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Symmetrically inequivalent partitions of a square array

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Abstract

An $n \times n \times p$ proper array is a three-dimensional array of directed cubes that obeys certain constraints. These constraints allow $n \times n \times p$ proper arrays to be enumerated via a transition matrix, $\boldsymbol{M_{n \times n}}$. The main goal of this paper is to compute a lower bound for the basis size of $\boldsymbol{M_{n \times n}}$. A lower bound is obtained from the exponential generating function which counts the symmetrically inequivalent partitions of a $n \times n$ array, i.e. the exponential generating function counts the symmetrically inequivalent $n \times n$ *letter representations*. The symmetry in question is that of a square, namely $D_4 = C_4 \times C_2$. The aforementioned exponential generating function is a linear combination of six exponential generating functions, each of which is associated with a particular symmetry of the $n \times n$ array.

1 Introduction

This paper is a continuation of the author's previous work on three-dimensional proper arrays [2]. In [2], the objects studied were $m \times n \times p$ proper arrays ($m \neq n$), where $m \times n \times p$ proper arrays are three-dimensional configurations of directed cubes which are enumerated by the transition matrix $\boldsymbol{M_{m \times n}}$. The main theorem of [2] involved the construction of an exponential generating function for the number of symmetrically inequivalent partitions of an $m \times n$ rectangle. This exponential generating function provided a lower bound for the basis size of $\boldsymbol{M_{m \times n}}$. The $m \times n$ partitions of the rectangle were defined to be the $m \times n$ *letter representations* associated with the $m \times n \times p$ proper array. Two $m \times n$ letter representations were said to be *symmetrically equivalent* if they mapped to one another via a symmetry operation of the $m \times n$ rectangle. For more details about proper arrays and the transition matrix, the reader is referred to [2].

We now let $m = n$. Thus, our object of study is an $n \times n \times p$ proper array. The main theorem of this paper, Theorem 2.1, provides an exponential generating function for the number of symmetrically inequivalent partition of an $n \times n$ square. This exponential generating function, itself a linear combination of six exponential generating functions, provides a lower bound for the basis size of $\boldsymbol{M_{n\times n}}$. We define the $n \times n$ partitions of a square to be the $n \times n$ *letter representations* associated with the $n \times n \times p$ proper array. Furthermore, we say two $n \times n$ letter representations are *symmetrically equivalent* if and only if the symmetry group of the square, $D_4 = C_4 \times C_2$, maps one into the other. Figure 1.1 provides an illustration of eight symmetrically equivalent $3 \times 3 \times 3$ proper arrays, along with their eight equivalent 3×3 letter representations. An auxillary result, given in Appendix B, provides a generating function for the $n \times n$ letter representations fixed by the two diagonal reflection maps of D_4.

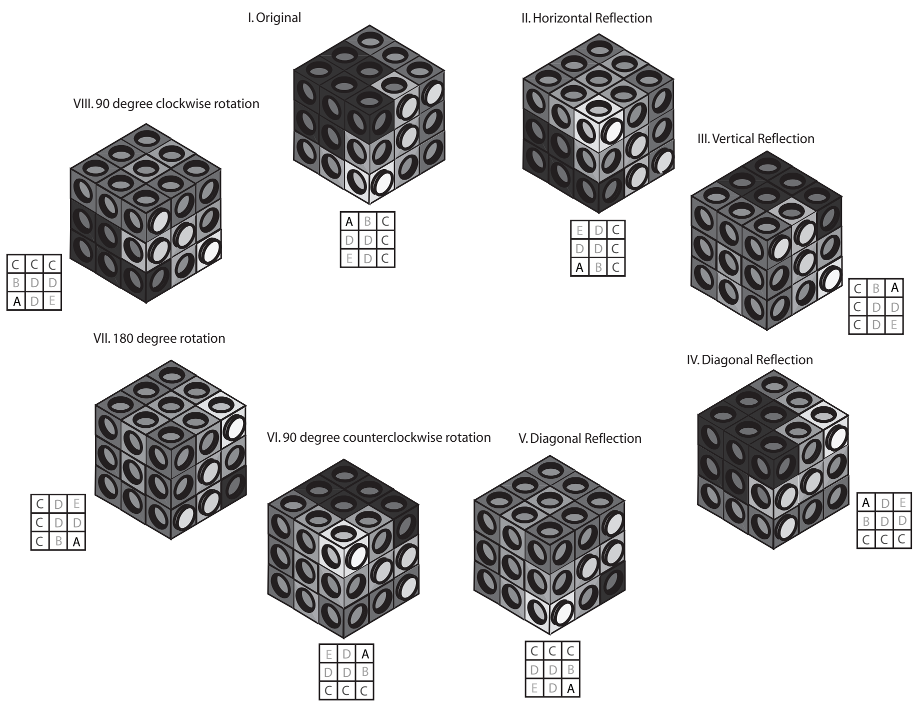

Figure 1.1: An example of $3 \times 3 \times 3$ proper array and its eight symmetrically equivalent versions. Beside each $3 \times 3 \times 3$ proper array is its associated 3×3 letter representation. The 3×3 letter representation uses letter to describe the partition structure of the outward pointing face of the $3 \times 3 \times 3$ proper array. Note, these eight images are only counted once in the enumeration procedure.

Remark 1.1 *We use the letters in the letter representations as computational device. In other words, the letters are just labels for the components and the labels are* *unimportant. For example, take the letter representations provided in Figure 1.1. If we replace the A with an X, the resulting letter representations are considered the same as the illustrated letter representations since the partition of the $n \times n$ array has not changed.*

2 Enumerating Letter Representations

With the preliminary information in place, we are ready to develop a formula that counts, modulo D_4 symmetry, the number of $n \times n$ letter representations associated with $n \times n \times p$ proper arrays. Fortunately, it is an easy exercise to show that the entire set of $n \times n$ letter representations has cardinality given by the Bell number $B(n^2)$, denoted B_n [1, 4, 5]. In general, if $n \neq m$, we will need double subscripts. For example, we denote $B(nm)$ as $B_{n,m}$. However, in our context, a single subscript suffices.

Define L_n to be the number of $n \times n$ letter representations modulo D_4 symmetry. Then, L_n is the lower bound for the basis size of $\boldsymbol{M_{n \times n}}$. We calculate L_n as follows.

1. Let SS_n count $n \times n$ letter representations that are fixed by all eight symmetry transformations of D_4.

2. Let S_n count the $n \times n$ letter representations that fixed by both horizontal and vertical reflections. The quantity $S_n - SS_n$ counts the $n \times n$ letter representations that have horizontal and vertical reflective symmetry without having 90° rotational symmetry.

3. Let H_n count the $n \times n$ letter representations fixed via horizontal reflection. The quantity $H_n - S_n$ counts the $n \times n$ letter representations that are fixed only by horizontal reflection.

