DEFICIENCIES OF r-REGULAR k-EDGE-CONNECTED GRAPHS

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ABSTRACT:

Let G be a simple graph having a maximum matching M. The deficiency def(G) of G is the number of M-unsaturated vertices in G. A problem that arises is that of determining the set of possible values of def(G). In this paper we present a solution for the case of r-regular k-edge-connected graphs.

1. INTRODUCTION

In this paper the graphs are finite, loopless and have no multiple edges. For the most part our notation and terminology follow Bondy and Murty [3]. Thus G is a graph with vertex set V(G), edge set E(G), $\nu(G)$ vertices and $\varepsilon(G)$ edges. However we denote the complement of G by $\bar{G}$.

A **matching** M in G is a subset of E(G) in which no two edges have a vertex in common. M is a **maximum matching** if $|M| \geq |M'|$ for any other matching M′ of G. A vertex v is **saturated** by M if some edge of M is incident with v; otherwise v is said to be **unsaturated**. A matching M is **perfect** if it saturates every vertex of the graph. The **deficiency** def(G) of G is the number of vertices unsaturated by a maximum matching

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M of G. Observe that def(G) = ν(G) - 2|M|. Consequently, def(G) has the same parity as ν(G), and def(G) = 0 if and only if G has a perfect matching.

Many problems concerning matchings and def(G) in graphs have been investigated in the literature - see, for example, Bollobás and Eldridge [2], Katerinis [8], Little, Grant and Holton [9] and Lovász and Plummer [10]. We have studied the function def(G) for the case when G is a tree with each vertex having degree 1 or k, k ≥ 2 [4] and for the case when G is a cubic graph [5].

In this paper we obtain the upper bound of def(G) and the set of possible values of def(G) when G is r-regular k-edge-connected. We find the set of possible values of def(G) by constructing the graphs.

2. THE UPPER BOUND

Let G be a connected graph on n vertices having a maximum matching M. Since def(G) = n - 2|M|, then clearly def(G) ≤ n - 2 for n ≥ 2. Thus we need to look at restricted classes of graphs to obtain more interesting results. In this paper we focus on the class of regular graphs. A well known result of Petersen states that every 3-regular connected graph with no more than two cut edges has a perfect matching.

When S ⊂ V(G), G-S denotes the graph formed from G by deleting all the vertices in S together with their incident edges. For E′ ⊆ E(G), G-E′ denotes the graph formed from G by deleting the edges of E′. An **edge-cut set** of a connected graph G is a subset E′ of E(G) such that G-E′ is disconnected, but G-E″ is connected for every proper subset E″ of E′. A **k-edge cut** is an edge-cut set having k elements.

A component of a graph G is **odd** or **even** according as it has an odd or even number of vertices. The number of odd components of a graph G is denoted by o(G). We can state Berge's formula ([1], p. 159) for a graph G as :

$$\operatorname{def}(G) = \max_{S \subset V(G)} \{o(G-S) - |S|\}. \tag{2.1}$$

Let G be an r-regular graph and $S \subset V(G)$. As a 1-regular graph is a perfect matching, we may suppose that $r \geq 2$. Let $G_1, G_2, \ldots, G_{o(G-S)}$ denote the odd components of G-S. The number of edges in G joining the vertices of G_i to the vertices of S is denoted by t_i. It is clear that

$$r\,\nu(G_i) = 2\varepsilon(G_i) + t_i. \tag{2.2}$$

A consequence of (2.2) is that t_i and r have the same parity. We let ℓ_t denote the number of odd components of G-S that are joined to S by exactly t edges. Observe that $\ell_t = 0$ when t and r have different parity.

Lemma 2.1 : Let G be an r-regular graph, $r \geq 2$. Then there exists a set $S \subset V(G)$ such that

$$r\,\operatorname{def}(G) \leq \begin{cases} \displaystyle\sum_{t=0}^{\frac{1}{2}(r-2)} (r-2t)\ell_{2t}, & \text{if } r \text{ is even} \\ \displaystyle\sum_{t=0}^{\frac{1}{2}(r-3)} (r-2t-1)\ell_{2t+1}, & \text{otherwise.} \end{cases}$$

Proof : Clearly $def(G) \geq o(G)$ and the result is true when $def(G) = o(G)$ since we can take $S = \phi$.

So suppose $def(G) > o(G)$. By (2.1) there exists a set $S \subset V(G)$ such that

$$o(G-S) = |S| + def(G).$$

Since $def(G) > o(G)$, then $S \neq \phi$. The number of odd components joined to S by at least r edges is

$$o(G-S) - \sum_{t=0}^{r-1} \ell_t .$$

Now since G is r-regular we have

$$r|S| \geq r\,(o(G-S) - \sum_{t=0}^{r-1} \ell_t\,) + \sum_{t=0}^{r-1} t\,\ell_t$$

$$= r(|S| + def(G) - \sum_{t=0}^{r-1} \ell_t) + \sum_{t=0}^{r-1} t\,\ell_t$$

and hence

$$r\,def(G) \leq r \sum_{t=0}^{r-1} \ell_t - \sum_{t=0}^{r-1} t\,\ell_t$$

$$= \sum_{t=0}^{r-1} (r-t)\,\ell_t .$$

The result follows since $\ell_t = 0$ when r and t have different parity. □

For connected graphs with deficiency not equal to one we have the following lemma.

Lemma 2.2 : Let G be an r-regular, connected graph having $def(G) \neq 1$. Suppose that for any $\phi \neq V_1 \subset V(G)$ every odd component of $G - V_1$ is joined to V_1 by not less than m edges, $1 \leq m \leq r - 2$ ($m \equiv r \pmod 2$). Then there exists a non-empty set $S \subset V(G)$ such that G-S has

$$\ell \geq \frac{r}{r-m} \, def(G)$$

odd components joined to S by at most r-2 edges.

