AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **41** (2008), Pages 15–28

Average firefighting on infinite grids

M. E. Messinger

Department of Mathematics & Statistics
Dalhousie University, Halifax
Nova Scotia, B3H 3J5
Canada

Abstract

In the Firefighter Problem, a fire breaks out at a vertex of a graph G, then f firefighters protect f vertices. At each subsequent time step, the fire spreads from each "burned" vertex to all of its unprotected neighbours, then f firefighters "protect" f unburned vertices. Once a vertex is protected or burned, it remains so from then onward. A common objective is to determine the minimum number f, such that if f vertices are protected at each time step, then the fire can be contained on a graph G. In this paper, *average firefighting* is introduced: the number of vertices protected in each time step is allowed to vary. If the number of firefighters used is periodic and the average number (per time step) is strictly greater than 3/2, then a fire on the Cartesian grid can be contained. Similar results are also determined for the triangular and strong grids.

1 Introduction

The Firefighter Problem, introduced by Hartnell [2] in 1995, is a deterministic discrete-time model of the spread of fire on vertices of a graph G. We suppose a fire breaks out at a vertex of a graph at time $t = 0$, then f firefighters protect f vertices. At each subsequent time step, the fire spreads from each burned vertex to all of its (previously unprotected and unburned) neighbours and then f (previously unburned and unprotected) vertices are protected. The fire and firefighters alternate until the fire can no longer spread. A **burned vertex** is a vertex to which the fire has spread; once a vertex is burned, it will remain a burned vertex for all subsequent time steps. A **protected vertex** is a vertex that cannot be burned; it has been "defended" by a firefighter and once protected, it will remain protected for all subsequent time steps. When the fire can no longer spread, any vertex that has not been burned is a **saved** vertex.

The spread of fire on vertices of a graph can also be viewed as the spread of an epidemic through a population [4]. If the graph is interpreted as a social network, then the Firefighter Problem can also be considered as modeling a perfectly contagious disease with no cure. The act of protecting vertices at each time step, could then be viewed as vaccinations.

Some objectives, relevant to both interpretations of the problem, include maximizing the number of saved vertices [3, 6]; minimizing the number of time steps to contain the fire [11]; determining if a specific set of vertices can be saved [5]; and minimizing the number of firefighters used to contain a fire [1, 8, 9, 7, 11]. On an infinite graph, a fire is **contained** once it is enclosed by protected vertices and can no longer spread. In this paper, we consider the problem of minimizing the number of firefighters used to contain a fire. As defined in [9], let $\mathcal{F}(G) = \{n :$ n firefighters contain any fire on graph $G\}$ and $f_G = \min\{n : n \in \mathcal{F}(G)\}$. That is, f_G is the minimum number of firefighters needed to contain a fire on graph G (where f_G vertices are protected at each time step).

The concept of **average firefighting** was originally introduced in 2004 in [7] and was examined with respect to several infinite graphs. More recently, Ng and Raff have independently introduced the same concept, which they have dubbed fractional firefighting in [10]. We refer to it in this paper as average firefighting for several reasons. First, we are specifically calculating the *average* number of firefighters used to contain a fire. Second, a fractional version of firefighting was introduced in [1] in 2003 where a fraction of a vertex could be burned or protected.

We suppose there are T teams of firefighters, where the number of firefighters in each team may vary. Then if a fire breaks out at a vertex of graph G, one team of firefighters protects vertices of G at each time step. More specifically, let a_i denote the number of vertices protected by team i. Then $[a_0, a_1, \ldots, a_{T-1}]$ is a periodic sequence with period T. For example, if $T = 3$ and $[a_0, a_1, a_2] = [3, 2, 1]$, then the number of vertices protected at time step t is $a_{(t \ mod \ T)}$. Once a fire has been contained, the total number of vertices protected divided by the total number time intervals yields the average number of vertices protected per time step, denoted by Af_G. Stemming from the interest in determining f_G for various graphs and classes of graphs [1, 8, 9, 7, 11], we wish to minimize Af_G with the restriction that $a_i \leq f_G$ for all i.

