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Examples of goal-minimally k-diametric graphs for some small values of k

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Abstract

A graph G with diameter k is said to be goal-minimally k-diametric if for every edge uv of G distance $d_{G-uv}(x,y) > k$ if and only if $\{x,y\} = \{u,v\}$. It is rather difficult to construct such graphs. In this paper we give examples of such graphs for several small values of k. In particular, we present infinitely many goal-minimally 4-diametric graphs and 6-diametric graphs.

1 Introduction

We consider finite, undirected, and simple graphs G with the vertex set $V(G)$ and the edge set $E(G)$. Our graph-theoretical terminology and notation are based on Chartrand and Lesniak [8]. The number of vertices [edges] of G is often referred to as the order [size] of G and denoted by n [m, respectively]. The degree of a vertex u is denoted by $\deg(u)$ and the maximum [minimum] degree by Δ [δ, respectively]. If G is a connected graph, then the distance $d(u,v)$ between two vertices u and v is defined as the length of a u-v geodesic (a shortest path from u to v). The eccentricity $ec(u)$ of a vertex u is the distance to a farthest vertex from u. For any integer i we denote $D_i(u) = \{x \in V(G) \mid d(u,x) = i\}$. Thus $D_0(u) = \{u\}$ and the neighborhood $N(u) = D_1(u)$. Clearly, the sets $D_0(u), \ldots, D_{ec(u)}(u)$ form the distance decomposition of $V(G)$ from u. The diameter $\operatorname{diam}(G)$ [radius $\operatorname{rad}(G)$] is the maximum [minimum, respectively] eccentricity among the vertices of G. The girth of G is the length of a shortest cycle in G.

A graph G with diameter k is also called a k-diametric graph. It is said to be minimal with respect to diameter or, more precisely, minimally k-diametric, if for any edge $e \in E(G)$ we have $\operatorname{diam}(G-e) > k$. The distance function in $G-e$ is allowed to exceed k in an arbitrary pair of vertices. If we restrict this to the ends of

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the edge e, then we get the following special class of minimally k-diametric graphs. A graph G with diameter k is said to be goal-minimal with respect to diameter or, more precisely, goal-minimally k-diametric (k-GMD for short), if for each edge uv of G the inequality $d_{G-uv}(x,y) > k$ holds if and only if $\{x,y\} = \{u,v\}$. Clearly, the complete graphs are precisely the 1-GMD graphs, but already the case $k = 2$ is interesting. The complete bipartite graphs $K_{r,s}$ with $r, s \geq 2$ are examples of 2-GMD graphs.

The minimal graphs with respect to diameter were studied under various names (e.g. graphs without superfluous edges, or critical, or edge-critical, or edge-diameter critical, or diameter-minimal) by many authors (see e.g. [2, 3, 4, 7, 10, 11, 12, 15, 16, 17, 18, 19, 22, 23, 26]). Note that so-called vertex-critical graphs (with respect to diameter) were also studied (see e.g. [5, 6, 9, 13, 25]). In this paper we deal with k-GMD graphs and present several such graphs. The goal-minimal graphs with respect to diameter were introduced by Kyš in [21] which called them "diameter strongly critical graphs". He gave several properties of such graphs. Further contributions were done by Gliviak and Plesník [14]. Some of known results are summarized below.

Theorem 1 [21, 14] *The girth of a k-GMD graph G of order at least 3 is $k+2$ and every edge of G lies in a cycle of length $k+2$.*

Theorem 2 [21] *For any two non-adjacent vertices u and v of a k-GMD graph there are at least two internally disjoint u-v paths of length not exceeding k.*

The class of all 2-GMD graphs is rather rich.

Theorem 3 [21] *Let G be a graph without 3-cycles. Then there is a 2-GMD graph containing G as an induced subgraph.*

By Theorem 1, in any k-GMD graph all cycle lengths $3, 4, \ldots, k+1$ are forbidden. Thus a result of Alon et al. [1] can be applied to receive the following bound on the size.

