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A note on non-jumping numbers

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Abstract

Let $r \geq 2$ be an integer. A real number $\alpha \in [0, 1)$ is a jump for r if there exists $c > 0$ such that for any $\epsilon > 0$ and any integer m, $m \geq r$, there exists an integer n_0 such that any r-uniform graph with $n > n_0$ vertices and density $\geq \alpha + \epsilon$ contains a subgraph with m vertices and density $\geq \alpha + c$. It follows from a theorem of Erdős and Stone that every $\alpha \in [0, 1)$ is a jump for $r = 2$. Erdős asked whether the same is true for $r \geq 3$. In the paper 'Hypergraphs do not jump' (*Combinatorica* 4 (1984), 149–159), Frankl and Rödl gave a negative answer by showing that $1 - 1/l^{r-1}$ is not a jump for r if $r \geq 3$ and $l > 2r$. Following a similar approach, we give some other non-jumping numbers for $r \geq 3$.

1 Introduction

For a finite set V and a positive integer r we denote by $\binom{V}{r}$ the family of all r-subsets of V. The graph $G = (V, E)$ is called an *r-uniform graph* if $E \subseteq \binom{V}{r}$. The *density* of G is defined by $d(G) = |E| \,/\, \left|\binom{V}{r}\right|$. Note that for an r-uniform graph G, the average on densities of all its induced subgraphs with $m \geq r$ vertices is $d(G)$ (c.f. [5]). Therefore, there exists a subgraph with m vertices and density $\geq d(G)$. A natural question is whether there exists a subgraph of given size with density $\geq d(G) + c$, where c is a constant? To be precise, the concept of 'jump' is given below.

Definition 1.1 *A real number $\alpha \in [0, 1)$ is a* jump *for r if there exists a constant $c > 0$ such that for any $\epsilon > 0$ and any integer m, $m \geq r$, there exists an integer $n_0(\epsilon, m)$ such that any r-uniform graph with $n > n_0(\epsilon, m)$ vertices and density $\geq \alpha + \epsilon$ contains a subgraph with m vertices and density $\geq \alpha + c$.*

The study of jump is closely related to the study of Turán density. Finding good estimates for Turán densities in hypergraphs ($r \geq 3$) is believed to be one of the most challenging problems in extremal set theory. For a family $\mathcal{F}$ of r-uniform graphs, the *Turán density* of $\mathcal{F}$, denoted by $t_r(\mathcal{F})$ is the limit of the maximum density of an r-uniform graph of order n not containing any member of $\mathcal{F}$ as $n \to \infty$, i.e.,

$$t_r(\mathcal{F}) = \lim_{n\to\infty} \frac{\max\{|E| : G = (V,E) \text{ is an } \mathcal{F}-\text{free } r-\text{uniform graph of order } n\}}{\binom{n}{r}}.$$

Let

$$\Gamma_r = \{t_r(\mathcal{F}) : \mathcal{F} \text{ is a family of } r-\text{uniform graphs}\}.$$

Note that α is a jump for r if and only if there exists $c > 0$ such that $\Gamma_r \cap (\alpha, \alpha + c) = \emptyset$ (c.f. [4]). Consequently, every $\alpha \in [0, 1)$ is a jump for r if and only if Γ_r is a well-ordered set.

For $r = 2$, Erdős and Stone [2] proved that every $\alpha \in [0, 1)$ is a jump for $r = 2$. For $r \geq 3$, Erdős [1] proved that every $\alpha \in [0, r!/r^r)$ is a jump. Furthermore, Erdős proposed the following jumping constant conjecture: Every $\alpha \in [0, 1)$ is a jump for every integer $r \geq 2$. He also offered \$1000 for a proof or disproof. In [4], Frankl and Rödl disproved the conjecture by showing that

Theorem 1.1 *For $r \geq 3$, $1 - \frac{1}{l^{r-1}}$ is not a jump if $l > 2r$.*

Using a similar approach, more non-jumping numbers were found in [3], [6], [7], [8], [9] and [10]. However, there are still a lot of unknowns in determining whether or not a number is a jump. Following the approach in [4], we prove the following result.