The **Cartesian grid** C is the resulting graph from the Cartesian product of two infinite paths; the **triangular grid** Tr is the infinite graph formed by tiling the plane regularly with equilateral triangles; and the **strong grid** S is the resulting graph from the strong product of two infinite paths. Let $G = (V, E)$ denote a graph G with vertex set $V(G)$ and edge set $E(G)$. Note that $V(C) = V(Tr) = V(S)$, $E(C) \subseteq E(Tr) \subseteq E(S)$, and we use the Cartesian coordinates to describe vertices of C, Tr, and S in the proofs below.

In this paper, we show that if the number of firefighters used at each time step is allowed to vary, but is at most f_G, then the average number of firefighters used (or equivalently, the number of vertices protected) per time step to contain a fire in the Cartesian, triangular, and strong grids will be less than f_C, f_{Tr} and f_S, respectively. Although there is a clear trade-off between the number of burned vertices and the number of firefighters used, in this paper we are only concerned with the number of firefighters used to contain a fire. Theorems 1 and 2 show that if

$$a_i = \begin{cases} f_G & i = 0 \\ f_G - 1 & i \geq 1, \end{cases}$$

then the average number of firefighters used to contain a fire tends to $f_G - 1$ (as $T \to \infty$) for $G = S$ and $G = Tr$. Theorem 3 shows that if

$$a_i = \begin{cases} f_G & \text{if } i \text{ is odd or } i = 0 \\ f_G - 1 & \text{if } i \text{ is even} \end{cases}$$

where $T = 2x + 1$ for nonnegative $x \in \mathbb{Z}$, then the average number of firefighters used to contain a fire on the Cartesian grid tends to $3/2 = f_C - 1/2$. We conjecture that an average of $3/2$ firefighters cannot contain a fire on the Cartesian grid. The result for the Cartesian grid is independently determined in [10].

2 Results

2.1 Triangular Grids

It was shown in [1] that $f_{Tr} = 3$. In this section, we give a configuration of firefighters such that $Af_{Tr} = 2 + \frac{1}{T+\frac{1}{2T+4}}$. It is interesting to note that

$$\lim_{T\to\infty} 2 + \frac{1}{T + \frac{1}{2T+4}} = 2,$$

which implies that 2 firefighters can *almost* contain a fire on the triangular grid. The reader may easily verify that two firefighters on the triangular grid can save half of the triangular grid, but it does not seem probable that two firefighters can do better.

Let Team Assignment 1 be defined as:

$$a_i = \begin{cases} f_G & i = 0 \\ f_G - 1 & i \geq 1. \end{cases}$$

Theorem 1 *If a fire breaks out on the triangular grid Tr and T teams of firefighters use Team Assignment* 1*, then*
a) $Af_{Tr} = 2 + \frac{1}{T+\frac{1}{2T+4}}$ *and the fire is contained by time* $t = 2T^2 + 4T$.
b) there are at most $T^4 + 5T^3 + 8T^2 + 5T + 1$ *vertices burned.*

Proof of a): Assume a fire breaks out at $(0, 0)$ and T teams of firefighters use Team Assignment 1. An algorithmic approach will be used to prove part a): the algorithm builds western and northern fire walls (Phase 1); an eastern fire wall (Phase 2 and Phase 3); and a southern fire wall (Phase 4). The result of firefighting using this algorithm is illustrated in Figure 1 with the time at which vertices are protected indicated.

Phase 1:
The firefighters begin by creating a western vertical fire wall and a northern horizontal fire wall both originating at $(-1, 1)$.

$t = 0$: protect $(-1, 0), (-1, 1), (0, 1)$

$t = 1$:

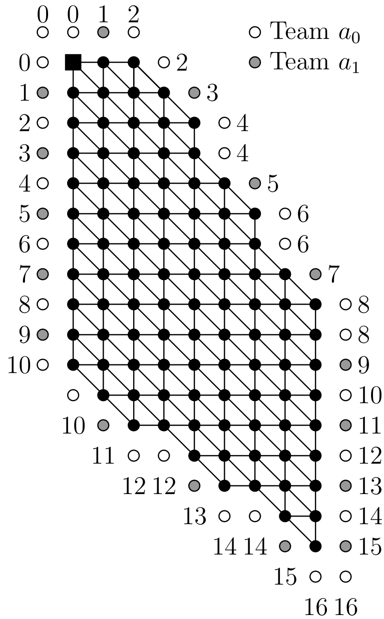

Figure 1: Triangular grid with $T = 2$, using Team Assignment 1.

```
while (t ≤ T − 1) {
    protect (t, 1), (−1, −t)
    t := t + 1
}
```