Theorem 4 [14] *For any k-GMD graph of order n and size m we have*

$$m \leq \frac{1}{2}(n^{1+1/\lfloor\frac{k+1}{2}\rfloor} + n)$$

In paper [14] another related bound was derived:

Theorem 5 [14] *Let G be a k-GMD graph with $k \geq 3$, minimum degree δ and order n. Then for any vertex $u \in V(G)$ we have:*

(a) If k is odd, then

$$n \geq \begin{cases} \deg(u)(k+1)/2 + \max\{2ec(u) - k - 1, 1\} & \text{if} \quad \delta = 2 \\ \deg(u)[(\delta-1)^{(k+1)/2} - 1]/(\delta - 2) + \max\{2ec(u) - k - 1, 1\} & \text{if} \quad \delta > 2 \end{cases}$$

(b) If k is even, then

$$n \geq \begin{cases} \deg(u)(k/2+1) + 2ec(u) - k & \text{if} \quad \delta = 2 \\ \deg(u)[(\delta-1)^{k/2} - 1]/(\delta - 2) + 2ec(u) - k & \text{if} \quad \delta > 2 \end{cases}$$

Since $\frac{k+1}{2} \leq ec(u)$ and we can take for u a vertex of maximum degree, we get:

Corollary 1 [14] *Let G be a k-GMD graph with $k \geq 3$, maximum degree Δ, minimum degree δ and order n. Then we have:*

(a) If k is odd, then

$$n \geq \begin{cases} \Delta(k+1)/2 + 1 & \text{if} \quad \delta = 2 \\ \Delta[(\delta-1)^{(k+1)/2} - 1]/(\delta - 2) + 1 & \text{if} \quad \delta > 2 \end{cases}$$

(b) If k is even, then

$$n \geq \begin{cases} \Delta(k/2 + 1) + 2 & \text{if} \quad \delta = 2 \\ \Delta[(\delta-1)^{k/2} - 1]/(\delta - 2) + 2 & \text{if} \quad \delta > 2 \end{cases}$$

This gives an upper bound for Δ immediately.

As to bounding the order from above, the well known Moore bound (see e.g. [8], p. 312) is applicable but a better one was derived:

Theorem 6 [14] *Let G be a k-GMD graph with $k \geq 3$, maximum degree Δ, minimum degree δ and order n. Then we have:*

$$n \leq 1 + \delta\Big[\frac{(\Delta-1)^{k-1} - 1}{\Delta - 2} + \frac{(\Delta-1)^{k-2}(\Delta-2)}{2}\Big]$$

For example, if $k = 3$ we get the following simple bounds.

Corollary 2 [14] *Let G be a 3-GMD graph with maximum degree Δ and minimum degree δ. Then the order n of G fulfills the following inequalities*

$$1 + \delta\Delta \leq n \leq 1 + \delta\Big[\Delta + \frac{(\Delta-1)(\Delta-2)}{2}\Big]$$

Kyš [21] presented two infinite classes of 4-GMD graphs. They are subdivisions of complete graphs and subdivisions of complete bipartite graphs without an edge, respectively. As to other diameters, he gave one example of a 3-GMD graph and one example of a 6-GMD graph. Moreover, he raised the conjecture that for every integer $k \geq 1$ there exists a k-GMD graph. This conjecture appeared rather difficult to prove. As reported in [14], a computer search gave about fifty examples of 3-GMD graphs (the maximum order was 38). To exclude isomorphic graphs the following approach was applied: For each generated graph a list of cardinalities of the distance decomposition for every vertex was produced and then lexicographically ordered lists were compared. (We admit that also some non-isomorphic graphs could be excluded.) It is the purpose of this paper to give further examples of k-GMD graphs. In Section 2 we cover a few odd diameters by presenting computer results for $k = 3, 5$ and 7. In Section 3 we present a construction of 4-GMD graphs which includes both Kyš's constructions [21] as special cases and produces infinitely many new graphs. Another construction is presented in Section 4 that provides infinitely many 6-GMD graphs.

