AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **39** (2007), Pages 259–272

Order properties of the Motzkin and Schröder families*

Antonio Bernini   Luca Ferrari

Dipartimento di Sistemi e Informatica
viale Morgagni 65, 50134 Firenze
Italy

`bernini@dsi.unifi.it`   `ferrari@math.unifi.it`

Abstract

Motivated by Barcucci et al., (*Order* 22 (2005), 311–328), we consider a natural distributive lattice structure on both Motzkin and Schröder paths of a given length and transfer it to suitable subsets of (coloured) noncrossing partitions and (coloured) generalized pattern avoiding permutations. Among our results there are some new order structures on partitions as well as the fact that the sets $S_n(31-2, k-(k-1)(k-2)\cdots 21)$ with the induced strong Bruhat order are distributive lattices, for any $k \geq 2$.

1 Introduction

The starting point of the present paper is the main result of [3] (also summarized in Section 2), i.e. the fact that the distributive lattice of Dyck paths of length $2n$ induced by the natural partial order considered in [11] can be alternatively described using noncrossing partitions and 312-avoiding permutations of an n-set. In the former case, a new lattice structure is obtained and then studied (a particular emphasis is put on the fact that the resulting lattice is distributive, whereas the lattice induced by the refinement order is not); instead, in the latter case, it comes out that the partial order on $S_n(312)$ coincides with the strong Bruhat order, so that $S_n(312)$ is a distributive sublattice of S_n (which does not possess a lattice structure as a whole). One of the main tool to get the above described results is a well-known bijection between Dyck paths and noncrossing partitions, which is reported, for example, in [15] and whose origins are probably to be traced back to some mathematical folklore. This bijection, which will be useful also in the present paper, can be described as follows: given a Dyck path, number its up steps in increasing order from left to right, then label each of its down steps with the number of the up step it is matched with and, finally, consider the partition whose blocks are given by the labels of consecutive

* This work was partially supported by MIUR project: *Automi e linguaggi formali: aspetti matematici e applicativi.*

sequences of down steps. Such a partition is easily seen to be noncrossing, and the above described construction can be immediately inverted, so that it is actually a bijection. Such a bijection is the essential link which allows us to transfer our order structure from Dyck paths to noncrossing partitions. On the other hand, in order to do the same thing from noncrossing partitions to 312-avoiding permutations, we make use of the following notation: each noncrossing partition $\pi = B_1|B_2\cdots|B_k$ is expressed by listing its blocks B_i in increasing order of their maxima, whereas the elements inside each block are listed in decreasing order. It is clear that every (noncrossing) partition can be uniquely written in this way, which will be called here the *standard notation* for (noncrossing) partitions. Now it is not difficult to observe that removing the vertical bars from a noncrossing partition $\pi = B_1|\cdots|B_k$ gives rise to a permutation which turns out to be 312-avoiding. This "bar-removing" bijection is precisely what we needed to transfer the order structure from noncrossing partitions to 312-avoiding permutations.

The main goal of the present paper is to find analogous results starting from the distributive lattices of Motzkin and Schröder paths. More precisely, we aim at finding suitable modifications of the above described bijections which allow us to obtain distributive lattice structures on some kind of noncrossing partitions and pattern avoiding permutations having some combinatorial relevance. In the Motzkin case, our results are reported in Section 3 and are strikingly similar to those of the Catalan family. Our basic tool is a bijection described in [10] which codifies Motzkin paths by means of a special kind of Dyck paths. Moreover, our main result is the fact that $S_n(31-2), k-(k-1)(k-2)\ldots 21$ is a distributive lattice (endowed with the strong Bruhat order) for every $k \geq 2$; to the best of our knowledge, this is a new result of order-theoretic flavor concerning classes of pattern avoiding permutations.

In the Schröder case, things are not so neat, and we need to introduce coloured objects to achieve some satisfactory results (which are described in Section 4). The last section is devoted to the presentation of some open problems (many of which are also scattered throughout the paper), as well as of some possible directions of future research.

At the end of this introduction, we give explanations concerning some notations we will use in the paper.

The word "bar" is used to denote both vertical and horizontal bars, so that its meaning depends on the context. When we speak of "bar-removing bijection", we mean the function which removes the *vertical* bars in the standard notation of a partition to obtain a permutation, whereas the terms "barring" and "unbarring" indicate the operation of putting and removing a horizontal bar over an element of a permutation. However, the choice between vertical and horizontal should be clear from the context.