At time $t = T - 1$ the fire has spread to burn the square of vertices enclosed by $(0, 0)$, $(T - 1, 0)$, $(0, -T + 1)$, and $(T - 1, -T + 1)$. The northern horizontal fire wall is on $(x, 1)$ with $-1 \leq x \leq T - 1$. The western vertical wall is on $(-1, y)$ with $1 \leq y \leq -T + 1$.

Phase 2:

At each time interval, the firefighters maintain the appropriate length of the western fire wall and create an eastern fire wall as an extension of the northern fire wall.

```
t = T:    protect (t, 1), (t + 1, 0), (−1, −t)
let i = −1
t = T + 1:
    while (t ≤ T^2 + 2T) {
        if (t is divisible by T) {
                protect (−1, −t), (t + 1, i), (t + 1, i − 1)
            t := t + 1
            i := i − 2
        }
        else {
            protect (−1, −t), (t + 1, i)
```

$t := t + 1$
$i := i - 1$
}
}

Phase 2 ends at time $t = T^2 + 2T$ when the firefighters control the spread of fire in the north, west, and east directions. The firefighters have created an eastern fire wall such that every time Team a_0 (the team with three firefighters) protects vertices, an extra vertex is protected. They have created an eastern fire wall which extends to $(T^2 + 2T + 1, -T^2 - 2T - 1)$. The western vertical fire wall is on $(-1, y)$ with $-T^2 - 2T \leq y \leq 1$ and the fire has spread as far east as $x = T^2 + 2T$ and south as $y = -T^2 - 2T$.

Phase 3:
In Phase 3, the firefighters continue to maintain the appropriate lengths of the eastern and western fire walls.

$t = T^2 + 2T + 1 = (T + 1)^2$:
while $(t \leq T^2 + 3T - 1)$ {
protect $(-1, -t), ((T + 1)^2, -t - 1)$
$t := t + 1$
}

In Phase 3, the firefighters protect vertices in two vertical columns southward at $x = -1$, $x = T^2 + 2T + 1$ until such time as the Team a_0 is the next team to protect vertices.

Phase 4:
The firefighters maintain the appropriate length of the eastern fire wall and create a southern fire wall as an extension of the western fire wall.

$t = T^2 + 3T$: protect $(-1, -t), (0, -t - 1), ((T + 1)^2, -t - 1)$
let $i = 1$
$t = T^2 + 3T + 1$:
while $(t \leq 2T^2 + 4T - 1)$ {
if (t is divisible by T) {
protect $(i, -t - 1), (i + 1, -t - 1), ((T + 1)^2, -t - 1)$
$t := t + 1$
$i := i + 2$
}
else {
protect $(i, -t - 1), ((T + 1)^2, -t - 1)$
$t := t + 1$
$i := i + 1$
}
}
$t = 2T^2 + 4T$: protect $((T + 1)^2 - 1, -t - 1), ((T + 1)^2, -t - 1)$

At time $t = 2T^2 + 4T$ the southern fire wall meets the eastern fire wall and the fire has been completely contained.

There were $2T^2 + 4T + 1$ time intervals where each team had $2T + 4$ moves and

Team a_0 had an extra move where only two vertices were protected. A total of $3(2T+4)+2+2(T-1)(2T+4) = 4T^2+10T+6$ vertices were protected and

$$Af_{Tr} = \frac{4T^2+10T+6}{2T^2+4T+1} = 2 + \frac{1}{T+\frac{1}{2T+4}}.$$

■

Proof of b): After using the above algorithm with Team Assignment 1 to contain a fire on the triangular grid, the fire has spread as far north as $y=0$, as far south as $y=-2T^2-4T$, as far west as $x=0$ and as far east as $x=T^2+2T$. If the rectangle with dimensions $(T^2+2T+1)\times(2T^2+4T+1)$ is imposed, then the number of vertices saved when the fire is contained can simply be counted. There are two regions of saved vertices in this rectangle.