Table 1: A 4-regular 3-GMD graph of order 20

1: 2 6 7 17	2: 3 8 20	3: 4 9 16	4: 5 6 10	5: 7 8 11
6: 12 19	7: 9 13	8: 12 14	9: 12 15	10: 13 14 15
11: 15 16 17	12: 18	13: 16 18	14: 17 19	15: 20
16: 19	17: 18	18: 20	19: 20	

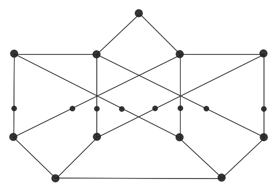

Figure 1: A 5-GMD graph of order 19

Finally Section 5 provides examples for other small even diameters: $k = 8, 10, 12$ and 14.

Some graphs are presented by figures where we have drawn standard diagrams, but some others are given by tables. Every table describes a graph G of order n where $V(G) = \{1, 2, \ldots, n\}$. Edges are defined by a list of vertex neighbors (a vertex followed by colon and adjacent vertices). However, each edge is given exactly once. Thus some vertices have reduced neighbor sets and those with empty neighbor sets are deleted. For example, a complete graph of order 4 (with vertices 1, 2, 3, 4) can be described as follows: 1: 2 3 4, 2: 3 4, 3: 4.

2 Small odd diameters

In paper [14] we have reported about fifty 3-GMD graphs obtained by a computer search; we presented 12 of them. Among those only one was regular (it has 12 vertices and degree 3). Since that the collection has been extended by a few new graphs found by a computer. Now we know sixty 3-GMD graphs in total (their order ranges from $n = 8$ to $n = 38$ and there are gaps: for example, there is no 3-GMD graph of order 10 or 11) and between them there are five 4-regular: they have 19, 20, 21, 22, and 23 vertices. That of order 19 appeared to be known Robertson's graph [24] and that of order 20 is given in Table 1.

Our computer search gave also a collection of forty five 5-GMD graphs. Their orders fulfill the interval from 19 to 30 excepting number 29. Two of them are in Figures 1 and 2. The collection contains only two 5-GMD graphs without vertices of degree two; they are 3-regular and have 26 and 28 vertices. That of order 28 is drawn in Figure 2.

Our computer search gave only four 7-GMD graphs; their orders are 35 (two graphs), 39, and 43; all have also some vertices of degree 2. One of them is in Table 2.

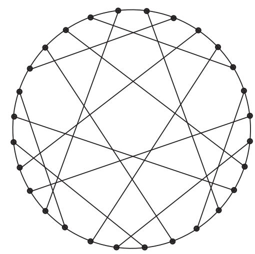

Figure 2: A 3-regular 5-GMD graph of order 28

Table 2: A 7-GMD graph of order 35

1: 2 9 35	2: 3 20	3: 4	4: 5 14	5: 6 29
6: 7	7: 8 23	8: 9	9: 10	10: 11
11: 12 31	12: 13	13: 14 25	14: 15	15: 16
16: 17 33	17: 18	18: 19 28	19: 20	20: 21
21: 22	22: 23 32	23: 24	24: 25	25: 26
26: 27 35	27: 28	28: 29	29: 30	30: 31
31: 32	32: 33	33: 34	34: 35	

Remark. Very recently Gyürki [20] discovered a few regular 3-GMD graphs and one 5-GMD graph of larger orders.

3 Diameter 4

In this section we design an infinite family of 4-GMD graphs. Let us consider a complete graph K_p with $p \geq 4$ vertices. Let V and E denote its vertex set and edge