Proof : The result is trivially true when $def(G) = 0$. So suppose $def(G) \geq 2$. From Lemma 2.1 we have $\phi \neq S \subset V(G)$ with

$$r \, def(G) \leq \sum_{t=0}^{r-2} (r-t) \, \ell_t$$

$$\leq \sum_{t=0}^{r-2} (r-m) \, \ell_t$$

$$= (r-m) \sum_{t=0}^{r-2} \ell_t$$

and hence

$$\ell = \sum_{t=0}^{r-2} \ell_t \geq \frac{r}{r-m} \, def(G)$$

as required. □

Lemma 2.2 has a number of corollaries when G is k-edge-connected. It is convenient to let $\mathcal{G}(n,r,k)$ denote the class of r-regular, k-edge-connected graphs on n vertices.

Corollary 1: Let $G \in \mathcal{G}(n,r,k)$, $1 \le k \le r-2$, be a graph with $def(G) \ne 1$. Then there exists a non-empty set $S \subset V(G)$ such that G-S has

$$\ell \ge \frac{r}{r-k'} \, def(G)$$

odd components each of which is joined to S by at most r-2 edges, where k′ is the least integer not less than k having the same parity as r. □

When $\nu(G)$ is even, def(G) is even and thus G has a perfect matching if $def(G) < 2$. We thus have the following corollary to Lemma 2.2.

Corollary 2 : Let $G \in \mathcal{G}(n,r,k)$, $1 \le k \le r-2$ and n even, and let k′ be the least integer not less than k having the same parity as r. If G has fewer than $2r/(r-k')$ disjoint edge-cut sets whose cardinality is of the same parity as r and at most r-2, then G has a perfect matching. □

When $k = r-2$, Corollary 2 reduces to the following result mentioned in Chartrand and Nebesky [7].

Corollary 3 : Let $G \in \mathcal{G}(n,r,r-2)$, $r \ge 3$ and n even. If G contains at most r-1 (r-2)-edge cuts, then G has a perfect matching. □

When $k = r-1$, we have the following well known result (see [1], p. 160).

Corollary 4 : Let $G \in \mathcal{G}(n,r,r-1)$, $r \ge 2$ and n even. Then G has a perfect matching. □

Corollary 3 is usually regarded as a generalization of Petersen's result. We remark that Corollary 1 is a generalization of a result (Theorem 2.1) proved in [5].

Bollobás and Eldridge [2] considered the problem of determining the minimum possible value of a maximium matching of a graph G with prescribed minimum and maximum degrees and prescribed edge or vertex connectivity. A consequence of their results (Theorems 4 and 5) is the following upper bound on def(G) for $G \in \mathcal{G}(n,r,k)$.

Theorem 2.1 : Let

$$d = \max\{\mathrm{def}(G) : G \in \mathcal{G}(n,r,k),\ k \le r \le n-1,\ r \ge 3$$
$$\text{and } n \text{ is even when } r \text{ is odd}\}.$$

Then

(a) $$2\left\lceil d_0 - \frac{5}{2} \right\rceil \le d \le 2\left\lfloor d_0 + \frac{1}{2} \right\rfloor, \text{ if } r \text{ is odd and } k = 1,$$

where

$$d_0 = \frac{n(r^2 - 3r + 2)}{2(r^3 - 3r)} \quad ;$$

(b) $$d \le \max\left\{1, \frac{n(r - k')}{r\,r^* + k'}\right\} \text{ and } d \equiv n (\mathrm{mod}\ 2), \text{ otherwise},$$

where k' is the least integer not less than k having the same parity as r and r^* is the least odd integer greater than r. □

The above result does not give an exact value of d for every n, r and k. We now extend Theorem 2.1 to obtain an exact value of d for $k \ge 2$. We need the following simple lemma.

Lemma 2.3 : Let G be an r-regular graph, $S \subset V(G)$ and G_0 be an odd component of G-S which is joined to S by fewer than r edges. Then $\nu(G_0) > r$. □

Theorem 2.2 : Let $G \in \mathcal{G}(n,r,1)$. If for any non-empty set $S \subset V(G)$ every odd component of G-S is joined to S by not less than m edges, where $1 \le m \le r - 2$ and $m \equiv r \pmod 2$, then

(a) $\operatorname{def}(G) \le 2 \left\lfloor \frac{r-m}{2r} \left\lfloor \frac{rn}{rr^* + m} \right\rfloor \right\rfloor$, if n is even ;

(b) $\operatorname{def}(G) = 1$, if $n < \frac{r^2 + r + m}{r} \left\lceil \frac{3r}{r-m} \right\rceil$ and n is odd;

(c) $\operatorname{def}(G) \le 1 + 2 \left\lfloor \frac{r-m}{2r} \left\lfloor \frac{rn}{r^2 + r + m} \right\rfloor - \frac{1}{2} \right\rfloor$,

otherwise;

where r* is the least odd integer greater than r.

Proof : The result is trivially true when def(G) = 0 or 1. So suppose def(G) ≥ 2. Lemma 2.2 implies that there exists a non-empty set $S \subset V(G)$ such that G-S has $\ell \ge \frac{r}{r-m}$ def(G) odd components, $G_1, G_2, \ldots, G_\ell$ say, joined to S by at most r-2 edges. Simple counting of edges between these odd components and S yields

$$r \mid S \mid \ge \ell m$$

and hence

$$|S| \ge \frac{\ell m}{r} .$$

Lemma 2.3 implies that $\nu(G_i) \ge r^*$ for $i = 1,2,\ldots,\ell$. Hence

$$n \geq |S| + \sum_{i=1}^{\ell} \nu(G_i)$$

$$\geq \frac{\ell m}{r} + \ell r^*.$$

Consequently,

$$\ell \leq \left\lfloor \frac{rn}{rr^* + m} \right\rfloor .$$

Now, since $\ell \geq \frac{r}{r-m}$ def(G) we have

$$\mathrm{def}(G) \leq \frac{r-m}{r} \left\lfloor \frac{rn}{rr^* + m} \right\rfloor .$$