Theorem 1.2 *Let $r \geq p \geq 3$ be integers. Then $(1 - \frac{1}{p^{p-1}})\frac{p^p}{p!}\frac{r!}{r^r}$ is not a jump for r.*

The proof of Theorem 1.2 is given in Section 3. In the next section, we introduce Lagrange functions and some other tools used in the proof.

2 Lagrange functions and other tools

We first give the definition of the Lagrange function for an r-uniform graph, proved to be a helpful tool in our approach.

Definition 2.1 *For an r-uniform graph G with vertex set $\{1, 2, \ldots, m\}$, edge set $E(G)$ and a vector $\vec{x} = (x_1, \ldots, x_m) \in R^m$, define*

$$\lambda(G, \vec{x}) = \sum_{\{i_1,\ldots,i_r\}\in E(G)} x_{i_1} x_{i_2} \ldots x_{i_r}.$$

The number x_i is called the *weight* of vertex i.

Definition 2.2 *Let $S = \{\vec{x} = (x_1, x_2, \ldots, x_m) : \sum_{i=1}^{m} x_i = 1, x_i \geq 0$ for $i = 1, 2, \ldots, m\}$. The Lagrange function of G, denoted by $\lambda(G)$, is defined as*

$$\lambda(G) = \max\{\lambda(G, \vec{x}) : \vec{x} \in S\}.$$

A vector $\vec{y} \in S$ is called an *optimal vector* of $\lambda(G)$ if $\lambda(G, \vec{y}) = \lambda(G)$.

The following fact is easily implied by the definition of the Lagrange function.

Fact 2.1 *Let G_1, G_2 be r-uniform graphs and $G_1 \subset G_2$. Then $\lambda(G_1) \leq \lambda(G_2)$.*

We also give a simple lemma which is useful in calculating the Lagrange function for certain graphs.

We call two vertices i, j of an r-uniform graph G *equivalent* if for all $f \in \binom{V(G)-\{i,j\}}{r-1}$, $f \cup \{j\} \in E(G)$ if and only if $f \cup \{i\} \in E(G)$.

Lemma 2.2 *(c.f. [4]) Let G be an r-uniform graph and $v_1, \ldots, v_t \in V(G)$ be all pairwise equivalent. Suppose $\vec{y} \in S$ is an optimal vector of $\lambda(G)$, i.e., $\lambda(G) = \lambda(G, \vec{y})$. If $\vec{z} \in S$ is obtained from $\vec{y}$ by setting the weights of the vertices $v_1, \ldots, v_t$ to be equal while leaving the other weights unchanged then $\lambda(G) = \lambda(G, \vec{z})$.*

We also introduce the *blow-up* of an r-uniform graph which will allow us to construct r-uniform graphs with arbitrary large number of vertices and density close to $r!\lambda(G)$ based on an r-uniform graph G.

Definition 2.3 *Let G be an r-uniform graph with $V(G) = \{1, 2, \ldots, m\}$ and $\vec{n} = (n_1, \ldots, n_m)$ be a positive integer vector. Define the $\vec{n}$ blow-up of G, $\vec{n} \otimes G$ as an m-partite r-uniform graph with vertex set $V_1 \cup \ldots \cup V_m$, $|V_i| = n_i$, $1 \leq i \leq m$, and edge set $E(\vec{n} \otimes G) = \{\{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\}$, where $\{i_1, i_2, \ldots, i_r\} \in E(G)$ and $v_{i_k} \in V_{i_k}$ for $1 \leq k \leq r\}$.*

We make the following easy remark.

Remark 2.3 *Let G be an r-uniform graph with m vertices and $\vec{y} = (y_1, y_2, \ldots, y_m)$ be an optimal vector of $\lambda(G)$. Then for any $\epsilon > 0$, there exists integer $n_0(\epsilon)$, such that for any integer $n \geq n_0$,*

$$d((\lceil ny_1 \rceil, \lceil ny_2 \rceil, \ldots, \lceil ny_m \rceil) \otimes G) \geq r!(\lambda(G) - \epsilon). \tag{1}$$

Proof.