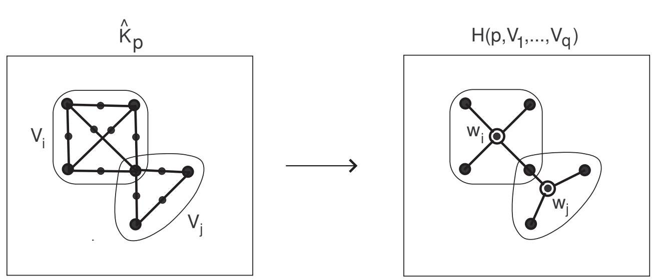

Figure 3: Constructing 4-GMD graphs from $\hat{K}_p$

set, respectively. Inserting into every edge $xy \in E$ one new vertex s_{xy} we obtain a subdivision $\hat{K}_p$ with vertex set $V \cup S$ of cardinality $|V| + |E|$ and edge set, say, T of cardinality $2|E|$. The original vertices (belonging to V)) are called basic vertices and those belonging to S are called subdividing vertices of $\hat{K}_p$. Graph $\hat{K}_p$ is a 4-GMD graph as observed by Kyš [21]. We are going to modify this graph further. Let $q \geq 0$ be an integer and sets $V_1, \ldots, V_q \subset V$ be such that:

(i) $|V_i| \geq 3$ for every $i = 1, \ldots, q$.

(ii) $|V_i \cap V_j| \leq 1$ whenever $i \neq j$ for all $i, j = 1, \ldots, q$.

(iii) There are 4 distinct basic vertices b_1, b_2, b_3, b_4 with the following property. Let $\tilde{V}(b_1, b_2)$ denote the set of basic vertices consisting of vertices b_1, b_2 and all vertices of a set V_i containing b_1 and b_2 (if any). Let $\tilde{V}(b_3, b_4)$ be defined similarly. Then we ask that $\tilde{V}(b_1, b_2) \cap \tilde{V}(b_3, b_4) = \emptyset$.

Note that (iii) is fulfilled e.g. if there is no set V_i containing b_1, b_2 and no set V_j containing b_3, b_4 or if $q \geq 2$ and the sets $V_1, \ldots, V_q$ are pairwise disjoint.

Following Figure 3, we take $\hat{K}_p$ and for every $i = 1, \ldots, q$ do:

(1) Delete all vertices s_{xy} with $x, y \in V_i$.

(2) Add a new vertex w_i and join it to every vertex $x \in V_i$ by a new edge.

The resulting graph is denoted by $H(p, V_1, \ldots, V_q)$. Clearly, if $q = 0$ then we have $\hat{K}_p$, which is the first class of 4-GMD graphs constructed by Kyš in [21]. If $q = 1$ and $|V_1| = p - 2$ we get a part of his second class of 4-GMD graphs. If $q = 2$ and $V_1 \cap V_2 = \emptyset$ then we get the remaining graphs of his second class of 4-GMD graphs. In general, we have:

Theorem 7 *Any graph $H(p, V_1, \ldots, V_q)$ is a 4-GMD graph.*

Proof: Let $H = H(p, V_1, \ldots, V_q)$. Put $p_i = |V_i|$ for all i. Our construction can be realized as follows. For every $i = 1, \ldots, q$ we merge all $p_i(p_i - 1)/2$ midvertices of the 2-paths connecting vertices of V_i into one group vertex w_i and thus all the edges of these 2-paths incident to the same basic vertex are merged into one group edge. According to the assumptions (i) and (ii) these group elements are determined uniquely. For a midvertex x symbol $\bar{x}$ denotes its group vertex if x was merged, else we put $\bar{x} = x$. Therefore for distances we have $d_H(\bar{x}, \bar{y}) \leq d_{\hat{K}_p}(x, y)$ and consequently, $\mathrm{diam}(H) \leq \mathrm{diam}(\hat{K}_p) = 4$. Further, we see that each edge of H lies in a 6-cycle and lies in no shorter cycle. Thus for any two vertices u, v of H and any edge e of H we have $d_{H-e}(u, v) = 5$ when $e = (u, v)$ and $d_{H-e}(u, v) \leq 4$ when $e \neq (u, v)$. To prove that H is of diameter 4 it suffices to find two vertices with distance 4. But this is ensured by the assumption (iii). More precisely, the subdividing vertex s_{b_1,b_2} or its group vertex and the subdividing vertex s_{b_3,b_4} or its group vertex have their distance equal to 4. ■

All the received 4-GMD graphs are of minimum degree 2. But other 4-GMD are possible. For example in Fig. 4 we have a 4-GMD graph of minimum degree 3. (This 3-regular vertex symmetric graph of order 16 is known as the Möbius-Kantor graph or generalized Petersen graph $P_{8,3}$ [27].)