Now when n is even, def(G) must be even and thus we can write

$$\mathrm{def}(G) \leq 2 \left\lfloor \frac{r-m}{2r} \left\lfloor \frac{rn}{rr^* + m} \right\rfloor \right\rfloor ,$$

proving (a). When n is odd, r is even and so $r^* = r + 1$. Further, def(G) must be odd. Hence

$$3 \leq \mathrm{def}(G) \leq \frac{r-m}{r} \left\lfloor \frac{rn}{rr^* + m} \right\rfloor .$$

Therefore

$$\left\lceil \frac{3r}{r-m} \right\rceil \leq \left\lfloor \frac{rn}{rr^* + m} \right\rfloor$$

and thus

$$n \geq \frac{r^2 + r + m}{r} \left\lceil \frac{3r}{r-m} \right\rceil .$$

So, if $n < \frac{r^2 + r + m}{r} \left\lceil \frac{3r}{r-m} \right\rceil$ is odd, then def(G) = 1. When $n \geq \frac{r^2 + r + m}{r} \left\lceil \frac{3r}{r-m} \right\rceil$ is odd, then def(G) is odd and we can write

$$\mathrm{def}(G) \le 1 + 2 \left\lfloor \frac{r-m}{2r} \left\lfloor \frac{rn}{r^2 + r + m} \right\rfloor - \frac{1}{2} \right\rfloor .$$

This completes the proof of the theorem. □

For the case when $G \in \mathcal{G}(n,r,k)$ we have the following two corollaries to Theorem 2.2.

Corollary I : Let $G \in \mathcal{G}(n,r,k)$, with $1 \le k \le r-2$. Then

(a) $\mathrm{def}(G) \le 2 \left\lfloor \frac{r-k'}{2r} \left\lfloor \frac{rn}{rr^* + k'} \right\rfloor \right\rfloor$, if n is even;

(b) $\mathrm{def}(G) = 1$, if n is odd and $n < \frac{r^2 + r + k'}{r} \left\lceil \frac{3r}{r-k'} \right\rceil$;

(c) $\mathrm{def}(G) \le 1 + 2 \left\lfloor \frac{r-k'}{2r} \left\lfloor \frac{rn}{r^2 + r + k'} \right\rfloor - \frac{1}{2} \right\rfloor$,otherwise;

where k' is the least integer not less than k which has the same parity as r and r^* is as in Theorem 2.2. □

Corollary II : Let $G \in \mathcal{G}(n,r,k)$, with $1 \le k \le r-2$ and n even. If G has no perfect matching, then

$$n \ge \frac{rr^* + k'}{r} \left\lceil \frac{2r}{r-k'} \right\rceil$$

where r^* and k' are as defined in Corollary I. □

Remark 1 : In the next section we will show, by construction, that the bounds given in Theorem 2.2 and Corollary I are sharp for $m \ne 1$. Further, the bounds given in Corollary II are sharp.

Remark 2 : Corollary II is a generalization of a result of Wallis [12].

3. CONSTRUCTIONS

We make use of the following notations in the description of our graphs. A matching of size t in a graph H is denoted by $M_t(H)$. The complement in H of a matching M of size t is denoted by $\bar{M}_t(H)$; that is $\bar{M}_t(H) = H\backslash M$. It is very well known that K_{2n+1} has a Hamiltonian cycle decomposition and that K_{2n} is the edge sum of n-1 Hamilton cycles plus a perfect matching. Let $H_1, H_2, \ldots, H_{\lfloor\frac{1}{2}(n-1)\rfloor}$ denote the Hamilton cycles of K_n.

We now define three classes of graphs. For $0 \le t \le \frac{1}{2}(n-1)$ we define

$$G_0(n,t) = K_n\backslash\{H_1, H_2, \ldots, H_t\}.$$

Observe that $G_0(n,t)$ is (n-2t-1)-regular and contains $\lfloor\frac{1}{2}(n-1)\rfloor - t$ Hamilton cycles. For $p \le \frac{1}{2}$ n we form the following graphs from $G_0(n,t)$;

$$G_p(n,t) = G_0(n,t) \cup M_p(H_t) \;;$$
$$G'_p(n,t) = G_0(n,t) \cup \bar{M}_p(H_t) \;.$$

Observe that each of these graphs has $\lfloor\frac{1}{2}(n-1)\rfloor - t$ Hamilton cycles. We make use of the above graphs in the proof of the following result.

Theorem 3.1 : For $2 \le k \le r-2 \le n-3$, let

$$D(n,r,k) = \{def(G) : G \in \mathcal{G}(n,r,k)\}.$$

Then

(a) $D(n,r,k) = \phi$, if n and r are odd;

(b) $D(n,r,k) = \{d : 0 \le d \le 2 \lfloor \frac{r-k'}{2r} \lfloor \frac{rn}{rr^* + k'} \rfloor\rfloor$, d is even}, if n is even;

(c) $D(n,r,k) = \{1\}$, if $n < \frac{r^2 + r + k'}{r} \lceil \frac{3r}{r-k'} \rceil$ is odd and r is even;

(d) $D(n,r,k) = \{d : 1 \le d \le 1 + 2 \lfloor \frac{r-k'}{2r} \lfloor \frac{rn}{r^2 + r + k'} \rfloor - \frac{1}{2} \rfloor$, d is odd}, otherwise;

where k′ is the least integer not less than k which has the same parity as r and r* is the least odd integer greater than r.

Proof : As the number of vertices of odd degree is even, part (a) is obvious. So suppose that at least one of n or r is even. The upper bound on def(G), $G \in \mathcal{G}(n,r,k)$, is determined in Corollary I of Theorem 2.2.