The boundaries of region R_1 are created by the vertices protected in Phase 2 and two sides of the rectangle: $y=0$ and $x=T^2+2T$. The boundaries of region R_2 are created by the vertices protected in Phase 4 and two sides of the rectangle: $y=-2T^2-4T$ and $x=0$. As the areas of R_1 and R_2 are equal, only the number of vertices in R_1 will be counted. An example of R_1 and R_2 for $T=2$ and $T=3$ is shown in Figure 2.

In Phase 2, the firefighters created an eastern fire wall. For each move of Teams $a_1,\cdots,a_{T-1}$, firefighters protected a vertex which was immediately to the south and immediately to the east of the last protected vertex. For each move of Team a_0, firefighters protected a vertex immediately to the south and immediately to the east of the last protected vertex and also a vertex immediately to the south of the last protected vertex.

The pattern is repeated every T moves, so R_1 can be divided up into $T+1$ smaller areas. Each of the $T+1$ smaller areas contains $[i(T+1)+1]+[i(T+1)+2]+\cdots+[i(T+1)+T]$ vertices for $i\in\{0,1,\cdots,T\}$, which is a total of $\frac{1}{2}T^4+\frac{3}{2}T^3+\frac{3}{2}T^2+\frac{1}{2}T$ vertices in R_1.

Subtracting the areas of R_1 and R_2 from the total number of vertices in the $(T^2+2T+1)\times(2T^2+4T+1)$ triangular grid, gives a total of $T^4+5T^3+8T^2+5T+1$ vertices burned. ■

2.2 Strong Grids

Although it is known that $f_S=4$ and that three firefighters cannot even contain a fire to one quadrant [8], in this section we give a configuration of firefighters such that $Af_S = 3+\frac{1}{T}$. Again, we note that

$$\lim_{t\to\infty} 3+\frac{1}{T} = 3,$$

which implies that an average of three firefighters can almost contain a fire on the strong grid. The introduction of an extra firefighter every T moves allows the firefighters to contain a fire no matter how large T is.

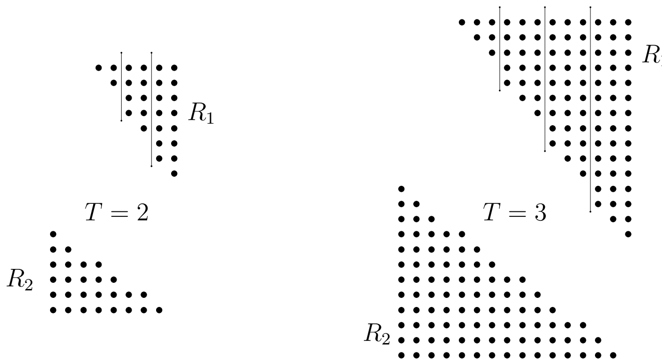

Figure 2: Saved vertices on the triangular grid using Team Assignment 1.

Theorem 2 *If a fire breaks out on the strong grid S and T teams of firefighters use Team Assignment* 1*, then*
a) $Af_S = 3 + \frac{1}{T}$ *and the fire can be contained by time* $t = 2T^2 + 6T - 1$.
b) there are at most $T^4 + 8T^3 + 17T^2 + 8T + 1$ *vertices burned.*

Proof of a): Assume the fire breaks out at $(0, 0)$ on the strong grid and T teams of firefighters use Team Assignment 1. An algorithmic approach is used to prove a): the algorithm builds western and southern fire walls (Phase 1); a northern fire wall (Phase 2); and an eastern fire wall (Phase 3). The result of firefighting using this algorithm is illustrated in Figure 3.

Phase 1:
The firefighters begin by creating a vertical fire wall at $x = -1$ to prevent the fire from spreading in the western direction. They also create a southern diagonal fire wall while maintaining the western wall.

$t = 0$: protect $(-1, 1), (-1, 0), (-1, -1), (0, -1)$
$t = 1$:
while $(t \leq T - 1)$ {
  protect $(-1, t + 1), (t - 1, -t - 1), (t, -t - 1)$
  $t := t + 1$
}
$t = T$: protect $(-1, t + 1), (t - 1, -t - 1), (t, -t - 1), (t + 1, -t - 1)$

At time $t = T$, the firefighters have created and maintained the western vertical fire wall on $(-1, y)$ with $0 \leq y \leq T + 1$. The southern diagonal fire wall protects vertices diagonally from $(-1, 0)$ down to $(T + 1, -T - 1)$ by time $t = T$. The fire has only spread as far south as $y = -T$ and as far east as $x = T$.