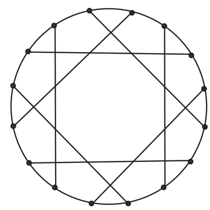

Figure 4: A 4-GMD graph with minimum degree 3

4 Diameter 6

Here we present an infinite class of 6-GMD graphs. Our construction depends on two parameters $p, q \geq 2$. In Figure 5 we have depicted such a 6-GMD graph with

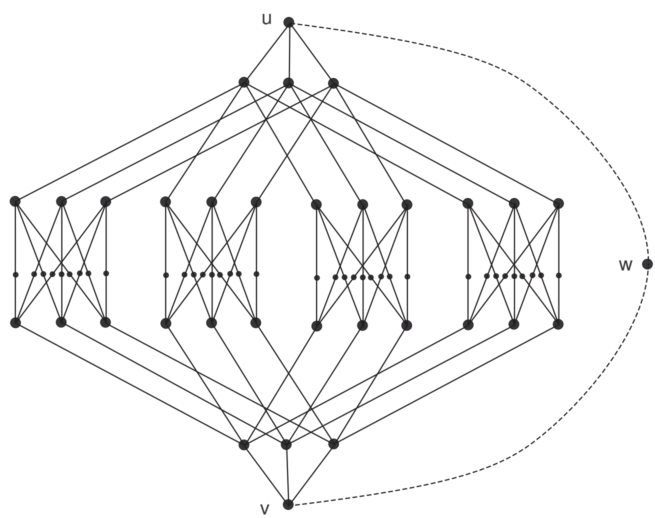

Figure 5: Illustrating the construction of 6-GMD graphs for $p = 4$ and $q = 3$

$p = 4$ and $q = 3$. In general, we take p copies of a complete bipartite graph $K_{q,q}$. Further we take two copies of a star $K_{1,q}$; let u and v be their central vertices. Then for each copy of $K_{q,q}$ we join by q edges the q vertices of one partite set to the q vertices of $N(u)$ (one-to-one) and symmetrically, the q vertices of the other partite set are joined to the q vertices of $N(v)$ by further q edges. Finally, in each copy of $K_{q,q}$ every edge is subdivided by inserting one new vertex. The reader can easily verify that the resulting graph is a 6-GMD graph of order $n = pq^2 + 2pq + 2q + 2$

and size $m = 2pq^2 + 2pq + 2q$. Moreover, if we add a further vertex w and two edges uw and vw, we get a 6-GMD graph too.

5 Further small even diameters

While our k-GMD graphs for odd k have been found by a computer, those with even diameters are handmade. We tried to generalize our construction of 6-GMD graphs.

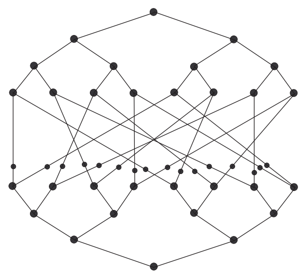

Figure 6: The first 8-GMD graph of order 46

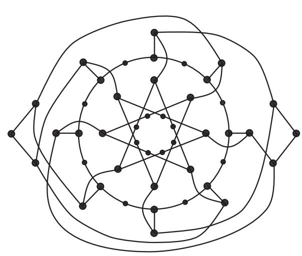

Figure 7: The second 8-GMD graph of order 46

Although no general construction has been found at least some isolated examples were produced. The idea was to connect the leaves of two binary trees of height $k/2 - 1$ by a 2-regular bipartite graph B, where one partite set is formed by the leaves of the first tree and the other partite set by the leaves of the second tree and finally to subdivide each edge of B. This yielded three 8-GMD graphs of order 46. They are in Figures 6, 7, and in Table 3.

Analogous considerations led to three 10-GMD graphs of order 94. Here we present just one of them in Table 4.