4. Let V_n count the $n \times n$ letter representations fixed via vertical reflection. The quantity $V_n - S_n$ counts the $n \times n$ letter representations that are fixed only by vertical reflection.

5. Let D_n count the $n \times n$ letter representations that are fixed by both diagonal reflections. The quantity $D_n - SS_n$ counts the $n \times n$ letter representations that are fixed by both diagonal reflections but do not have 90° rotational symmetry.

6. Let I_n count the $n \times n$ letter representations that are fixed by a single diagonal reflection. The quantity $I_n - D_n$ counts the $n \times n$ letter representations that are fixed only by one diagonal reflection.

7. Let R_n count the $n \times n$ letter representations fixed via 180° rotation. The quantity $R_n - S_n - D_n - N_n + 2SS_n$ counts the $n \times n$ letter representation that are fixed only by 180° rotation.

8. Let N_n count the $n \times n$ letter representations that are symmetrical with respect to $90°$ rotational symmetry. The quantity $N_n - SS_n$ counts the $n \times n$ letter representations that are fixed only by $90°$ rotation.

9. Let $C_n = B_n - H_n - V_n - R_n - 2I_n + 2D_n + 2S_n$ count the $n \times n$ letter representations that have eight distinct symmetry images. In otherwords, C_n is the number of $n \times n$ letter representations that are not fixed by any symmetry tranformation.

Theorem 2.1 *Let $L_n, C_n, B_n, V_n, H_n, R_n, I_n, D_n, S_n, N_n$ and SS_n be as previously defined. Then*

$$L_n = \frac{B_n + H_n + V_n + R_n + 2I_n + 2N_n}{8}$$

Proof of Theorem 2.1: To calculate the number of $n \times n$ letter representations modulo D_4 symmetry, we first determine whether a given $n \times n$ letter representation, called A, is fixed via any of the eight symmetry transformations. If A is not fixed by any symmetry, it has eight equivalent images. However, if A is fixed under a symmetry transformation, it has at most four symmetry images. It follows that

$$\begin{aligned} L_n = {} & \frac{C_n + 8SS_n}{8} + \frac{N_n + D_n + S_n - 3SS_n}{2} \\ & + \frac{H_n + V_n + R_n + 2I_n + 2SS_n - 3S_n - 3D_n - N_n}{4} \\ = {} & \frac{B_n + H_n + V_n + R_n + 2I_n + 2N_n}{8} \qquad \square \end{aligned}$$

Remark 2.1 *Theorem 2.1 can be considered to be an immediate consequence of Burnside's Lemma.*

2.1 A Numerical Example

In order to understand how Theorem 2.1 provides a lower bound for the basis size of the transition matrix $\boldsymbol{M_{n\times n}}$, look at the following example. Let $n = 2$. It can be shown that $\boldsymbol{M_{2\times 2}}$ is a 28×28 matrix [2]. Theorem 2.1 counts the number of symmetrically distinct $n \times n$ letter representations. Recall that $B_2 = 15$. Next, we calculate $H_2 = 7, V_2 = 7, R_2 = 7, I_2 = 7$, and $N_2 = 3$. Theorem 2.1 implies that modulo D_4 symmetry, the number of 2×2 letter representations is $\frac{15+7+7+7+14+6}{8} = 7$. Thus, the transition matrix associated with the $2\times 2\times p$ proper arrays must be at least as large as a 7×7 matrix. The goal of the author's research is to obtain a formula that calculates the actual basis size of $\boldsymbol{M_{n\times n}}$. As this example demonstrates, Theorem 2.1 provides not the actual basis size, but a lower bound on the basis size.

3 Generating Function for Diagonal Symmetry

In order to use Theorem 2.1, we need to find generating functions for H_n, V_n, R_n, I_n, and N_n. Since previous papers [2],[3] have discussed the generating functions for

$H_{m,n}$, $V_{m,n}$, and $R_{m,n}$, we will use the remainder of this paper to determine generating functions for I_n, N_n, and SS_n. The main technique for determining these generating functions involves the subdivision of the $n \times n$ array into either two halves or four quadrants. In either case, we can arbitrarily fill one of the halves/quadrants with any arrangement of letters, and then use symmetry to fill the remaining half/quadrants. The trick to this technique is to carefully subdivide around any row and column that will be fixed under the symmetry transformations. Thus, we must take into account whether n is an even or odd integer.

Remark 3.1 *Recall that* $\left\{ \begin{matrix} p \\ q \end{matrix} \right\}$ *is a Stirling number of the second kind which counts the partitions of a set of size p with q blocks. Thus, the arbitrary arrangement of letters describe in the preceding paragraph is* $\left\{ \begin{matrix} p \\ q \end{matrix} \right\}$*, where p is the number of squares in the half/quarter and q is the number of letters used to fill those squares.*

We begin by calculating the generating functions associated with I_n. Assume that the center of the $n \times n$ array is the origin and that the reflection in question is over the line $y = -x$. In this case, we subdivide the $n \times n$ array into two halves, one above and one below the line $y = -x$. In particular, define a **diagonal square** to be a square whose center lies on $y = -x$; i.e. a square fixed by the reflection. Define an **off-diagonal layer** to be two squares, each of which is the image of the other via reflection. Define the **bottom half of an off-diagonal layer** to be the square whose upper right vertex lies on the line $y = -x$. The other square is said to be the **upper half of the off-diagonal layer**. The **bottom half of the $n \times n$ array** is the union of all the squares which occur in the bottom half of an off diagonal layer. The **upper half of the $n \times n$ array** is the union of all the squares which occur in the top half of an off diagonal layer. The geometric strategy for constructing I_n is as follows.

I: Fill the diagonal squares with an arbitrary arrangement of letters.

II: Determine the number of off diagonal layers that are *completely* filled by a letter that occurs along the diagonal.

III: Fill the bottom halves of the remaining off diagonal layers with an arbitrary arrangement of letters and use reflective symmetry to determine the top halves of these off diagonal layers.

There are three types of letters that can fill the bottom half of an off-diagonal layer. The three possiblities consist of those letters that go to themselves under reflection, letters that reflect to a letter that does not appear in the bottom half of the array, and letters that reflect to another letter that has appeared in the bottom half of the array. In later this case, we say an **interchange** has occured. These possiblities are illustrated in Figure 3.1.

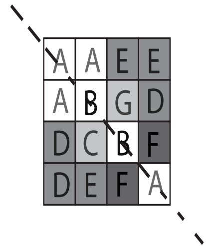

Figure 3.1: An example of a 4×4 letter representation that is fixed by diagonal reflection over $y = -x$. Note that A and B occur along the diagonal. One off-diagonal layer is filled with A. The remaining off-diagonal layers illustrate the three cases described in previous paragraph. The dark gray off-diagonal layer is filled by F, where F goes to itself via reflection. The medium gray layers demonstrate how D and E interchange positions when reflected over $y = -x$. The light gray layer demonstrates how C reflects to G, where G does not appear in the bottom half of the array.