Phase 2:
In Phase 2, the firefighters are now in a position to maintain the appropriate length of the southern horizontal fire wall and create a northern fire wall as an extension of

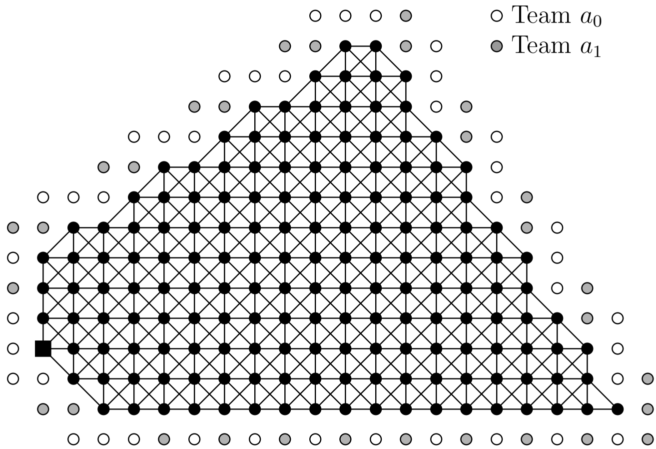

Figure 3: Strong grid with $T = 2$, using Team Assignment 1.

the western fire wall.

let $i = -1$
$t = T + 1$:
while ($t \leq T^2 + 3T$) {
if (t is divisible by T) {
protect $(t + 1, -T - 1)$, $(i, t + 1)$, $(i + 1, t + 1)$, $(i + 2, t + 1)$
$t := t + 1$
$i := i + 2$
}
else {
protect $(t + 1, -T - 1)$, $(i, t + 1)$, $(i + 1, t + 1)$
$t := t + 1$
$i := i + 1$
}
}

At time $t = T^2 + 3T$, the firefighters have created a northern fire wall while maintaining the southern fire wall. The fire has spread as far north as $y = T^2 + 3T$ while the firefighters have remained one step ahead of the fire, protecting vertices as far north as $y = T^2 + 3T + 1$. Both the northern and southern walls have been extended as far east as $x = T^2 + 3T + 1$ while the fire is as far east as $x = T^2 + 3T$.

Phase 3:
The firefighters have contained the fire with north, west, and south boundaries. They now maintain the length of the southern horizontal fire wall and create an eastern fire wall, extending from the northern fire wall to meet the southern fire wall.

$i = T^2 + 3T + 1$

```
t = T^2 + 3T + 1:
    while (t ≤ 2T^2 + 6T − 1) {
        if (t divisible by T) {
            protect (t + 1, −T − 1), (t + 1, i), (t + 1, i − 1), (t + 1, i − 2)
            t := t + 1
            i := i − 2
        }
        else {
            protect (t + 1, −T − 1), (t + 1, i), (t + 1, i − 1)
            t := t + 1
            i := i − 1
        }
    }
```

At time $t = 2T^2 + 6T - 1$ the eastern fire wall meets the southern horizontal fire wall and the firefighters have completely contained the fire. The average number of firefighters used to contain the fire is

$$Af_S = \frac{3T+1}{T} = 3 + \frac{1}{T}.$$

∎

Proof of b): After using the algorithm from part a) to contain a fire, the fire has spread as far north as $y = T^2 + 3T$, as far south as $y = -T$, as far west as $x = 0$, and as far east as $x = 2T^2 + 6T - 1$. If the rectangle with dimensions $(2T^2 + 6T) \times (T^2 + 4T + 1)$ that contains the fire is imposed, then the number of vertices saved can be easily counted. There are 3 regions of vertices as shown in Figure 4, for $T = 2$.

Region R_1 is formed by the vertices protected in Phase 1 that created the southern diagonal fire wall, along with the boundaries $x = 0$ and $y = -T$. As, R_1 is simply a triangle, it contains $\frac{T(T+1)}{2}$ vertices.