$$\begin{aligned} d((\lceil ny_1 \rceil, \lceil ny_2 \rceil, \ldots, \lceil ny_m \rceil) \otimes G) &= \frac{\sum_{\{i_1, i_2, \ldots, i_r\} \in E(G)} \lceil ny_{i_1} \rceil \cdot \lceil ny_{i_2} \rceil \cdots \lceil ny_{i_r} \rceil}{\binom{\lceil ny_1 \rceil + \lceil ny_2 \rceil + \cdots + \lceil ny_m \rceil}{r}} \\ &\geq \sum_{\{i_1, i_2, \ldots, i_r\} \in E(G)} \frac{n^r y_{i_1} y_{i_2} \cdots y_{i_r}}{\binom{n}{r}} - \epsilon \\ &> r!(\sum_{\{i_1, i_2, \ldots, i_r\} \in E(G)} y_{i_1} y_{i_2} \cdots y_{i_r} - \epsilon) \\ &= r!(\lambda(G) - \epsilon) \end{aligned}$$

for $n \geq n_0(\epsilon)$. ∎

Let us also state a fact which follows directly from the definition of Lagrange function.

Fact 2.4 *(c.f. [4]) Let $\vec{n} = (n, n, \ldots, n), n \geq 1$. Then for every r-uniform graph G and every integer n, $\lambda(\vec{n} \otimes G) = \lambda(G)$ holds.*

The following lemma proved in [4] gives a necessary and sufficient condition for a number α to be a jump.

Lemma 2.5 *(c.f. [4]) The following two properties are equivalent.*

1. *α is a jump for r.*
2. *There exists a finite family $\mathcal{F}$ of r-uniform graphs satisfying $\lambda(F) > \frac{\alpha}{r!}$ for all $F \in \mathcal{F}$ and $t_r(\mathcal{F}) \leq \alpha$.*

We also need the following lemma from [4].

Lemma 2.6 *[4] For any $\sigma \geq 0$ and any integer $k \geq r$, there exists $t_0(k, \sigma)$ such that for every $t > t_0(k, \sigma)$, there exists an r-uniform graph A satisfying:*

1. *$|V(A)| = t$,*
2. *$|E(A)| \geq \sigma t^{r-1}$,*
3. *For all $V_0 \subset V(A), r \leq |V_0| \leq k$, we have $|E(A) \cap \binom{V_0}{r}| \leq |V_0| - r + 1$.*

The approach in proving Theorem 1.2 is sketched as follows: Let α be the non-jumping number stated in the Theorem. We construct an r-uniform graph with Lagrange function slightly smaller than $\alpha/r!$, then use Lemma 2.6 to add an r-uniform graph with enough number of edges but sparse enough (see properties 2 and 3 in this Lemma) and obtain an r-uniform hypergraph with Lagrange function slightly over $\alpha/r!$. Then we 'blow up' this hypergraph to a hypergraph, say H with large enough number of vertices and density $> \alpha + \epsilon$ for some positive ϵ (see Remark 2.3). If α is a jump, then by Lemma 2.5, $t_r(\mathcal{F}) \leq \alpha$ for a finite family $\mathcal{F}$ of r-uniform graphs satisfying $\lambda(F) > \alpha/r!$ for all $F \in \mathcal{F}$. So the hypergraph H must contain some member of $\mathcal{F}$ as a subgraph. On the other hand, we will show that any subgraph of hypergraph H with the number of vertices not greater than $\max\{|V(F)|, F \in \mathcal{F}\}$ has Lagrange function $\leq \alpha/r!$ and derive a contradiction.

We give the proof of the main result in next section.

3 Proof of Theorem 1.2

In fact, we will prove the following extension of Theorem 1.2.

Theorem 3.1 *Let $r \geq p \geq 3, l \geq 2$ be integers. Then $(1 - \frac{1}{l^{p-1}})\frac{r!}{p!}\frac{l^p}{(l+r-p)^r}$ is not a jump for r provided $\frac{p^p}{r^r} \leq \frac{l^p}{(l+r-p)^r}$.*

Note that when $l = p$, the above result is Theorem 1.2. Taking $r = p$ in this theorem, we obtain that $1 - \frac{1}{l^{r-1}}$ is not a jump for r if $l \geq 2$ and this implies Theorem 1.1 (there was the condition $l > 2r$ in Theorem 1.1. We note that this condition can be relaxed to $l \geq 2$ due to the proof given later).