Then

$$I_n = \sum_{\substack{d,w,p,m \\ q,T,r=0}}^{\infty} \frac{(\frac{n^2-n}{2})! \left\{ {n \atop d} \right\} \left\{ {p \atop 2w} \right\} \left\{ {q \atop m} \right\} \left\{ {r \atop T} \right\} (2w)! d^{\frac{n^2-n}{2}-p-q-r}}{2^w p! q! r! w! (\frac{n^2-n}{2} - p - q - r)!}$$

where,

1. d counts the letters that fill the diagonal squares.
2. $2w$ is the number of letters in the bottom half of the array that interchange among themselves when reflected to the top half of the array.
3. p counts the squares in the bottom half of the array that are occupied by these $2w$ letters.
4. m is the number of letters in the bottom half of the array that reflect to a letter that does not occur in either the bottom half or the diagonal.
5. q counts the squares in the bottom half of the array that are occupied by these m letters.
6. T is the number of letters in the bottom half of the array that reflect to themselves.
7. r counts the squares in the bottom half of the array that are occupied by these T letters.

Remark 3.2 *In the previous sum, we sum only over a range of values than ensure nonnegative factorials. Instead of explicitly writing the range of summation for each index, we use the shorthand notation of summing each variable from zero to infinity. This convention will be used throughout the paper.*

Before going further, we should recall the following representation of the Second Stirling numbers which will be useful in proving the theorems in Sections 3 through 5 [1, 4, 5].

Remark 3.3

$$\sum_{r=0}^{\infty} \frac{\left\{ \begin{matrix} r \\ t \end{matrix} \right\} y^r}{r!} = \frac{(e^y - 1)^t}{t!},$$

Theorem 3.1 *Let I_n be as previously defined. Then, I_n is $n!\left(\frac{n^2-n}{2}\right)!$ times the coefficient of $t^{\frac{n^2-n}{2}} v^n$ in the expansion of $\exp(e^t(e^v-1) + \frac{1}{2}(e^t-1)(e^t+3))$.*

Proof of Theorem 3.1: Define $I_n(t,v) =$

$$\sum_{\substack{n,d,w,p,\frac{n^2-n}{2} \\ m,q,T,r=0}}^{\infty} \frac{\left\{ \begin{matrix} n \\ d \end{matrix} \right\} \left\{ \begin{matrix} p \\ 2w \end{matrix} \right\} \left\{ \begin{matrix} q \\ m \end{matrix} \right\} \left\{ \begin{matrix} r \\ T \end{matrix} \right\} (2w)! d^{\frac{n^2-n}{2}-p-q-r} v^n t^p t^q t^r t^{\frac{n^2-n}{2}-p-q-r}}{2^w (\frac{n^2-n}{2} - p - q - r)! n! p! q! r! w!}.$$

Sum over $\frac{n^2-n}{2}$. Then, use Remark 3.3 to sum over n, p, q, and r. Finally, sum over d, w, m, and T to obtain the desired result. For a more detailed calculation, see the proof of Theorem 5.1. □

In Section 5, we need I_n^i, where I_n^i calculates the $n \times n$ letter representations which are fixed by diagonal reflection and have exactly i letters. In particular,

$$I_n^i = \sum_{\substack{d,w,p \\ m,q,r=0}}^{\infty} \frac{(\frac{n^2-n}{2})! \left\{ \begin{matrix} n \\ d \end{matrix} \right\} \left\{ \begin{matrix} p \\ 2w \end{matrix} \right\} \left\{ \begin{matrix} q \\ m \end{matrix} \right\} \left\{ \begin{matrix} r \\ i-d-2w-2m \end{matrix} \right\} (2w)! d^{\frac{n^2-n}{2}-p-q-r}}{2^w p! q! r! w! (\frac{n^2-n}{2} - p - q - r)!} \tag{3.1}$$

Using the same techniques we used to prove Theorem 3.1, we can easily prove the following corollary.

Corollary 3.1 *Let I_n^i be as defined previously. Then, I_n^i is $i!n!\left(\frac{n^2-n}{2}\right)!$ times the coefficient of $x^i t^{\frac{n^2-n}{2}} v^n$ in the expansion of $\exp(x(x+1)(e^t-1) + xe^t(e^v-1) + \frac{x^2}{2}(e^t-1))$.*

Remark 3.4 *We should note that Theorem 3.1 could be proven using the context of involutions acting on a finite set. In particular, suppose the involution is acting on a finite set in manner which provides t matching pairs and u fixed points. Then, by the argument used to derive $H_{2m+1,n}$ in [2], the number of partitions of the set that are inequivalent under the involution is $\exp(e^{y+x} + \frac{1}{2}e^{2y} - \frac{3}{2})$, where the powers of x correspond to u and the powers of y correspond to t. The following table provides the substitutions which allow us to transform $\exp(e^{y+x} + \frac{1}{2}e^{2y} - \frac{3}{2})$ into the generating functions for H_n, V_n, R_n, and I_n.*

	u	t	y	x
H_{2n}	*0*	$2n^2$	y	*0*
V_{2n+1}	n	$2n^2+n$	y	x
R_{2n+1}	*1*	$2n^2+n+[\frac{2n+1}{2}]$	y	$\ln(1+te^{-t})$
I_n	n	$\frac{n^2-n}{2}$	y	x

Table 3.1: The substitutions used to transform $e^{(e^{y+x}+\frac{1}{2}e^{2y}-\frac{3}{2})}$ into the associated generating functions. The u enumerates the fixed points while the t enumerates the matching pairs.

4 Generating Function for 90° Rotational Symmetry

Our next step is to find a generating function associated with N_n. When analyzing rotational symmetry, it is necessary to consider the case of n even as separate from the case of n odd. When n is even, we partition the $n \times n$ array into four quadrants, fill the upper left quadrant with any arbitrary letter configuration and use 90° clockwise rotation symmetry to complete the remaining three quadrants. Let A be a letter in the upper left quadrant. Define **the four-letter cycle of A** to be A, the letter in the upper right quadrant that is the 90° clockwise rotational image of A, the letter in the lower right quadrant that is the 180° image of A, and the letter in the lower left quadrant that is the 90° counterclockwise rotational image of A. There are four types of four-letter cycles.

Type 1: The four-letter cycle contains only one letter that appears in the upper left quadrant. In this case, the letter from the upper left quadrant is called a **singleton**.

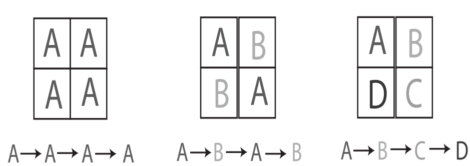

Figure 4.1: The three ways a singleton letter A can be transformed under 90° rotation.

Type 2: The four-letter cycle is composed of *three* letters from the upper left quadrant and *one* letter that does not occur in the upper left quadrant.