Region R_2 is a polygon formed by the vertices of the northern fire wall protected in Phase 2, by $x = 0$, and $y = T^2 + 3T$. It is easy to count the number of vertices in R_2 due to the regularity at which vertices of the northern fire wall were protected in Phase 2. Region R_2 can be divided into $T+1$ rectangles of vertices and $T+2$ triangles of vertices. Note that the height of the longest rectangle is $T^2 + T$ as it ranges from $y = T^2+3T$ to $y = 2T+1$. Each rectangle is $(T+1)\times(hT)$ where $h \in \{1, 2, \cdots, T+1\}$. Thus, there are $T(T+1)[1+2+\cdots+T+1] = \frac{1}{2}T(T+1)^2(T+2) = \frac{1}{2}T^4+2T^3+\frac{5}{2}T^2+T$ vertices in the rectangles.

There are $T + 2$ triangles of vertices, each with base and height equal to $T - 1$. Thus, there are $(T + 2)\frac{T(T-1)}{2} = \frac{1}{2}T^3 + \frac{1}{2}T^2 - T$ vertices in the triangles. The total number of vertices in R_2 is $\frac{1}{2}T^4 + \frac{5}{2}T^3 + 3T^2$.

Region R_3 is a polygon formed by the vertices of the eastern fire wall protected in Phase 3, $y = T^2 + 3T$, and $x = 2T^2 + 6T$. It is also easy to count the number of vertices in R_3 due to the regularity at which vertices of the eastern fire wall were protected in Phase 3. Region R_3 can be divided into $T + 2$ rectangles of vertices and $T + 2$ small polygons of vertices.

Let h be the length of the longest rectangle. Each rectangle is $T \times (h - x(T+1))$ where $x \in \{1, \cdots, T+1\}$. So the number of vertices in rectangles is $T(T+2)(h - \frac{1}{2}(T+1)^2)$ where $h = T^2 + 3T$. There are $T+2$ regular polygons and note that if one vertex was added to a regular polygon, it would be a triangle. Therefore the number of vertices in the regular polygons is $(T+2)(\frac{T(T+1)}{2} - 1)$. Finally, there is one extra polygon which is another triangle. It has $\frac{(T-2)(T-1)}{2}$ vertices. Thus, the total number of vertices in R_3 is $T(T+2)(H - \frac{1}{2}(T+1)^2) + (T+2)(\frac{T(T+1)}{2} - 1) + \frac{(T-2)(T-1)}{2} = \frac{1}{2}T^4 + \frac{7}{2}T^3 + \frac{11}{2}T^2 - \frac{5}{2}T - 1$.

The total number of vertices of all three regions is $T^4 + 6T^3 + 9T^2 - 2T - 1$. Subtracting that from the total number of vertices in the grid yields $T^4 + 8T^3 + 17T^2 + 8T + 1$ burned vertices. ∎

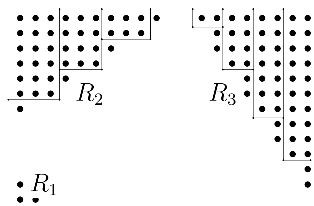

Figure 4: Saved vertices with $T = 2$ on the strong grid using Team Assignment 1.

2.3 Cartesian Grids

Although it was determined that $f_C = 2$ [11], in this section we give a configuration of firefighters for which $Af_C = \frac{3}{2} + \frac{1}{3x+2}$. It is interesting to note that

$$\lim_{x\to\infty} \frac{3}{2} + \frac{1}{3x+2} = \frac{3}{2}$$

.

Let Team Assignment 2 be defined as:

$$a_i = \begin{cases} f_G & \text{if } i \text{ is odd or } i = 0 \\ f_G - 1 & \text{if } i \text{ is even} \end{cases}$$

where $T = 2x + 1$ for nonnegative $x \in \mathbb{Z}$.

Theorem 3 *If a fire breaks out on the Cartesian grid C and $T = 2x + 1$ teams of firefighters use Team Assignment 2, then*
a) $Af_C = \frac{3}{2} + \frac{1}{3x+2}$ and the fire can be contained by time $t = 12x + 7$.
b) there are at most $18x^2 + 42x + 18$ vertices burned.