We have to note that the only solutions to $\frac{p^p}{r^r} \leq \frac{l^p}{(l+r-p)^r}$ and $r \geq p \geq 3, l \geq 2$ are $l = p \geq 3$ (Theorem 1.2) or $r = p \geq 3$ (Theorem 1.1 with the condition $l > 2r$ replaced by $l \geq 2$). If $r > p$, let

$$f(l) = \frac{l^p}{(l+r-p)^r}.$$

Then

$$f'(l) = \frac{l^{p-1}(r-p)(p-l)}{(l+r-p)^{r+1}},$$

and $f'(l) > 0$ when $l < p$ and $f'(l) < 0$ when $l > p$. Therefore, $f(l)$ has the maximum value only if $l = p$. In other words, $\dfrac{p^p}{r^r} \geq \dfrac{l^p}{(l+r-p)^r}$ and the equality holds only if $l = p$.

Now we are going to prove Theorem 3.1.

Proof of Theorem 3.1. Let l, p, r be any fixed integers satisfying the conditions in Theorem 3.1. Based on a p-uniform hypegraph given in [4], we construct an r-uniform graph. Details are given below.

In [4], it was proved that $1 - \dfrac{1}{l^{p-1}}$ is not a jump for $p \geq 3$ (with the condition $l > 2p$ there), i.e., the case $r = p$ in Theorem 3.1. This number is related to the following p-uniform hypegraph $G^{(p)}$ defined on l pairwise disjoint sets $V_1, V_2, \ldots, V_l$, each of them of size t, where t is a fixed integer determined later. The edge set of $G^{(p)}$ is $\binom{\cup_{i=1}^{l} V_i}{p} - \cup_{i=1}^{l} \binom{V_i}{p}$.

We assume that $(1 - \dfrac{1}{l^{p-1}})\dfrac{r!}{p!}\dfrac{l^p}{(l+r-p)^r}$ is a jump for r. In view of Lemma 2.5, there exists a finite collection $\mathcal{F}$ of r-uniform graphs satisfying the following:

i) $\lambda(F) > (1 - \dfrac{1}{l^{p-1}})\dfrac{1}{p!}\dfrac{l^p}{(l+r-p)^r}$ for all $F \in \mathcal{F}$, and

ii) $t_r(\mathcal{F}) \leq (1 - \frac{1}{l^{p-1}})\frac{r!}{p!}\frac{l^p}{(l+r-p)^r}$.

Set $k = \max_{F \in \mathcal{F}}\{|V(F)|, p\}$ and $\sigma = \dfrac{2l^p + \binom{p}{2}(l^{p-1} - l)}{p!}$. Let $t_0(k, \sigma)$ be given as in Lemma 2.6. Fix an integer $t > \max(t_0, t_1)$, where t_1 is determined later (in (2)).

Now take a p-uniform graph A satisfying the conditions in Lemma 2.6 with $V(A) = V_1$ and the p-uniform graph $G_*^{(p)}$ is obtained by adding A to $G^{(p)}$. Note that the edge number in $G_*^{(p)}$ is

$$\begin{aligned} |E(G_*^{(p)})| &= \binom{lt}{p} - l\binom{t}{p} + \sigma t^{p-1} \\ &= \frac{(lt)^p}{p!} - \frac{\binom{p}{2} l^{p-1}t^{p-1}}{p!} - \frac{lt^p}{p!} + \frac{\binom{p}{2} lt^{p-1}}{p!} + \sigma t^{p-1} + o(t^{p-1}) \\ &= \frac{(lt)^p}{p!}(1 - \frac{1}{l^{p-1}} - \frac{\binom{p}{2}(l^{p-1}-l)}{tl^p} + \frac{\sigma p!}{tl^p} + o(\frac{1}{t})) \\ &= \frac{(lt)^p}{p!}(1 - \frac{1}{l^{p-1}} + \frac{2}{t} + o(\frac{1}{t})) \\ &\geq \frac{(lt)^p}{p!}(1 - \frac{1}{l^{p-1}} + \frac{1}{t}) \end{aligned} \tag{2}$$

for $t \geq t_1$.