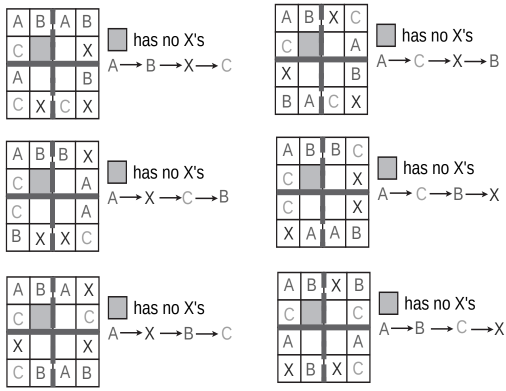

Figure 4.2: The six types of four-letter cycles where A, B, and C are the three letters that appear in the upper left quadrant, and X is a letter that is new relative to the upper left quadrant.

Type 3: The four-letter cycle is composed of *two* letters that appear in the upper left quadrant and *two* letters that are new relative to the upper left quadrant.

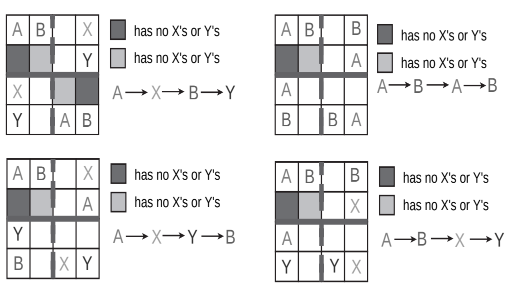

Figure 4.3: The four types of four-letter cycles where A and B are from the upper left quadrant, and X and Y are new with respect to the upper left quadrant.

Type 4: The four-letter cycle is composed of *four* letters from the upper left quadrant.

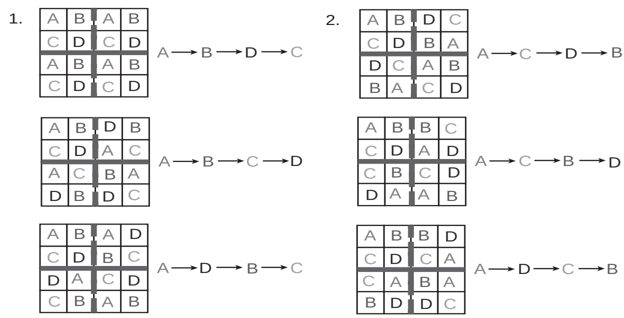

Figure 4.4: The six types of four-letter cycles composed of four letters that appear in the upper left quadrant.

Let $n = 2m$. The previous discussion implies that

$$N_{2m} = \sum_{\substack{j,q,s\\t,w=0}}^{\infty} \frac{\left\{ {m^2 \atop j} \right\} j! 3^q 2^{t-2w}}{q!s!t!w!(j-3s-2t-q-4w)!}$$

where,

1. j counts the letters that appear in the upper left quadrant. Note that $j \equiv q + 3s + 2t + 4w$.
2. q counts how many of these j letters are singletons.
3. $3s$ counts how many of these j letters are used in four-letter cycles of Type 2.
4. $2t$ counts how many of these j letters are used in four-letter cycles of Type 3.
5. $4w$ counts how many of these j letters are used in four-letter cycles of Type 4.

Using the techniques of Theorem 3.1, we easily prove Theorem 4.1

Theorem 4.1 *Let N_{2m} be as previously defined. Then, N_{2m} is $(m^2)!$ times the coefficient of x^{m^2} in the expansion of* $\exp(\frac{1}{4}(e^{4x} + 2e^{2x} + 4e^x - 7))$.

Our next step is to compute N_{2m+1}. The only difference between this situation and N_{2m} is the occurrence of a fixed middle square. Otherwise, the $(2m+1) \times (2m+1)$ array is divided into four $(m+1) \times m$ rectangular quadrants. Using a strategy similiar to that of N_{2m}, we obtain

$$N_{2m+1} = \sum_{\substack{j,k,q\\s,t,w=0}}^{\infty} \frac{\left\{ {m^2+m \atop j} \right\} j! 3^q 2^{t-2w}}{k!q!s!t!w!(j-3s-2t-4w-q-k)!}$$

where,

1. j, q, t, s, and w are as defined for N_{2m}.
2. k counts how many of these j letters that could occur the fixed middle square.

By using a strategy similiar to the one that proved Theorem 4.1, we can prove Theorem 4.2. The difference between Theorem 4.1 and Theorem 4.2 is a factor of e^x. This factor corresponds to the number of ways to fill the fixed middle square.

Theorem 4.2 *Let N_{2m+1} be as previously defined. Then, N_{2m+1} is $(m^2+m)!$ times the coefficient of x^{m^2+m} in the expansion of* $\exp(x + \frac{1}{4}(e^{4x} + 2e^{2x} + 4e^x - 7))$.

5 Fully Symmetrical Letter Representations

We say an $n \times n$ letter representation is **fully symmetric** if and only if it is fixed via horizontal reflection *and* $90°$ rotation; i.e. the letter representation is counted by SS_n. Once again, it is necessary to consider the case of n even separate from the case of n odd. We begin our discussion with SS_{2n}. In this case, we implement a two step process.

I: We partition the $n \times n$ array into four quadrants and fill the upper left hand quadrant with any letter configuration that is symmetrical with respect to reflection over the line $y = -x$.

II: We take letters from the upper left hand quadrant and using the horizontal reflection, along with the $90°$ clockwise rotational symmetry, complete the other three quadrants.

Step 2 is complex, since we first must classify the upper left quadrant letter via diagonal symmetry. In other words, given any letter in the upper left quadrant, it is either a **diagonal letter**, that is, a letter that appears along the diagonal, or an **off-diagonal letter**, that is, a letter that does not appear in the diagonal. Then, based on the type of letter that has been choosen, we use $90°$ clockwise rotation to determine its four-letter cycle. The number of four-letter cycles is restricted by the condition that the completed array must obey horizontal reflection. Hence, Step 2 must combine the techniques used to compute I_n and N_{2n}.

In particular,

$$SS_{2n} = \sum_{i=1}^{n^2} I_n^i T_2 T_3 T_4 S$$

where,

1. I_n^i counts the $n \times n$ letter representations which are fixed via reflection over $y = -x$ and have exactly i letters. These are the letters representations that may fill the upper left quadrant.
2. T_4 counts how many of the i letters form fully symmetrical four-letter cycles of Type 4.
3. T_3 counts how many of the i letters form fully symmetrical four-letter cycles of Type 3.
4. T_2 counts how many of the i letters form fully symmetrical four-letter cycles of Type 2.
5. S counts how many of the i letters are singletons.