Also three 12-GMD graphs of order 190 were found; one of them is given in Table 5. Finally two 14-GMD graphs of order 382 were produced and Table 6 shows one of

Table 3: The third 8-GMD graph of order 46

1: 2 3	2: 4 5	3: 6 7	4: 8 9	5: 10 11
6: 12 13	7: 14 15	8: 31 32	9: 33 34	10: 35 36
11: 37 38	12: 39 43	13: 41 44	14: 40 45	15: 42 46
16: 17 18	17: 19 20	18: 21 22	19: 23 24	20: 25 26
21: 27 28	22: 29 30	23: 31 39	24: 35 40	25: 33 41
26: 37 42	27: 32 45	28: 36 43	29: 34 46	30: 38 44

Table 4: A 10-GMD graph of order 94

1: 2 3	2: 4 5	3: 6 7	4: 8 9	5: 10 11
6: 12 13	7: 14 15	8: 16 17	9: 18 19	10: 20 21
11: 22 23	12: 24 25	13: 26 27	14: 28 29	15: 30 31
16: 63 64	17: 65 66	18: 67 68	19: 69 70	20: 71 72
21: 73 74	22: 75 76	23: 77 78	24: 79 80	25: 81 82
26: 83 84	27: 85 86	28: 87 88	29: 89 90	30: 91 92
31: 93 94	32: 33 34	33: 35 36	34: 37 38	35: 39 40
36: 41 42	37: 43 44	38: 45 46	39: 47 48	40: 49 50
41: 51 52	42: 53 54	43: 55 56	44: 57 58	45: 59 60
46: 61 62	47: 63 79	48: 71 87	49: 67 83	50: 75 91
51: 65 81	52: 73 89	53: 69 85	54: 77 93	55: 64 84
56: 72 92	57: 68 80	58: 76 88	59: 66 86	60: 74 94
61: 70 82	62: 78 90			

them. Unfortunately, we did not succeed in continuation of this sequence and thus 14 is the maximum diameter of a k-GMD graph we have found so far.

Table 5: A 12-GMD graph of order 190

1: 2 3	2: 4 5	3: 6 7	4: 8 9	5: 10 11
6: 12 13	7: 14 15	8: 16 17	9: 18 19	10: 20 21
11: 22 23	12: 24 25	13: 26 27	14: 28 29	15: 30 31
16: 32 33	17: 34 35	18: 36 37	19: 38 39	20: 40 41
21: 42 43	22: 44 45	23: 46 47	24: 48 49	25: 50 51
26: 52 53	27: 54 55	28: 56 57	29: 58 59	30: 60 61
31: 62 63	32: 127 128	33: 129 130	34: 131 132	35: 133 134
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46: 155 156	47: 157 158	48: 159 160	49: 161 162	50: 163 164
51: 165 166	52: 167 168	53: 169 170	54: 171 172	55: 173 174
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86: 109 110	87: 111 112	88: 113 114	89: 115 116	90: 117 118
91: 119 120	92: 121 122	93: 123 124	94: 125 126	95: 127 159
96: 143 175	97: 135 167	98: 151 183	99: 131 163	100: 147 179
101: 139 171	102: 155 187	103: 129 177	104: 145 161	105: 137 185
106: 153 169	107: 133 181	108: 149 165	109: 141 189	110: 157 173
111: 128 188	112: 152 164	113: 136 180	114: 144 172	115: 132 184
116: 156 160	117: 140 176	118: 148 168	119: 130 166	120: 154 190
121: 138 174	122: 146 182	123: 134 162	124: 158 186	125: 142 170
126: 150 178				

Table 6: A 14-GMD graph of order 382

1: 2 3	2: 4 5	3: 6 7	4: 8 9	5: 10 11
6: 12 13	7: 14 15	8: 16 17	9: 18 19	10: 20 21
11: 22 23	12: 24 25	13: 26 27	14: 28 29	15: 30 31
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251: 270 360	252: 294 380	253: 286 352	254: 310 372	

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