Based on the p-uniform graph $G_*^{(p)}$, we construct an r-uniform graph $G_*^{(r)}$ on $l + r - p$ pairwise disjoint sets $V_1, V_2, \ldots, V_l, V_{l+1}, \ldots, V_{l+r-p}$, each of them of size t. An r-element set $\{u_1, u_2, \ldots, u_p, u_{p+1}, \ldots, u_r\}$ is an edge of $G_*^{(r)}$ if and only if $\{u_1, u_2, \ldots, u_p\}$ is an edge of $G_*^{(p)}$ and for each $j, p+1 \leq j \leq r, u_j \in V_{l+j-p}$. Notice that

$$|E(G_*^{(r)})| = t^{r-p}|E(G_*^{(p)})|. \tag{3}$$

Now we give a lower bound of $\lambda(G_*^{(r)})$. Corresponding to the $(l+r-p)t$ vertices of this r-uniform graph, let us take the vector $\vec{x} = (x_1, \ldots, x_{(l+r-p)t})$, where $x_i = \frac{1}{(l+r-p)t}$ for each $i, 1 \leq i \leq (l+r-p)t$. Then

$$\begin{aligned} r!\lambda(G_*^{(r)}) &\geq r!\lambda(G_*^{(r)}, \vec{x}) = r!\frac{|E(G_*^{(r)})|}{((l+r-p)t)^r} \\ &\overset{(2),(3)}{\geq} (1 - \frac{1}{l^{p-1}} + \frac{1}{t})\frac{r!}{p!}\frac{l^p}{(l+r-p)^r}. \end{aligned}$$

Now suppose $\vec{y} = (y_1, y_2, \ldots, y_{(l+r-p)t})$ is an optimal vector of $\lambda(G_*^{(r)})$. Take $\epsilon = \frac{1}{2tp!}\frac{l^p}{(l+r-p)^r}$. Then by Remark 2.3, the r-uniform graph $S_n = (\lceil ny_1 \rceil, \ldots, \lceil ny_{(l+r-p)t} \rceil) \otimes G_*^{(r)}$ has density larger than $(1 - \frac{1}{l^{p-1}})\frac{r!}{p!}\frac{l^p}{(l+r-p)^r} + \epsilon$ for $n \geq n_0(\epsilon)$.

On the other hand, in view of Lemma 2.5, some member F of $\mathcal{F}$ is a subgraph of S_n for $n > n_0(\epsilon)$. For such $F \in \mathcal{F}$, there exists a subgraph $M^{(r)}$ of $G_*^{(r)}$ with $|V(M^{(r)})| \leq k$ so that $F \subset \vec{\boldsymbol{n}} \otimes M^{(r)}$.

By Fact 2.1 and Fact 2.4, we have

$$\lambda(F) \leq \lambda(\vec{\boldsymbol{n}} \otimes M^{(r)}) = \lambda(M^{(r)}).$$

Theorem 3.1 will follow from the following Lemma. The proof of this Lemma will be given in Section 3.1.

Lemma 3.2 *Let $M^{(r)}$ be a subgraph of $G_*^{(r)}$ with $|V(M^{(r)})| \leq k$. Then*

$$\lambda(M^{(r)}) \leq (1 - \frac{1}{l^{p-1}})\frac{1}{p!}\frac{l^p}{(l+r-p)^r}. \tag{4}$$

Assuming the validity of this Lemma, we have

$$\lambda(F) \leq (1-\frac{1}{l^{p-1}})\frac{1}{p!}\frac{l^p}{(l+r-p)^r}$$

which contradicts to our choice of F, i.e., contradicts to the fact that $\lambda(F) > (1-\frac{1}{l^{p-1}})\frac{1}{p!}\frac{l^p}{(l+r-p)^r}$ for all $F \in \mathcal{F}$. This proves Theorem 3.1. ■

The only thing we owe to the proof of Theorem 3.1 is to show Lemma 3.2. Now let us turn to it.