We will now describe how to calculate T_4. We should note that T_4 consists of subcases, each of which is determined by the number of diagonal letters and the number of off-diagonal letters that occur in the given four cycle. In particular, we have $T_4 = T_{41}T_{42}T_{43}T_{44}$, where,

1. T_{41} counts the fully symmetrical four-letter cycles of consisting of four letters from the upper left quadrant distributed as follows: two of the letters are located in one off-diagonal layer while the other two letters occur along the diagonal. Examples of these four cycles are illustrated by Diagrams 1 and 2 of Figure 4.4.

2. T_{42} counts the fully symmetrical four-letter cycles consisting of four letters from the upper left quadrant distributed as follows: one letter occurs in the diagonal while the other three occur in off-diagonal layers. One of the off-diagonal layers must contain two letters while the other off-diagonal layer has only one letter.

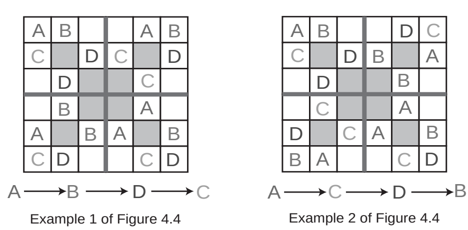

Figure 5.1: The two forms of a four-letter cycle enumerated by T_{42}. The dark gray (peach) squares do not contain D, B, or C. Note that B and C fill an off-diagonal layer, the D fills an entire off-diagonal layer, and the A is on the diagonal.

3. T_{43} counts the fully symmetrical four-letter cycles consisting of four letters from the upper left quadrant which fill three off-diagonal layers. Two of these off-diagonal layers have exactly one letter, while the third off-diagonal layer has two letters.

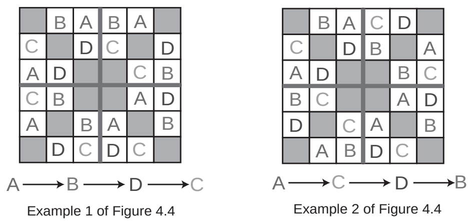

Figure 5.2: The two forms of a four-letter cycle enumerated by T_{43}. The dark gray (peach) squares do not contain A, B, C or D. Note that B and C fill one off-diagonal layer while A and D each fill an entire off-diagonal layer.

4. T_{44} counts the fully symmetrical four-letter cycles consisting of four letters from the upper left quadrant which fill two off-diagonal layers. Each of these off-diagonal layers has *two* letters.

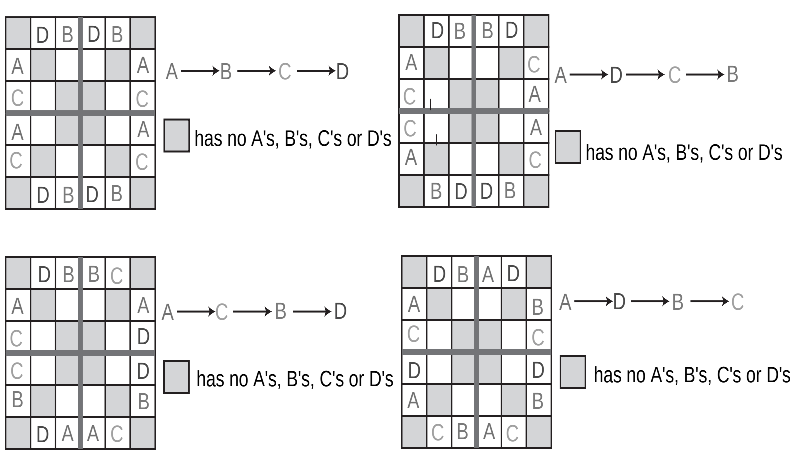

Figure 5.3: The four forms of a four-letter cycle enumerated by T_{44}. Note that A and D fill one off-diagonal layer while B and C fill the second off-diagonal layer.

By using similar arguments, we can show that $T_3 = T_{31}T_{32}T_{33}T_{34}$ and $T_2 = T_{21}T_{32}$, where,

1. T_{31} enumerates the fully symmetrical four-letter cycles which have the property that exactly two of the four letters occur along the diagonal of the upper left quadrant.

2. T_{32} enumerates the fully symmetrical four-letter cycles which have the property that exactly two of the letters occur in the upper left quadrant. Futhermore, each such letter completely fills an off-diagonal layer.

3. T_{33} enumerates the fully symmetrical four-letter cycles which have the property that exactly two of the letters occur in the upper left quadrant. One of these letters is along the diagonal while the other letter completely fills an off-diagonal layer.

4. T_{34} enumerates the fully symmetrical four-letter cycles which have the property that exactly two of the letters occur the upper left quadrant. Both of these letters occur in a single off-diagonal layer.

5. T_{21} enumerates the fully symmetrical four-letter cycles which have the property that exactly three of the letters come from the upper left quadrant. One of these letters occurs along the diagonal while the other two occur in a single off-diagonal layer.

6. T_{22} enumerates the fully symmetrical four-letter cycles which have the property that exactly three of the letters come from the upper left quadrant. These three letters occur in off-diagonal layers. One of the off-diagonal layers has one letter while the other off-diagonal layer has two letters.

By combining the expression for I_n^i in (3.1), along with algebraic expressions for $S, T_{41}, T_{42}, T_{43}, T_{44}, T_{31}, T_{32}, T_{33}, T_{34}, T_{21}$, and T_{22}, we obtain the following sum.

$$
\begin{aligned}
SS_{2n} = & \sum_{\substack{i,d,w,m,P,Q,R,k,j \\ l,a,q,r,u,g,h,p=0}}^{\infty} \frac{\left(\frac{n^2-n}{2}\right)! 2^{a+j+2g+h+p+u} 3^{i-w-m-3(j+k+l)-2(a+h+p+r+u+q)-g}}{(i-d-2w-2m-j-2l-2r-u-p)!k!j!l!a!g!q!r!u!h!p!} \\
& * \frac{\left\{ \begin{matrix} n \\ d \end{matrix} \right\} \left\{ \begin{matrix} P \\ 2w \end{matrix} \right\} \left\{ \begin{matrix} Q \\ m \end{matrix} \right\} \left\{ \begin{matrix} R \\ i-d-2w-2m \end{matrix} \right\}}{w!2^w P!Q!R!(\frac{n^2-n}{2}-P-Q-R)!(d-2k-j-2q-h-u)!} \\
& * \frac{d^{\frac{n^2-n}{2}-P-Q-R}(2w)!(w+m)!d!(i-d-2w-2m)!}{(w+m-k-j-l-2a-g-h-p)!}
\end{aligned}
$$

where,

1. i is the number of letters in the upper left quadrant.

2. d is the number of letters along the upper left quadrant diagonal.

3. $2w$ is the number of letters in the bottom half of the upper left quadrant that interchange among themselves when reflected over the line $y = -x$.

4. P is the number of spaces in the bottom half of the upper left quadrant that are filled by $2w$ letters.

5. m is the number of letters in the bottom half of the upper left quadrant that, when reflected over $y = -x$, go to a letter that does not appear in the bottom half.