Proof of a): Assume the fire breaks out at $(0, 0)$ on the Cartesian grid and T teams of firefighters use Team Assignment 2. An algorithmic approach is used to prove a) and the result of firefighting using this algorithm with $x = 2$, $T = 5$ is illustrated in Figure 5.

Phase 1:
The firefighters begin by restricting the fire to an eighth of the plane by creating a northern fire wall and a southwestern diagonal fire wall.
$t = 0$: protect $(-1, 0), (0, 1)$
$t = 1$: protect $(-1, -1), (0, -2)$
$t = 2$:

```
while (t ≤ 2x) {
        if (t odd) protect (t − 1, 2), ((t−1)/2, (−t−3)/2)
        else protect (t − 1, 2)
        t := t + 1
}
```

Phase 2:
The firefighters now maintain southwestern diagonal wall and the northern horizontal fire wall.
$t = 2x + 1$: protect $(t - 1, 2), (x, \frac{-t-3}{2})$
$t = 2x + 2$:

```
while (t ≤ 4x + 1) {
        if (t odd) protect ((t−1)/2, (−t−3)/2)
        else protect (t − 1, 2), (t, 1)
        t := t + 1
}
```

Phase 3:
The firefighters then maintain the length of the southwestern diagonal wall and the northern fire wall.
$t = 4x + 2$: protect $(t - 1, 2), (t, 1)$
$t = 4x + 3$:

```
while (t ≤ 6x + 2) {
        if (t odd ) protect (t, 1), ((t−1)/2, (−t−3)/2)
        else protect (t, 1)
        t := t + 1
}
```

Phase 4:
The firefighters maintain the southwestern diagonal wall and create an eastern fire wall as an extension of the northern fire wall.
$t = 6x + 3$: protect $(t, 1), (\frac{t-1}{2}, \frac{-t-3}{2})$
$t = 6x + 4$: protect $(t, 1), (t + 1, 0)$
$j := -1$
$t = 6x + 5$:

```
while (t ≤ 8x + 3) {
        if (t odd): protect ((t−1)/2, (−t−3)/2)
        else {
            protect (6x + 6, j), (6x + 5, j − 1)
```

$j := j - 2$
}
$t := t + 1$
}

Phase 5:
The firefighters now maintain the southwestern diagonal wall and the eastern wall.
$t = 8x + 4$: protect $(6x + 6, j), (6x + 5, j - 1)$
$j := j - 2$
$t = 8x + 5$:
while $(t \leq 10x + 4)$ {
if (t odd): protect $(6x + 5, j), (\frac{t-1}{2}, \frac{-t-3}{2})$
else protect $(6x + 5, j)$
$j := j - 1$
$t := t + 1$
}

Phase 6:
The firefighters continue to maintain the southwestern diagonal wall and the eastern wall.
$t = 10x + 5$: protect $(6x + 5, j), (\frac{t-1}{2}, \frac{-t-3}{2})$
$j := j - 1$
$t = 10x + 6$:
while $(t \leq 12x + 5)$ {
if (t odd): protect $(\frac{t-1}{2}, \frac{-t-3}{2})$
else {
protect $(6x + 5, j), (6x + 4, j - 1)$
$j := j - 2$
}
$t := t + 1$
}

Phase 7:
Firefighters connect the southwestern diagonal wall and eastern walls with the final protected vertices.
$t = 12x + 6$: protect $(6x + 5, j), (6x + 4, j - 1)$
$j := j - 2$
$t = 12x + 7$: protect $(6x + 4, j), (6x + 3, j - 1)$

At time $t = 12x + 7$, the fire has been completely contained.
In each of Phase 1 to Phase 6, a total of $3x+2$ vertices were protected and 4 vertices were protected in Phase 7. Thus, in $12x + 8$ time intervals,

$$Af_C = \frac{6(3x + 2) + 4}{12x + 8} = \frac{3}{2} + \frac{1}{3x + 2}.$$

■

Proof of b): Assume T teams of firefighters use the previous algorithm to contain the fire. The pattern created by burned vertices *almost* forms a triangle. As it is easy to count the number of vertices in a triangle, we impose the triangle bound by $(-1,0),(6x+4,0),(6x+4,-6x-5)$. The number of vertices in the triangle is $18x^2+39x+21$. It is also easy to count the burned vertices in each phase which are outside the triangle and the saved or protected vertices inside the triangle:

Phase 1: Note that $(-1,0)$ is protected. There are $2x-1$ (for $x>0$) or 0 ($x=0$) vertices burned.
Phase 2: There are $x+1$ (for $x>0$) or 0 ($x=0$) vertices burned.
Phase 3: There is 1 vertex burned.
Phase 4: There are $x-1$ (for $x>0$) or 0 ($x=0$) vertices burned.
Phase 5: There is 1 (for $x>0$) or 0 ($x=0$) vertices burned.
Phase 6: There are x vertices protected.
Phase 7: There is 1 vertex saved and 2 vertices protected.

The total number of burned vertices is $18x^2+42x+18$ ($\forall x \geq 0$) where $T=2x+1$.

■

Conjecture 4 *If a fire breaks out on the Cartesian grid C, an average of $\frac{3}{2}$ firefighters cannot contain the fire.*

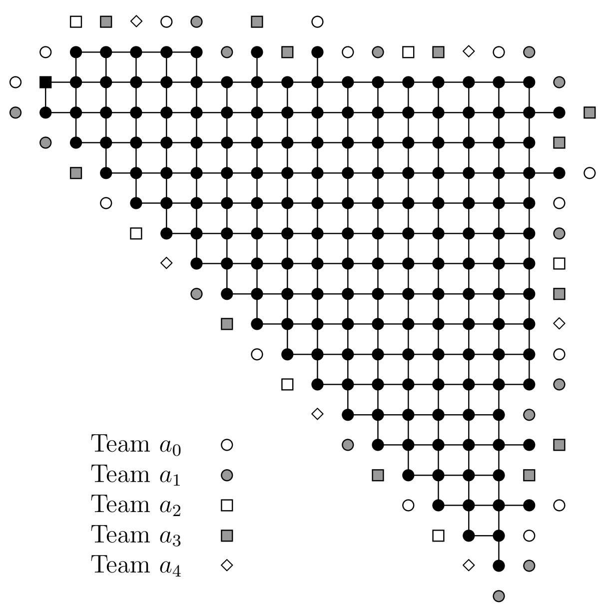

Figure 5: Cartesian grid with $x=2, T=5$, using Team Assignment 2.

3 Acknowledgements

We would like to thank Richard Nowakowski for his contributions to this paper.

References

[1] P. Fogarty, *Catching the Fire on Grids*, M. Sc. Thesis, University of Vermont (2003).

[2] B. L. Hartnell, *Firefighter! An Application of Domination*, presentation at the Twentieth Conference on Numerical Mathematics and Computing (University of Manitoba, 1995).

[3] B. L. Hartnell and Q. Li, Firefighting on trees: How bad is the greedy algorithm?, Proceedings of the Thirty-first Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, Fl., 2000) *Congr. Numer.* **145** (2000), 187–192.

[4] S. G. Hartke and M. Develin, Fire Containment in Grids of Dimension Three and Higher, submitted.

[5] A. King and G. MacGillivray, The Firefighter Problem for Cubic Graphs, *Discrete Math.* to appear.

[6] G. MacGillivray and P. Wang, On The Firefighter Problem, *J. Combin. Math. Combin. Comput.* **47** (2003) 83–96.

[7] M. E. Messinger, *Firefighting on Infinite Grids*, M.Sc. Thesis, Dalhousie University (2004).

[8] M. E. Messinger, Firefighting on the Strong Grid, submitted.

[9] M. E. Messinger, Firefighting on the Triangular Grid, *J. Combin. Math. Combin. Comput.* to appear.

[10] K. L. Ng and P. Raff, Fractional Firefighting in the Two Dimensional Grid, *DIMACS Technical Report 2005-23* (2005).

[11] P.Wang and S. A. Moeller, Fire Control on Graphs, *J. Combin. Math. Combin. Comput.* **41** (2000) 19–34.

[12] D. B. West, *Introduction to Graph Theory*, 2nd edition, Prentice-Hall, 2001.

(Received 19 June 2006; revised 16 Oct 2007)