3.1 Proof of Lemma 3.2

By Fact 2.1, we may assume that $M^{(r)}$ is an induced subgraph of $G_*^{(r)}$. Define $U_i = V(M) \cap V_i$ for $1 \leq i \leq l+r-p$. Let $M^{(p)}$ be the p-uniform graph defined on $\cup_{i=1}^{l} U_i$. The edge set of $M^{(p)}$ consists of all edges in the form of $e \cap (\cup_{i=1}^{l} U_i)$, where e is an edge of the r-uniform graph $M^{(r)}$. Let $\vec{\xi} = (x_1, x_2, ..., x_{k'})$ be an optimal vector of $\lambda(M)$. Let $\vec{\xi^{(p)}} = (x_1, x_2, \ldots, x_{k'_1})$ be the restriction of $\vec{\xi}$ in $\cup_{i=1}^{l} U_i$. Let a_i be the sum of the weights in $U_i, 1 \leq i \leq l+r-p$ respectively. In view of the relationship between $M^{(r)}$ and $M^{(p)}$, we have

$$\lambda(M^{(r)}) = \lambda(M^{(p)}, \vec{\xi^{(p)}}) \times \prod_{i=l+1}^{l+r-p} a_i. \tag{5}$$

The following lemma will imply Lemma 3.2. The proof of it will be given later.

Lemma 3.3

$$\lambda(M^{(p)}, \vec{\xi^{(p)}}) \leq \frac{1}{p!}(1-\frac{1}{l^{p-1}})(1-\sum_{i=l+1}^{l+r-p} a_i)^p.$$

Assuming the validity of Lemma 3.3, we have

$$\lambda(M) \leq \frac{1}{p!}(1-\frac{1}{l^{p-1}})(1-\sum_{i=l+1}^{l+r-p} a_i)^p \prod_{i=l+1}^{l+r-p} a_i = \frac{1}{p!}(1-\frac{1}{l^{p-1}})p^p(\frac{1-\sum_{i=l+1}^{l+r-p} a_i}{p})^p \prod_{i=l+1}^{l+r-p} a_i.$$

Since geometric mean is no more than arithmetic mean, we obtain that

$$\begin{aligned}\lambda(M) &\leq \frac{1}{p!}(1-\frac{1}{l^{p-1}})p^p(\frac{1}{r})^r \\ &\leq \frac{1}{p!}(1-\frac{1}{l^{p-1}})\frac{l^p}{(l+r-p)^r}\end{aligned}$$

since $\frac{p^p}{r^r} \leq \frac{l^p}{(l+r-p)^r}$ (see Theorem 3.1). ■

What remains is to prove Lemma 3.3.

Proof of Lemma 3.3. Define $M_1^{(p)} = (U_1, E(M^{(p)}) \cap \binom{U_1}{r})$ i.e., the subgraph of $M^{(p)}$, induced on U_1. Again, by Fact 2.1, it is enough to show Lemma 3.2 for the case $E(M_1^{(p)}) \neq \emptyset$ since otherwise we can add p vertices to $M_1^{(p)}$. Let us assume that $|V(M_1^{(p)})| = p-1+d$ with d a positive integer. Since $|V(M_1^{(p)})| \leq |V(M^{(r)})| \leq k$, in view of Lemma 2.6, $M_1^{(p)}$ has at most d edges. Let $V(M_1^{(p)}) = \{v_1, v_2, \cdots, v_{p-1+d}\}$ and we assume that $x_1, x_2, \cdots, x_{p-1+d}$ are the components of $\vec{\xi}$ corresponding to $v_1, v_2, \cdots, v_{p-1+d}$. We may assume that $x_1 \geq x_2 \geq \cdots \geq x_{p-1+d}$. The following claim was proved in [4] (See Claim 4.4 there).

Claim 3.4
$$\sum\{x_{i_1}x_{i_2}\cdots x_{i_p}, \{v_{i_1}, v_{i_2}, \ldots, v_{i_p}\} \in E(M_1^{(p)})\} \leq \sum_{1\leq j\leq d} x_1x_2\cdots x_{p-1}x_{p-1+j}.$$

By Claim 3.4, we may assume that

$$E(M_1^{(p)}) = \{\{v_1, \ldots, v_{p-1}, v_j\}, p \leq j \leq p-1+d\}.$$

Since $v_1, v_2 \ldots, v_{p-1}$ are pairwise equivalent, in view of Lemma 2.2, we may assume that $x_1 = x_2 = \cdots = x_{p-1} \overset{\text{def}}{=} \rho_0$. Notice that

$$\begin{cases} \sum_{i=1}^{l+r-p} a_i = 1 \\ a_i \geq 0, 1 \leq i \leq l+r-p \\ 0 \leq \rho_0 \leq \frac{a_1}{p-1}. \end{cases} \tag{6}$$