6. Q is the number of spaces in the bottom half of the upper left quadrant occupied by the m letters.

7. $t = i - d - 2w - 2m$ is the number of letters in the bottom half of the upper left quadrant that reflect to themselves over the line $y = -x$.

8. R counts the spaces in the bottom half of the upper left quadrant occupied by the t letters.

9. k is the number of letters from the upper left quadrant that occur in the four-letter cycles enumerated by T_{41}.

10. j is the number of letters from the upper left quadrant that occur in the four-letter cycles enumerated by T_{42}.

11. l is the number of letters from the upper left quadrant that occur in the four-letter cycles enumerated by T_{43}.

12. a is the number of letters from the upper left quadrant that occur in the four-letter cycles enumerated by T_{44}.

13. q is the number of letters from the upper left quadrant that occur in the four-letter cycles enumerated by T_{31}.

14. r is the number of letters from the upper left quadrant that occur in the four-letter cycles enumerated by T_{32}.

15. u is the number of letters from the upper left quadrant that occur in the four-letter cycles enumerated by T_{33}.

16. g is the number of letters from the upper left quadrant that occur in the four-letter cycles enumerated by T_{34}.

17. h is the number of letters from the upper left quadrant that occur in the four-letter cycles enumerated by T_{21}.

18. p is the number of letters from the upper left quadrant that occur in the four-letter cycles enumerated by T_{22}.

Theorem 5.1 *Let SS_{2n} be as previously defined. Then, SS_{2n} is $n!(\frac{n^2-n}{2})!$ times the coefficient of $x^nT^{\frac{n^2-n}{2}}$ in the expansion of* $\exp(\frac{-9}{2}+\frac{1}{2}e^{2T+2x}+e^{T+x}+\frac{1}{2}e^{4T}+2e^{2T}+\frac{1}{2}e^{4T+2x})$.

Before we prove Theorem 5.1, we should recall the following version of Taylor's Theorem.

Remark 5.1

$$e^{a\frac{d^n}{dw^n}}e^{bw}=e^{ab^n}e^{bw}$$

Proof of Theorem 5.1: Define $SS_{2n}(x,T)$ as the following two variable generating function.

$$\sum_{\substack{i,d,w,m,P,Q,R,k,j\\ l,a,q,r,u,g,h,p=0}}^{\infty}\frac{2^{a+j+2g+h+p+u}3^{i-w-m-3(j+k+l)-2(a+h+p+r+u+q)-g}(i-d-2w-2m)!}{(i-d-2w-2m-j-2l-2r-u-p)!k!j!l!a!g!q!r!u!h!p!P!Q!R!}$$
$$*\frac{(2w)!(w+m)!d!\left\{\begin{matrix}n\\ d\end{matrix}\right\}\left\{\begin{matrix}R\\ i-d-2w-2m\end{matrix}\right\}d^{\frac{n^2-n}{2}-P-Q-R}x^nT^{\frac{n^2-n}{2}}}{(w+m-k-j-l-2a-g-h-p)!}$$
$$*\frac{\left\{\begin{matrix}P\\ 2w\end{matrix}\right\}\left\{\begin{matrix}Q\\ m\end{matrix}\right\}}{w!2^wn!(\frac{n^2-2}{2}-P-Q-R)!(d-2k-j-2q-h-u)!}.$$

Summing over $\frac{n^2-n}{2}$ gives us

$$\sum_{\substack{i,d,w,m,P,Q,R,k,j\\ l,a,q,r,u,g,h,p=0}}^{\infty}\frac{2^{a+j+2g+h+p+u}3^{i-w-m-3(j+k+l)-2(a+h+p+r+u+q)-g}(i-d-2w-2m)!e^{dT}}{(i-d-2w-2m-j-2l-2r-u-p)!k!j!l!a!g!q!r!u!h!p!P!Q!R!}$$
$$*\frac{(2w)!(w+m)!d!\left\{\begin{matrix}n\\ d\end{matrix}\right\}\left\{\begin{matrix}R\\ i-d-2w-2m\end{matrix}\right\}\left\{\begin{matrix}P\\ 2w\end{matrix}\right\}\left\{\begin{matrix}Q\\ m\end{matrix}\right\}x^nT^{P+Q+R}}{w!2^wn!(d-2k-j-2q-h-u)!(w+m-k-j-l-2a-g-h-p)!}.$$

We now simplfy $SS_{2n}(x,T)$ in the following manner.

$$\sum_{\substack{i,d,w,m,k,j,l\\a,q,r,u,g,h,p=0}}^{\infty} \frac{2^{a+j+2g+h+p+u}3^{i-w-m-3(j+k+l)-2(a+h+p+r+u+q)-g}e^{dT}}{k!j!l!a!g!q!r!u!h!p!(i-d-2w-2m-j-2l-2r-u-p)!}$$
$$* \frac{(e^x-1)^d(e^T-1)^{i-d-m}(w+m)!}{w!2^w!m!(w+m-k-j-l-2a-g-h-p)!(d-2k-j-2q-h-u)!}$$

$$= \sum_{\substack{w,m,k,j,l\\a,q,r,u,g,h,p=0}}^{\infty} \frac{e^{3(e^T-1)+3e^T(e^x-1)}2^{a+j+2g+h+p+u}3^{w+m-j-k-l-2a-h-p-g}}{w!2^w k!j!a!g!q!r!u!h!p!}$$
$$* \frac{(e^T(e^x-1))^{2k+j+2q+h+u}(e^T-1)^{2w+m+j+2r+u+p+2l}(w+m)!}{w!2^w!m!(w+m-k-j-l-2a-g-h-p)!}$$

$$= \sum_{\substack{w,m,k,j,l\\a,g,h,p=0}}^{\infty} \frac{e^{-2+e^{2T+2x}+e^{x+T}}2^{a+j+2g+h+p}3^{w+m-j-k-l-h-p-j-2a-g}}{k!j!l!a!g!h!p!}$$
$$* \frac{(e^T(e^x-1))^{2k+j+h}(e^T-1)^{2w+m+j+2l+p}(w+m)!}{w!2^w m!(w+m-k-j-l-2a-h-g-p)!}$$

$$= \sum_{w,m,a=0}^{\infty} \frac{e^{-2+e^{2T+2x}+e^{x+T}}2^a(e^T-1)^{2w+m}(e^{2T+2x}+6)^{w+m-2a}(w+m)!}{a!m!w!2^w(w+m-2a)!}$$

$$= \sum_{w,m,a=0}^{\infty} \frac{e^{-2+e^{2T+2x}+e^{x+T}}2^a(e^T-1)^{2w+m}(e^{2T+2x}+6)^{w+m-2a}\left[\frac{d}{d\alpha}\right]_{\alpha=1}^{2a}\alpha^{w+m}}{m!a!w!2^w!}$$

$$= e^{-2+e^{2T+2x}+e^{x+T}}e^{\frac{2}{(6+e^{2T+2x})^2}\frac{d^2}{d^2\alpha}}e^{\frac{\alpha}{2}(6+e^{2T+2x})(e^T-1)(e^T+1)}\Big|_{\alpha=1}$$

By applying Remark 5.1 to the previous line (with $a=\frac{2}{(6+e^{2T+2x})^2}$ and $b=\frac{1}{2}(6+e^{2T+2x})(e^T-1)(e^T+1)$), we obtain the desired result. □.

