 1972, pp. 23-37.

[10] T.A. Redl, Graceful Graphs and Graceful Labelings: Two Mathematical programming Formulations and Some Other New Results, *Congr. Numer.* **164** (2003), 17–31.

[11] G. Ringel, Problem 25, in: *Theory of Graphs and its Applications*, Proc. Symp. Smolenice, 1963, G/A Czech. Acad. Sci. **162** (1964).

[12] A. Rosa, On certain valuations of the vertices of a graph, *Theory of Graphs (Internat. Sympos. Rome, 1966)* pp. 349–355, Gordon and Breach, New York; Dunod, Paris, 1967.

[13] F. Van Bussel, Relaxed Graceful Labellings of Trees, *Electr. J. Combin.* **9(1)** (2002).

[14] A. Vietri, Graceful labellings for an infinite class of Generalised Petersen Graphs, *Ars Combin.* **81** (2006), 247–255.

[15] A. Vietri, Sailing towards, and then against, the Graceful Tree Conjecture: some promiscuous results, *Bull. I.C.A.* (to appear).

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