Now we give an upper bound for $\lambda(M^{(p)}, \vec{\xi^{(p)}})$. Observing that each term in $\lambda(M^{(p)}, \vec{\xi^{(p)}})$ appears $p!$ times in the expansion

$$(x_1 + x_2 + \cdots + x_{k_1'})^p = (\underbrace{\rho_0 + \rho_0 + \cdots + \rho_0}_{p-1 \text{ times}} + a_1 - (p-1)\rho_0 + a_2 + \cdots + a_l)^p,$$

but this expansion contains lots of terms not appearing in $\lambda(M^{(p)}, \vec{\xi^{(p)}})$ as well. Since the only edges of $M^{(p)}$ in $\cup_{i=1}^{l}\binom{U_i}{p}$ are the edges in the form of $\{v_1, \ldots, v_{p-1}, v_j\}$ where $v_j \in U_1 - \{v_1, v_2, \ldots, v_{p-1}\}$, $a_1^p + \ldots + a_l^p$ should be subtracted and $p!\rho_0^{p-1}(a_1 - (p-1)\rho_0)$ will be added in this expansion. Also note that $\{v_i, v_i, v_{i_3}, v_{i_4}, \ldots, v_{i_{p-1}}, v\}$ is not an edge in $M^{(p)}$, where $1 \leq i \leq p-1$, and $\{i_3, i_4, \ldots, i_{p-1}\}$ is an $(p-3)$-subset of $\{1, 2, \ldots, p-1\} - \{i\}$ and v is any vertex in $\cup_{j=2}^{l} U_j$. Since each of the corresponding terms appears $p!/2$ times in the expansion, then $(p-1)\frac{p!}{2}\rho_0^{p-1}(a_2 + a_3 + \cdots a_l) \geq p!\rho_0^{p-1}(a_2 + a_3 + \cdots a_l) = p!\rho_0^{p-1}(\sum_{i=1}^{l} a_i - a_1)$ should be subtracted from the expansion. Therefore,

$$\lambda(M^{(p)}, \vec{\xi^{(p)}}) \leq \frac{1}{p!}\{(\sum_{i=1}^{l} a_i)^p - \sum_{i=1}^{l} a_i^p + p!\rho_0^{p-1}[a_1 - (p-1)\rho_0 - (\sum_{i=1}^{l} a_i - a_1)]\}. \tag{7}$$

Lemma 3.3 follows directly from the following claim.

Claim 3.5 *Let*

$$f(a_1, a_2, \ldots, a_l, \rho_0) = (\sum_{i=1}^{l} a_i)^p - \sum_{i=1}^{l} a_i^p + p!\rho_0^{p-1}[2a_1 - (p-1)\rho_0 - \sum_{i=1}^{l} a_i]$$

and c be a positive constant. Then

$$f(a_1, a_2, \ldots, a_l, \rho_0) \le f(c/l, c/l, ..., c/l, 0) = (1 - \frac{1}{l^{p-1}})c^p \tag{8}$$

holds under the constraints

$$\begin{cases} \sum_{i=1}^{l} a_i = c, \\ a_i \ge 0, 1 \le i \le l, \\ 0 \le \rho_0 \le \frac{a_1}{p-1}. \end{cases} \tag{9}$$

Proof of Claim 3.5. Since every term in $f(a_1, a_2, \ldots, a_l, \rho_0)$ has degree p, it is sufficient to show that this claim holds for the case $c = 1$. **So we assume that $c = 1$ throughout the proof of this claim**, i.e. $\sum_{i=1}^{l} a_i = 1$. Now function $f(a_1, a_2, \ldots, a_l, \rho_0)$ can be simplified as

$$f(a_1, a_2, \ldots, a_l, \rho_0) = 1 - \sum_{i=1}^{l} a_i^p + p!\rho_0^{p-1}[2a_1 - 1 - (p-1)\rho_0],$$

and we prove that

$$f(a_1, a_2, \ldots, a_l, \rho_0) \le 1 - \frac{1}{l^{p-1}} \tag{10}$$

under the constraints

$$\begin{cases} \sum_{i=1}^{l} a_i = 1, \\ a_i \ge 0, 1 \le i \le l, \\ 0 \le \rho_0 \le \frac{a_1}{p-1}. \end{cases} \tag{11}$$

We consider two cases.