First we consider the case when n is even. We will exhibit for each even d, $0 \le d \le 2 \lfloor \frac{r-k'}{2r} \lfloor \frac{rn}{rr^* + k'} \rfloor\rfloor$, a graph $G \in \mathcal{G}(n,r,k)$ with def(G) = d. For d = 0 we take the graph $G_{n/2}(n,\frac{1}{2}(n-r))$ if r is even, and the graph $G_0(n,\frac{1}{2}(n-r-1))$ if r is odd. Now consider d ≥ 2. Then $n \ge \frac{rr^* + k'}{r} \lceil \frac{2r}{r-k'} \rceil$. Define

$$\ell = \left\lceil \frac{rd}{r-k'} \right\rceil ,$$

$$s = \left\lceil \frac{k'\ell}{r} \right\rceil ,$$

$$p = n - s - (\ell-1)r^* ,$$

and

$$k'' = rs - k'(\ell-1) .$$

Making use of the fact that for any non-negative real numbers a, b and c with $b \neq 0$ and $\frac{a}{b} < 1$,

$$\left\lceil \frac{ac}{b-a} \right\rceil = \left\lceil \frac{a}{b} \left\lceil \frac{bc}{b-a} \right\rceil \right\rceil ,$$

we have

$$\ell - s = \left\lceil \frac{rd}{r-k'} \right\rceil - \left\lceil \frac{k'}{r} \ell \right\rceil$$

$$= \left\lceil d + \frac{k'd}{r-k'} \right\rceil - \left\lceil \frac{k'}{r} \left\lceil \frac{rd}{r-k'} \right\rceil \right\rceil$$

$$= d + \left\lceil \frac{k'd}{r-k'} \right\rceil - \left\lceil \frac{k'd}{r-k'} \right\rceil$$

$$= d.$$

We claim that $p \geq r^*$. Suppose that $p < r^*$. Then

$$n < s + \ell r^*$$

$$= \left\lceil \frac{k'\ell}{r} + \ell r^* \right\rceil$$

$$= \left\lceil \left(\frac{rr^* + k'}{r} \right) \ell \right\rceil$$

$$= \left\lceil \frac{rr^* + k'}{r} \left\lceil \frac{rd}{r-k'} \right\rceil \right\rceil$$

$$\le \left\lceil \frac{rr^* + k'}{r} \left\lfloor \frac{rn}{rr^* + k'} \right\rfloor \right\rceil \quad \text{(using the bound on d)}$$

$$\le n, \text{ a contradiction.}$$

Thus $p \ge r^*$. Furthermore, since n and d are even, r^* is odd and

$$p = n - s - (\ell-1)r^*$$

$$= n + d + r^* - (r^* + 1)\ell,$$

p must be odd. Also

$$\begin{aligned} k'' &= rs - k'(\ell-1) \\ &= r(\ell-d) - k'(\ell-1) \\ &= -rd + \ell(r-k') + k' \\ &= -rd + \left\lceil \frac{rd}{r-k'} \right\rceil (r-k') + k'. \end{aligned}$$

Hence k'' has the same parity as r and

$$k' \le k'' < r.$$

The required graphs are constructed as follows. Take an empty graph $\bar{K}_s$ with vertices $u_1, u_2, \dots, u_s$. When r is even we take $\ell-1$ copies G_1, $G_2, \dots, G_{\ell-1}$ of $G'_{\frac{1}{2}k'}(r+1,1)$ and one copy G_ℓ of $G'_{\frac{1}{2}k''}(p,\frac{1}{2}(p-r+1))$. Observe that G_i, $1 \le i \le \ell-1$, has k′ vertices $v_{i1}, v_{i2}, \dots, v_{ik'}$ say, of degree r-1 and r+1-k′ vertices of degree r. Further, G_ℓ has k″ vertices $v_{\ell 1}, v_{\ell 2}, \dots, v_{\ell k''}$ say, of degree r-1 and p-k″ vertices of degree r. Each G_i, $1 \le i \le \ell$, contains $\frac{1}{2}(r-2)$ Hamilton cycles and thus is k′-edge-connected. We form a graph $\hat{G}_1 \in \mathcal{G}(n,r,k)$ by adding the following edges. For $1 \le i \le s$ and $1 \le j \le k'$, join v_{ij} to u_z if $i + j - 1 \equiv z \pmod{s}$. Join v_{ij} to u_z if $(i-1)k' + j \equiv z \pmod{s}$, where $s + 1 \le i \le \ell$ and $1 \le j \le k'$ if $i < \ell$ and $1 \le j \le k''$ if $i = \ell$. Since $k' \ge k$ our $\hat{G}_1$ is k-edge connected. Since each G_i, $1 \le i \le \ell$, has a Hamilton cycle it follows that $\operatorname{def}(\hat{G}_1) \le \ell - s = d$. On the other hand, by choosing $S = \{u_1, u_2, \dots, u_s\}$ (2.1) implies that $\operatorname{def}(\hat{G}_1) \ge \ell - s = d$. Thus $\operatorname{def}(\hat{G}_1) = d$.

When r is odd the required graph $\hat{G}_2$ can be obtained by following the above construction taking G_i as the graph :

$$G_{\frac{1}{2}(r-k'+2)}(r + 2, 1) \quad \text{for } 1 \le i \le \ell-1;$$

and $G_{\frac{1}{2}(p-k'')}(p, \frac{1}{2}(p-r))$ for $i = \ell$. Note that k′ and k″ are defined relative to r. This proves part (b).

Now consider the case when n is odd. Then r is, of course, even. For def(G) = 1, the graph $G_0(n,\frac{1}{2}(n-r-1))$ has the required properties.