We end this paper by calculating the generating function for SS_{2q+1}. We begin by subdiving the $(2q+1)\times(2q+1)$ array into four quadrants, each a $q\times q$ array that avoids the central row and central column. The union of central row and central column is called the **central cross**. We can then implement the following five step process to arrive at the geometric sum associated with SS_{2q+1}.

I: We fill in the top q spots of the central column with an arbitrary arrangement of letters.

II: We use 90° clockwise rotation and vertical reflection to fill in all the spaces of the central cross except for the fixed middle position, where the **fixed middle position** is the square that occurs both in the central column and the central row. There are five ways to fill the central cross minus the fixed middle position. These possiblities are illustrated in Figure 5.4.

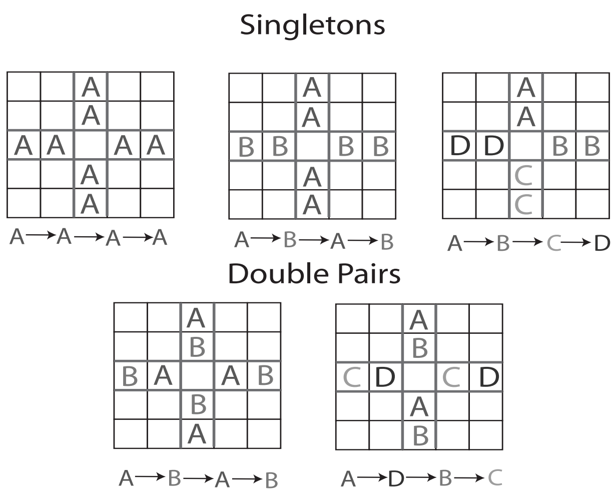

Figure 5.4: The five ways to complete the central cross minus the fixed middle position in a fully symmetrical manner. Below each diagram, we record the four letter cycle. In the first row, only A occurs in the top part of the central cross and is considered to be a singleton. In the second row, the image of A under the rotation and reflections contains B, where B also occurs in the top part of the central cross. Hence, A and B form a double pair.

III: We take letters that occur in the central cross and place them either in the diagonal spaces of the upper left quadrant or in the bottom half of the upper left quadrant. The only letters that occur in diagonal spaces are those singletons that go to themselves under rotation. The 90° clockwise rotation and vertical reflection uniquely determine how the letters appear in the remaining three quadrants.

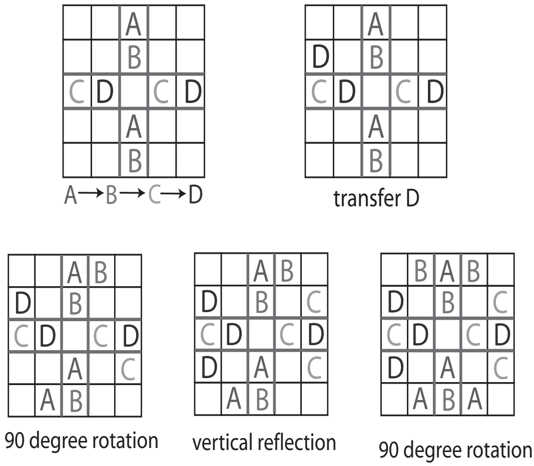

Figure 5.5: An illustration of how a letter that appears in the cross may uniquely occur in the quadrants. We begin by transfering D into the bottom half of the upper left quadrant. Then, using 90° rotation and vertical reflection, we determine the image of D in the other three quadrants.

IV: We now complete the free spaces of the quadrants with letters that do not occur in the central cross. This is done by determining T_4, T_3, T_2, and S, where T_4, T_3, T_3, and S are as defined previously for SS_{2n}.

V: We fill the fixed middle position of the central cross. There are four ways to fill this position: by a singleton from the cross that goes to itself under rotation and reflection, by a singleton in the diagonal of the upper left quadrant that goes to itself under rotation and reflection, by a letter in the upper left quadrant that completely fills an off diagonal layer and goes to itself under the symmetry mappings, or a letter that does not previously occur.

Following the steps outlined above, we arrive at a summation of 38 variables. The derivation of the exponential generating function utilizes the techniques of Theorem 5.1. Details are available, upon request, from the author.

Theorem 5.2 *Let SS_{2q+1} be as previously defined. Let $C = \frac{3}{2}e^{2y} + \frac{1}{2}e^{2y+2z} + \frac{1}{2}e^{4y+2z} - \frac{9}{2} + \frac{1}{2}e^{2x+4y} + e^{x+y+z} + \frac{1}{2}e^{2y+2x}$. Then, SS_{2q+1} is $(q!)^2 \left(\frac{q^2-q}{2}\right)!$ times $y^{\frac{q^2-q}{2}} z^q x^q$ in the expansion of $\exp(C)\exp(x+y+z)$.*

6 Open Questions

By using a particular decomposition of the Bell numbers [1, 4, 5], and applying various symmetry transformations to $n \times n$ arrays, we discovered a formula which calculates the number of letter representations modulo D_4 symmetry. The number only provides a lower bound for the basis size of the transition matrix $\boldsymbol{M_{n\times n}}$. It is an open question to find a formula, similiar in nature to Theorem 2.1, that exactly calculates the basis size of $\boldsymbol{M_{n\times n}}$. The author is attempting to find such a formula by extrapolating the results of this paper to $n \times n$ arrays of circled letters.

Another promising area of research involves exploring the connections between $n \times n \times p$ proper arrays and percolation theory. At the present time, the author has not explored the connection in any depth but realizes that the stochastic and probablisitic techniques of percolation theory could, when applied to the representation of an $n \times n \times p$ proper array as a bond percolation on Z^3 with an open cluster at the origin (see [2]), give rise to a whole new catagory of results.

Appendix A: Numerical Data

The following table provides, for small integer values of n, numerical values of I_n, N_n, and SS_n. All the values came from the generating functions given by either Sections 3 through 5 or Sections 3 and 4 of [2]. The results were verified by a short Maple program.

$n \times n$	H_n	V_n	R_n	I_n	N_n	SS_n
2×2	7	7	7	7	3	3
3×3	549	549	339	549	23	17
4×4	428131	428131	428131	874571	931	195
5×5	43537978637	43537978637	12126858113	43537978637	170765	8999

Table 1: Numerical Data for certain $n \times n$ letter representations

Appendix B: Generating Function for $\boldsymbol{D_n}$

In this appendix, we will enumerate those $n \times n$ letter representations which are symmetrical with respect to both diagonal reflections, i.e. those letter representations enumerated by D_n. The technique used to calculate D_n is similar to the technique used in the computation of $S_{m,n}$ [2], namely, subdividing the $n \times n$ array into an X and four quadrants. The X is the union of the squares whose centers lie on the line $y = -x$ and the line $y = x$. We must analyze the case of n even as separate from n odd since, if n is odd, the X contains a central square which is fixed by the two diagonal reflections.