Case 1. If $a_1 \le \frac{1}{2}$, then $f(a_1, a_2, \ldots, a_l, \rho_0) \le 1 - \sum_{i=1}^{l} a_i^p$ and the right hand side reaches maximum $1 - \dfrac{1}{l^{p-1}}$ when $a_1 = a_2 = \cdots = a_l = \frac{1}{l}$. Therefore (10) holds.

Case 2. If $a_1 \ge \frac{1}{2}$, since geometric mean is no more than arithmetic mean, then $\rho_0^{p-1}[2a_1 - 1 - (p-1)\rho_0] \le (\frac{2a_1-1}{p})^p$. So it is sufficient to show that

$$\begin{aligned} h(a_1, a_2, ..., a_l) &\stackrel{\text{def}}{=} 1 - \sum_{i=1}^{l} a_i^p + p!(\frac{2a_1 - 1}{p})^p \\ &\le 1 - \frac{1}{l^{p-1}}. \end{aligned} \tag{12}$$

Since $\displaystyle\sum_{i=2}^{l} a_i^p \ge (l-1)(\frac{\sum_{i=2}^{l} a_i}{l-1})^p = \frac{(1-a_1)^p}{(l-1)^{p-1}}$, we have

$$h(a_1, a_2, ..., a_l) \le 1 - a_1^p - \frac{(1-a_1)^p}{(l-1)^{p-1}} + p!(\frac{2a_1 - 1}{p})^p \stackrel{\text{def}}{=} h(a_1). \tag{13}$$

So it is sufficient to show that $h(a_1) \le 1 - \dfrac{1}{l^{p-1}}$ if $\frac{1}{2} \le a_1 \le 1$. Notice that

$$h'(a_1) = -pa_1^{p-1} + \frac{p(1-a_1)^{p-1}}{(l-1)^{p-1}} + \frac{2(p-1)!}{p^{p-2}}(2a_1-1)^{p-1} \tag{14}$$

and

$$h''(a_1) = -p(p-1)a_1^{p-2} - \frac{p(p-1)(1-a_1)^{p-2}}{(l-1)^{p-1}} + \frac{4(p-1)(p-1)!}{p^{p-2}}(2a_1-1)^{p-2}. \tag{15}$$

Note that $(2a_1-1)^{p-2} \le a_1^{p-2}$ when $\frac{1}{2} \le a_1 \le 1$. Also note that $\dfrac{4(p-1)!}{p^{p-1}} < 1$ when $p \ge 3$ since the expression in the left hand side decreases as p increases and it is $\frac{8}{9}$ when $p = 3$. Therefore,

$$\frac{4(p-1)(p-1)!}{p^{p-2}}(2a_1-1)^{p-2} \le p(p-1)a_1^{p-2}. \tag{16}$$

By (15) and (16), $h''(a_1) < 0$ when $\frac{1}{2} \le a_1 \le 1$. So

$$h'(a_1) \le h'(1/2) \overset{(14)}{=} \frac{-p}{2^{p-1}} + \frac{p}{2^{p-1}(l-1)^{p-1}} \le 0$$

since $l \ge 2$. Hence $h(a_1)$ decreases when $\frac{1}{2} \le a_1 \le 1$. So

$$\begin{aligned} h(a_1) \le h(1/2) &= 1 - \frac{1}{2^p} - \frac{1}{2^p(l-1)^{p-1}} \\ &\le 1 - \frac{1}{l^{p-1}} \end{aligned}$$

when $l \ge 2$ and $p \ge 3$. The last inequality is true because of the following: when $l = 2$, $\dfrac{1}{2^p} + \dfrac{1}{2^p(l-1)^{p-1}} = \dfrac{1}{l^{p-1}}$. If $l \ge 3$ and $p \ge 3$, then

$$\frac{1}{2^p} \ge \frac{1}{3^{p-1}} \ge \frac{1}{l^{p-1}}.$$

The proof of Claim 3.5 is completed. ∎

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