So suppose def(G) ≥ 3. Then part (c) of Corollary I to Theorem 2.2 implies that $n \geq \frac{r^2 + r + k'}{r} \lceil \frac{3r}{r-k'} \rceil$. For each odd d, $3 \leq d \leq 2 \lfloor \frac{r-k'}{2r} \lfloor \frac{rn}{r^2 + r + k'} \rfloor - \frac{1}{2} \rfloor$, a graph $\hat{G}_3 \in \mathcal{G}(n,r,k)$ with $def(\hat{G}_3) = d$ can be obtained by following the description used in defining $\hat{G}_1$. Note that here $r^* = r + 1$. This completes the proof of the theorem. □

Remark : Consider the graphs $\hat{G}_1$ and $\hat{G}_2$ defined in the above proof. If we set $p = r^*$ and $d = 2$, then $n = \lceil \frac{rr^* + k'}{r} \lceil \frac{2r}{r-k'} \rceil \rceil$. Consequently the bound given in Corollary II of Theorem 2.2 is sharp. Note that $\hat{G}_1$ and $\hat{G}_2$ are well defined when $k = k' = 1$.

4. REGULAR GRAPHS WITH PRESCRIBED DEFICIENCY

In the previous section we established that the bounds given in Theorem 2.2 are sharp for $m \neq 1$. In this section we consider the case $m = 1$. In our first result we establish a lower bound on n for a graph $G \in \mathcal{G}(n,r,1)$ having def(G) = d.

Theorem 4.1 : Suppose $G \in \mathcal{G}(n,r,1)$, where r is an odd integer greater than 1 and n is an even integer greater than r. Let d = def(G), and suppose that $d = t(r-1) + q + 2$ where t and q are integers, $0 \leq q \leq r - 3$.

Then

(a) $n \geq (r+2)d + (r+3) \lceil \frac{d}{r-1} \rceil + r + 1,$

if $r-q-2 \leq t \leq \frac{1}{2}(r-3)$;

(b) $n \geq (r+2)d + (r+1) \lceil \frac{d}{r-1} \rceil + 2r,$

if $\max\{\frac{1}{2}(r-1), r-q-2\} \leq t \leq r-3;$

(c) $n \geq (r+2)d + (r+1) \lceil \frac{d}{r-1} \rceil + 2 \lceil \frac{d-1}{r-2} \rceil$,

otherwise.

Proof : Since n is even, d is even. The result is trivially true when d = 0. So suppose d ≥ 2. By (2.1) there is a non-empty set $S \subset V(G)$ such that $o(G-S) = |S| + d$. Following the proof of Lemma 2.2 we conclude that G-S has $\ell \geq \lceil \frac{rd}{r-1} \rceil$ odd components, $G_1, G_2, \ldots, G_\ell$ say, joined to S by at most r-2 edges. Lemma 2.3 implies that $\nu(G_i) \geq r+2$ for every $1 \leq i \leq \ell$. Denote the remaining components of G-S by $G_{\ell+1}$, $G_{\ell+2}, \ldots, G_p$.

We have

$$r|S| \geq \varepsilon(G) - \sum_{i=1}^{P} \varepsilon(G_i)$$

$$\geq o(G-S) + |S| - 1$$

(since G is connected)

$$= (d + |S|) + |S| - 1$$

and hence

$$|S| \geq \lceil \frac{d-1}{r-2} \rceil .$$

Further

$$\begin{aligned} n &\geq (r+2)\ell + o(G-S) - \ell + |S| \\ &= (r+1)\ell + 2|S| + d \\ &\geq (r+1) \lceil \frac{rd}{r-1} \rceil + d + 2 \lceil \frac{d-1}{r-2} \rceil \\ &= (r+2)d + (r+1) \lceil \frac{d}{r-1} \rceil + 2 \lceil \frac{d-1}{r-2} \rceil \end{aligned}$$

This proves (c).

We need to consider the case when $r - q - 2 \le t \le r - 3$. Then $\lceil \frac{d}{r-1} \rceil = t + 1$ and $r \ge 5$. We have

$$
\begin{aligned}
|S| &\ge \lceil \frac{d-1}{r-2} \rceil \\
&= \lceil \frac{t(r-1) + q + 1}{r - 2} \rceil \\
&= t + 2.
\end{aligned}
$$

Hence

$$
\begin{aligned}
o(G-S) &= |S| + d \\
&\ge t + 2 + d \\
&= \lceil \frac{rd}{r-1} \rceil + 1 .
\end{aligned}
$$

We distinguish two cases according to the value of $\nu(G_i)$, where $\lceil \frac{rd}{r-1} \rceil + 1 \le i \le o(G-S)$. Suppose that $\nu(G_i) \ge r$ for such some i. Then

$$
\begin{aligned}
n &\ge (r+2) \lceil \frac{rd}{r-1} \rceil + r + o(G-S) - \lceil \frac{rd}{r-1} \rceil - 1 + |S| \\
&\ge (r+1) \lceil \frac{rd}{r-1} \rceil + r + d - 1 + 2 (\lceil \frac{d}{r-1} \rceil + 1) \\
&= (r+2)d + (r+3) \lceil \frac{d}{r-1} \rceil + r + 1 \\
&= n_1 . \qquad (4.1)
\end{aligned}
$$

If, on the other hand, $\nu(G_i) \le r - 2$ for every i, then $\ell = \lceil \frac{rd}{r-1} \rceil$ and each G_i has at least r edges going to S. If $\nu(G_i) \ge 3$ for some i, then there are at least

$$
\begin{aligned}
&r\, \nu(G_i) - (\nu(G_i)(\nu(G_i) - 1)) \\
&\ge 3r - 6
\end{aligned}
$$

edges between G_i and S. Consequently

$$r|S| \geq \ell + 3r - 6 + r(o(G-S) - \ell - 1)$$
$$= \ell + 3r - 6 + r(|S| + d - \ell - 1)$$

and hence

$$\ell \geq \frac{rd + 2r - 6}{r - 1}$$
$$\geq 1 + \frac{rd}{r-1} .$$

But this contradicts the fact that $\ell = \lceil \frac{rd}{r-1} \rceil$. Therefore $\nu(G_i) = 1$ for every $\lceil \frac{rd}{r-1} \rceil + 1 \leq i \leq o(G-S)$. Hence $|S| \geq r$ and

$$n \geq (r+2)\ell + o(G-S) - \ell + |S|$$
$$= (r+1) \lceil \frac{rd}{r-1} \rceil + 2|S| + d$$
$$\geq (r+2)d + (r+1) \lceil \frac{d}{r-1} \rceil + 2r$$
$$= n_2 . \qquad (4.2)$$

Inequalities (4.1) and (4.2) imply that

$$n \geq \min\{n_1, n_2\} .$$

Now we have

$$n_1 - n_2 = 2 \lceil \frac{d}{r-1} \rceil + 1 - r$$
$$= 2t + 3 - r.$$

Hence $n_1 \le n_2$ when $t \le \frac{1}{2}(r-3)$. This proves (a) and (b) and completes the proof of the theorem. □

That the above bounds are sharp follows from our next result. We make use of some of the graphs defined in the previous section.