First, we will work with a $2n \times 2n$ array. Assume the center of this array is at the origin. In order to calculate those $2n \times 2n$ letter representations fixed by the two diagonal mapping, we employ the following four part strategy.

I. Fill in the top half of the $y = -x$ diagonal with an arbitrary arrangement of letters, where by **top half of the $\boldsymbol{y = -x}$ diagonal**, we mean those squares, occuring in the first n rows, whose centers lie on $y=-x$. Then, by reflecting over $y=x$, we can fill in the remainder of the $y=-x$ diagonal, which we will call the **bottom half of the diagonal**. There are three possible ways a letter in the top half of the diagonal is transformed, via reflection, into the bottom half. It can reflect to itself; it can reflect to a letter which does not appear in the top half; it can reflect to another letter which appears in the top half. These are the same three possiblities that occured in the computation of I_n.

II. Fill in the top half of the $y=x$ diagonal. There are two possiblities. The first possiblity involves letters occuring in the $y=-x$ diagonal; these letters *must* be fixed by reflection over the $y=x$ diagonal. The second possiblity involves letters which do not occur in $y=-x$ diagonal. For these letters, we use the argument of Step I to complete the bottom half of the $y=x$ diagonal. The only difference is that reflect occurs over $y=-x$.

III. Transfer the letters that occur along the diagonal into the left quadrant, where the **left quadrant** consists of those n^2-n squares lying in the region determined by squares whose upper right corners lie on $y=-x$ and squares lower left corners lie on $y=x$. Then, by using the two diagonal reflections, we are able to *uniquely* determine the images which occur in the other three quadrants.

IV. Fill in the remaining squares of the left quadrant with an arbitrary arrangement of letters, none of which occur along the diagonals. Such a letter is either a singleton or part of a double pair. A **singleton letter** is a letter whose image, in the remaining three quadrants, is never another letter that appears in the left quadrant. There are five possible ways a letter can be a singleton. These five ways are illustrated in Figure 1.

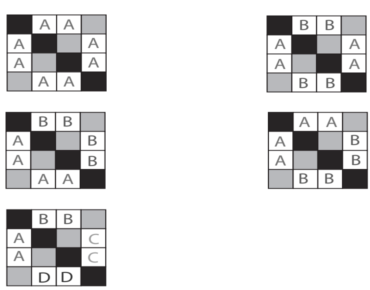

Figure 1: The five ways a singleton letter can be transformed under symmetry. None of the letters occur in the dark gray (pink) diagonal ($y=-x$) or the light gray (orange) diagonal ($y=x$).

The second type of letter present in the left quadrant can be considered to be part of a double pair. A letter is part of **double pair** when its image in one of the other three quadrants is another letter orginally present in the left quadrant. Figure 2 illustrates the six ways double pairs transform in a manner fixed by the two diagonal reflections.

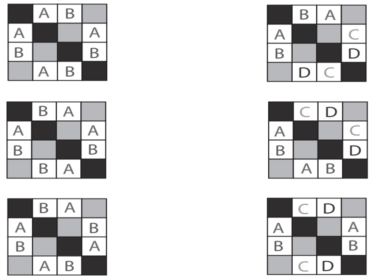

Figure 2: The six ways two letters, each appearing in the left quadrant, form double pairs. Once again, these letters do not appear in the shaded squares.

By utilizing this four step process, we obtain

$$D_{2n} = n!(n^2-n)! \sum_{\substack{q,s,w,t,p\\ S,W,v,r,u=0}}^{\infty} \frac{q!p!r!\left\{ \begin{array}{c} n \\ q \end{array} \right\}\left\{ \begin{array}{c} n-t \\ p \end{array} \right\}\left\{ \begin{array}{c} n^2-n-v \\ r \end{array} \right\}}{(q-2s-w)!(n-t)!(p-2S-2W)!(r-2u)!}$$
$$* \frac{w^t 6^u 5^{r-2u}(2q+2p-2s-w-2S-W)^v}{2^{s+S+u}(n^2-n-v)!s!w!t!S!W!v!u!}$$

where,

1. q counts the letters in the top half of the $y=-x$ diagonal.
2. s counts the interchanges which occur among the q letters. For definition of interchange, see page 6.
3. w counts those q letters that are fixed by reflection over the $y=x$ diagonal.
4. t counts the squares in the top half of the $y=x$ diagonal filled by the w letters.
5. p counts the letters which occur in the top half of the $y=x$ diagonal, but *not* in the $y=-x$ diagonal.
6. S counts the interchanges which occur among the p letters.
7. W counts those p letters that are fixed by reflection over the $y=-x$ diagonal.
8. v counts the squares in the left quadrant filled by letters which occur along the diagonal.
9. r counts the letters which occur in the left quadrant, but *not* in the two diagonals.
10. u counts the double pairs which occur among the r letters.

If we multiply the previous sum by $\frac{x^n}{n!}$, $\frac{z^n}{n!}$, $\frac{y^{n^2-n}}{(n^2-n)!}$, apply Remark 3.2 to sum over $n-t, n, n^2-n-v$, and then note that the remaining indicies are exponential sums, we can prove Theorem 6.1. For more detail, see the proof of Theorem 5.1 or [2, p. 17].

Theorem 6.1 *Let D_{2n} be as previously defined. Then, D_{2n} is $(n!)^2(n^2-n)!$ times the coefficient of $x^n z^n y^{n^2-n}$ in the expansion of* $\exp(2e^{2y}-2e^y-2+e^{x+y+z}+\frac{1}{2}e^{2y+2z}+\frac{1}{2}e^{2y+2x})$.

The strategy for calculating D_{2n+1} is akin to that of calculating D_{2n}. The only difference is that we must contend with a fixed middle square. This middle square can be filled with a letter that maps to itself under both diagonal reflections or a letter that appears only in that square. In either case, a slight adjustment to the previous sum allows us to prove the following theorem. Upon request, details are available from the author.

Theorem 6.2 *Let D_{2n+1} be as previously defined. Then, D_{2n+1} is $(n!)^2(n^2)!$ times the coefficient of $x^n z^n y^{n^2}$ in the expansion of* $\exp(2e^{2y} - 2e^y - 2 + e^{x+y+z} + \frac{1}{2}e^{2y+2z} + \frac{1}{2}e^{2y+2x} - 2)\exp(x + y + z)$.

n	2	3	4	5
D_n	5	79	7567	3301667

Table 2: Numerical Data for D_n

Acknowledgments

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