Theorem 4.2 : Suppose $d = t(r-1) + q + 2$ is an even non-negative integer, where q,r and t are integers, r is odd and $0 \le q \le r - 3$. Let

$$n_1 = \begin{cases} r + 1 , & \text{if } d = 0 \\ (r+2)d + (r+3)\left\lceil \frac{d}{r-1} \right\rceil + r + 1, & \text{if } r-q-2 \le t \le \frac{1}{2}(r-3) \\ (r+2)d + (r+1)\left\lceil \frac{d}{r-1} \right\rceil + 2r , & \text{if } \max\{\frac{1}{2}(r-1),\ r-q-2\} \le t \le r-3 \\ (r+2)d + (r+1)\left\lceil \frac{d}{r-1} \right\rceil + 2\left\lceil \frac{d-1}{r-2} \right\rceil , & \text{otherwise.} \end{cases}$$

Then for every even $n \ge n_1$ there exists a $G \in \mathcal{G}(n,r,1)$ with $def(G) = d$.

Proof : Assume that $n \ge n_1$ is even. First we observe that the graph $G_0(n,\frac{1}{2}(n-r-1)) \in \mathcal{G}(n,r,1)$ and has a perfect matching. This proves the result for $d = 0$. For $d \ge 2$ we consider four cases.

Case 1 : $r - q - 2 \le t \le \frac{1}{2}(r-3)$.

Recall that the graph $G_p(n,h)$, $p \le \frac{1}{2}n$ and $1 \le h < \frac{1}{2}n$ has 2p vertices of degree $n - 2h$, $n - 2p$ vertices of degree $n - 2h - 1$ and is hamiltonian. We form a graph $G_1^* \in \mathcal{G}(n,r,1)$ with $def(G_1^*) = d$ as follows. Take the empty graph $\bar{K}_{t+2}$ with vertices $u_1,u_2,\ldots,u_{t+2}$, the complete graph K_r with vertices $v_1,v_2,\ldots,v_r$, $rt + q + 2$ copies

$G_1, G_2, \ldots, G_{rt+q+2}$ of $G_{\frac{1}{2}(r+1)}(r + 2,\ 1)$ and the graph $G_{rt+q+3} = G_{\frac{1}{2}(n-n_1+q+4)}(r + 2 \quad + n - n_1,\ \frac{1}{2}(n - n_1 + 2))$. Observe that G_i, $1 \le i \le rt + q + 2$, has exactly one vertex, v_{r+i} say, of degree $r - 1$ and the graph G_{rt+q+3} has $(r - q - 2)$ vertices, $v_{r(t+1)+q+3}$, $v_{r(t+1)+q+4}, \ldots, v_{rt+2r}$ say, of degree $r-1$. Add the edges u_iv_j if $i \equiv j \pmod{t+2}$. This defines the graph G_1^*. Observe that

$$\nu(G_1^*) = t + 2 + r + (rt + q + 2)(r + 2) + r + 2 + n - n_1$$

$$= (r + 2)(rt - t + q + 2) + (r + 3)(t + 1) + r + 1 + n - n_1$$

$$= (r + 2)d + (r + 3) \left\lceil \frac{d}{r-1} \right\rceil + r + 1 + n - n_1$$

$$= n.$$

It is easy to verify that G_1^* is connected and r-regular. Thus $G_1^* \in \mathcal{G}(n,r,1)$. Further, taking $S = \{u_1, u_2, \ldots, u_{t+2}\}$, $G - S$ has $rt + q + 4$ odd components each having a Hamilton cycle and hence

$$\operatorname{def}(G) = rt + q + 4 - t - 2$$
$$= d,$$

as required.

Case 2 : $\max\{\frac{1}{2}(r-1),\ r-q-2\} \le t \le r-3$.

We form a graph $G_2^* \in \mathcal{G}(n,r,1)$ with $\mathrm{def}(G_2^*) = d$ as follows. Take the empty graph $\bar{K}_{2r-t-1}$ with vertices $u_1, u_2, \ldots, u_{2r-t-1}$, $rt + q + 2$ copies $G_1, G_2, \ldots, G_{rt+q+2}$ of $G_{\frac{1}{2}(r+1)}(r + 2,\ 1)$ and the graph $G_{rt+q+3} = G_{\frac{1}{2}(n-n_1+q+4)}(r + 2 + n - n_1,\ \frac{1}{2}(n - n_1 + 2))$. As in Case 1, above, the graph G_i, $1 \le i \le rt + q + 2$, has exactly one vertex, v_i say, of degree $r - 1$, and the graph G_{rt+q+3} has $(r - q - 2)$ vertices, v_{rt+q+3}, $v_{rt+q+4}, \ldots, v_{rt+r}$ say, of degree $r - 1$. Add the edges : u_iu_j for every $1 \le i \le r$ and $r + 1 \le j \le 2r - t - 1$; v_iu_j, if $i \equiv j \pmod r$, $1 \le i \le rt + r$, $1 \le j \le r$. This defines the graph G_2^* . Observe that

$$\nu(G_2^*) = 2r - t - 1 + (rt + q + 2)(r + 2) + r + 2 + n - n_1$$

$$= (r + 2)(rt - t + q + 2) + (r + 1)(t + 1) + 2r + n - n_1$$

$$= (r + 2)d + (r + 1)\left\lceil \frac{d}{r-1} \right\rceil + 2r + n - n_1$$

$$= n.$$

Again it is easy to verify that G_2^* is connected and r-regular. Thus $G_2^* \in \mathcal{G}(n,r,1)$. Further, taking $S = \{u_1, u_2, \ldots, u_r\}$, $G - S$ has $(rt + q + 3)$ odd components each having a Hamilton cycle and $r - t - 1$ components each a single vertex. Hence

$$\operatorname{def}(G_2^*) = rt + q + 3 + r - t - 1 - r$$
$$= d,$$

as required.

Case 3 : $t \le r - q - 3.$

We construct the required graph G_3^* as follows. Take the empty graph $\bar{K}_{t+1}$ with vertices $u_1, u_2, \ldots, u_{t+1}$ and the graphs $G_1, G_2, \ldots, G_{rt+q+3}$ defined in Case 2. Labelling the vertices of G_i, $1 \le i \le rt + q + 3$ as in Case 2, add the edges : $v_i u_j$ if $i \equiv j(\bmod(t+1))$. It is easy to verify that G_3^* is connected and r-regular. Further

$$\nu(G_3^*) = t + 1 + (r + 2)(rt + q + 2) + r + 2 + n - n_1$$

$$= (r + 2)(rt - t + q + 2) + (r + 1)(t + 1) + 2(t + 1) + n - n_1$$

$$= (r + 2)d + (r + 1) \left\lceil \frac{d}{r-1} \right\rceil + 2\left\lceil \frac{d-1}{r-2} \right\rceil + n - n_1$$

$$= n,$$

and so $G_3^* \in \mathcal{G}(n,r,1)$. Taking $S = \{u_1, u_2, \ldots, u_{t+1}\}$, G-S has $rt + q + 3$ odd components each having a Hamilton cycle. Hence

$$\operatorname{def}(G_3^*) = rt + q + 3 - t - 1$$
$$= d ,$$

as required.

Case 4 : $t \geq r - 2$.

Let $t = a(r - 2) + b$, where a and b are integers with $0 \leq b \leq r - 3$. We have

$$n_1 = (r + 2)d + (r + 1) \left\lceil \frac{d}{r-1} \right\rceil + 2 \left\lceil \frac{d-1}{r-2} \right\rceil$$

$$= (r + 2)d + (r + 1)(t + 1) + 2t + 2 \left\lceil \frac{t+q+1}{r-2} \right\rceil$$

$$= (r + 2)d + (r + 1)(a(r - 2) + b + 1) + 2a(r - 2) + 2b$$
$$+ 2a + 2 \left\lceil \frac{b+q+1}{r-2} \right\rceil .$$

Substituting for d and simplifying we get

$$n_1 = ar^3 + br^2 + (2b - 3a + q + 3)r + b + 2q + 2\alpha + 5,$$

where

$$\alpha = \begin{cases} 1, & \text{if } b \leq r - q - 3 \\ 2, & \text{otherwise.} \end{cases}$$

We begin our constructions by defining a graph $T_r(p)$ for a positive integer p. This graph will form the basic building block in our constructions to follow. Take $p - 1$ disjoint copies, $G_1, G_2, \ldots, G_{p-1}$ say, of the star $K_{1,r-1}$ and one copy, G_p say, of the star $K_{1,r-2}$. Let x_i be the centre of G_i. The graph $T_r(p)$ is formed by adding a new vertex, y say, and joining y to each x_i, $1 \leq i \leq p$. Observe that for $p > 1$, $T_r(p)$ has pr vertices of which $(r - 2) + (p - 1)(r - 1)$ have degree 1. In using $T_r(p)$ as a building block the vertices y and x_p need to be identified. For convenience we relabel x_p as z.

We consider two subcases according to the value of b. First suppose that $b \le r - q - 3$. Take a copies $T_1, T_2, \ldots, T_a$ of $T_r(r - 1)$, and one copy T_{a+1} say, of $T_r(2\lfloor \frac{1}{2}b \rfloor + 1)$. We relabel the vertices y and z of T_i by y_i and z_i, respectively. Now add the edges $y_i z_{i+1}$ for $1 \le i \le a$. We form the graph G' as follows : if b is even, add a new vertex u_0 and join it to z_1; if b is odd, add the star $K_{1,r-1}$ and join its centre to z_1. Observe that

$$\nu(G') = \begin{cases} ar(r - 1) + r(b + 1) + 1, & \text{if } b \text{ is even} \\ ar(r - 1) + br + r, & \text{otherwise.} \end{cases} \qquad (4.3)$$

From the graph G' we form the graph G'' as follows. Let $\lambda = r - 2 - 2(b - \lfloor \frac{1}{2} b \rfloor) - q$. Observe that λ is odd and $\lambda \le r - 2$. Recall that the graph $H_1 = G_{\frac{1}{2}(r+2+n-n_1-\lambda)}(r+2+n-n_1, \frac{1}{2}(n-n_1+2))$ defined in Section 3 has λ vertices, $u_1, u_2, \ldots, u_\lambda$ say, of degree $r - 1$ and all other vertices have degree r. Further, the graph $H_2 = G_{\frac{1}{2}(r+1-2\lfloor\frac{1}{2}b\rfloor)}(r+2, 1)$ has $2\lfloor \frac{1}{2}b\rfloor + 1$ vertices, $v_1, v_2, \ldots, v_{2\lfloor\frac{1}{2}b\rfloor+1}$ say, of degree $r - 1$ and all other vertices of degree r. Take λ vertices, $u'_1, u'_2, \ldots, u'_\lambda$ say, of G' that are adjacent to z_1 and have degree one in G' (note that there are at least $r - 2 \ge \lambda$ vertices of G' adjacent to z_1 that have degree 1 in G') and the $2\lfloor \frac{1}{2}b \rfloor + 1$ neighbours, $v'_1, v'_2, \ldots, v'_{2\lfloor\frac{1}{2}b\rfloor+1}$ say, of y_{a+1}. G'' is formed from $G' - \{u'_1, u'_2, \ldots, u'_\lambda, y_{a+1}\}$ by adding the graphs H_1 and H_2 along with the edges :

$u_i z_1$ for $1 \le i \le \lambda$; $v_i v_i'$ for $1 \le i \le 2 \lfloor \frac{1}{2} b \rfloor + 1$.

Observe that

$$\nu(G'') = \nu(G') + r + 2 + n - n_1 - \lambda + r + 1. \qquad (4.4)$$

Of these, there are f vertices of degree one, where

$$\begin{aligned} f &= ar(r - 2) + (r - 2) + b(r-1) + 2 \lfloor \tfrac{1}{2} b \rfloor - b + 1 - \lambda \\ &= ar^2 + (b - 2a)r + q + 1. \qquad (4.5) \end{aligned}$$

Identify these vertices as $w_1, w_2, \ldots, w_f$.

We form the graph G_4^* from G'' as follows. Take f copies, $G_1, G_2, \ldots, G_f$ say, of the graph $G_{\frac{1}{2}(r+1)}(r + 2, 1)$. Observe that each G_i, $1 \le i \le f$, has exactly one vertex, $\bar{w}_i$ say, of degree $r - 1$ and all other vertices of degree r. Let the neighbour of w_i, in G'', be w_i'. The graph G_4^* is now formed from $G'' - \{w_1, w_2, \ldots, w_f\}$ by adding the graphs $G_1, G_2, \ldots, G_f$ and the edges $\bar{w}_i w_i'$ for $1 \le i \le f$. Observe that

$$\nu(G_4^*) = \nu(G'') + (r + 1)f .$$

Now (4.3) and (4.4) together with a little algebra yield $\nu(G_4^*) = n \ge n_1$. It is immediate from our construction that G_4^* is r-regular and connected.

We now show that $\mathrm{def}(G_4^*) = d$, as required. Letting $S = \{v : v$ is adjacent to a vertex of degree 1 in $G'\}$, we have

$$|S| = a(r - 1) + b + 1$$

and

$$o(G_4^* - S) = f + a + 2$$
$$= ar^2 + (b - 2a)r + a + q + 3$$
$$\text{(using (4.5))}.$$

Further, every odd component of $G_4^* - S$ which is not a single vertex, has a Hamilton cycle.

Hence,

$$\begin{aligned} \operatorname{def}(G_4^*) &= o(G_4^* - S) - |S| \\ &= ar^2 + (b - 2a)r + a + q + 3 - a(r - 1) - b - 1 \\ &= (a(r - 2) + b)(r - 1) + q + 2 \\ &= d, \text{ as required.} \end{aligned}$$

The only case that remains is that when $b \geq r - q - 2$. Take $T_1, T_2, \ldots, T_a$ as above and let T_{a+1} be the graph $T_r(b + 2)$. Label the vertices y_i and z_i in T_i, $1 \leq i \leq a + 1$, as before. Now add the edge $y_i z_{i+1}$ for every $1 \leq i \leq a$ and the edge $z_1 y_{a+1}$. Call the resulting graph $\hat{G}'$. The graph $\hat{H} = G_{\frac{1}{2}(q+4)}(r + 2, 1)$ has $r - q - 2$ vertices, $v_1, v_2, \ldots, v_{r-q-2}$ say, of degree $r - 1$ and all other vertices of degree r. Let $v'_1, v'_2, \ldots, v'_{r-q-2}$ be $(r - q - 2)$ vertices of $\hat{G}'$ that are adjacent to z_1 and have degree one in $\hat{G}'$. Note that $d_{\hat{G}'}(z_1) = r$. We now form the graph $\hat{G}''$ from $\hat{G}'$.

The graph $\hat{G}'$ contains at least $(r - b - 3)$ vertices, $u_1, u_2, \ldots, u_{r-b-3}$ say, in T_1 of degree 1 having distinct neighbours $u'_1, u'_2, \ldots, u'_{r-b-3}$ none of which are z_1. We form $\hat{G}''$ from $\hat{G}' - \{u_1, u_2, \ldots, u_{r-b-3}, v'_1, v'_2, \ldots, v'_{r-q-2}\}$ by adding the graph $\hat{H}$ together with the edges : $v_i z_1$, $1 \le i \le r - q - 2$; $u_i' \, y_{a+1}$, $1 \le i \le r - b - 3$. Observe that

$$\nu(\hat{G}'') = \nu(\hat{G}') - (r - q - 2) - (r - b - 3) + r + 2$$

$$= ar(r - 1) + (b + 2)r - r + q + b + 7$$

$$= ar^2 + (b - a + 1)r + b + q + 7.$$

Further, each vertex of $\hat{G}''$ has degree 1 or r. The number of vertices f' of degree 1 is

$$f' = ar(r - 2) + (r - 2) + (b + 1)(r - 1) - (r - q - 2) - (r - b - 3)$$

$$= ar^2 + (b - 2a)r + q + 2.$$

We form the graph G_5^* from $\hat{G}''$ in the same way as we formed the graph G_4^* from G'' except that here we take G_i to be the graph $G_{\frac{1}{2}(r+1)}(r + 2,1)$ for $1 \le i \le f' - 1$ and $G_{f'}$ to be the graph $G_{\frac{1}{2}(r+1+n-n_1)}(r + 2 + n - n_1, \frac{1}{2}(n - n_1 + 2))$. Following the same argument we can establish that $G_5^* \in \mathcal{G}(n,r,1)$ and $\mathrm{def}(G_5^*) = d$ (here taking S as above, we have $|S| = a(r-1) + b+2$ and $o(G_5^* - S) = f' + a + 2$). This completes the proof of the theorem. □

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