The sequences of Schröder and Narayana numbers will be denoted by $(R_n)_{n\in\mathbf{N}}$ and $(N(n,k))_{n,k\in\mathbf{N}}$, respectively.

The up, horizontal and down steps in Dyck, Motzkin and Schröder paths will be denoted u, h, d, respectively. A Dyck path of length $2n$ is a lattice path consisting of u and d steps, from $(0,0)$ to $(2n,0)$ which never pass below the x-axis. A Motzkin path of length n is a lattice path which uses u, d and h steps, from $(0,0)$ to $(n,0)$, never passing below the x-axis. A Schröder path of length $2n$ is a lattice path starting from $(0,0)$ to $(2n,0)$ consisting of u, d and hh (double horizontal) steps, never going below the x-axis.

$[n]=\{1,2,\ldots,n\}$ is the set of the first n positive integers.

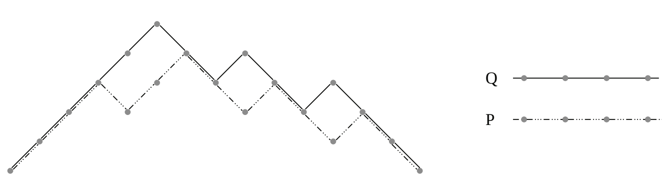
A set partition is said to be *noncrossing* when, given four elements, $1\leq a<b<c<d\leq n$, such that a,c are in the same block and b,d are in the same block, then the two blocks coincide. The set of all noncrossing partitions of an n-set will be denoted $NC(n)$. For a given noncrossing partition $\pi\in NC(n)$, its *max-vector* [3] is the vector $\max(\pi)=(\mu_1,\ldots,\mu_n)$ such that μ_i is the maximum of the first i elements of π (written in standard notation).

The symmetric group on n elements will be denoted by S_n, whereas the set of coloured permutations on n elements will be denoted by $\overline{S}_n$. If $\pi=\pi_1\cdots\pi_n$ and $\sigma=\sigma_1\cdots\sigma_k$ are two permutations of S_n and S_k, respectively, then π *avoids* the pattern σ if there are no indexes $i_1<i_2<\cdots<i_k$ such that $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ is in the same relative order as $\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}$. The subset of σ-avoiding permutations of S_n is denoted $S_n(\sigma)$. A *generalized* pattern τ [1] is a pattern with some dashes inserted, so that two consecutive elements of τ are adjacent if there is no dash between them (e.g. $261-4-35$ is a generalized pattern of length 6). A permutation π *contains* the generalized pattern τ if the elements of π corresponding to the elements of τ are in the same relative order and any pair of elements of π corresponding to two adjacent elements of τ must be adjacent in π as well. A permutation π *avoids* a generalized pattern τ if π does not contain τ. For instance, the permutation 35241 contains 321 but not $32-1$. More generally, if T is a set of patterns, $S_n(T)$ denotes the set of permutations of $[n]$ avoiding each pattern of T. Some references on generalized pattern avoidance, also useful for this paper, are [5, 7].

2 Dyck paths

Let τ be a permutation of length 3. In this section we describe the structure of $S_n(\tau)$ when such a set is a lattice with respect to the strong Bruhat order. This problem has been solved in [3] for $\tau=312$, however it is immediate to extend such a result to any other permutation of length 3. We start by recalling the poset structures we need for our purposes.

The set $\mathcal{D}_n$ of Dyck paths of length $2n$ can be endowed with a natural partial order by declaring $P\leq Q$ when P lies weakly below Q, that is, for any $k\leq 2n$, the ordinate of P is less than or equal to the ordinate of Q at k (see fig. below).



It is possible to show that such a poset is indeed a distributive lattice, whose rank function is essentially given by the area, i.e.

$$r(P) = \frac{A(P) - n}{2} ,$$

where $A(P)$ is the area of the region included between P and the x-axis. We will refer to this lattice as the *Dyck lattice of order* n. In $NC(n)$ define a partial order by setting $\pi \leq \rho$ when $\max(\pi) \leq \max(\rho)$ in the coordinatewise order. It turns out that such a poset is a distributive lattice, called the *Bruhat noncrossing partition lattice of order* n [3].

The main result of [3] is summarized in the following theorem.

Theorem 2.1 *For any* $n \in \mathbf{N}$*, the following order structures are isomorphic:*

1. *the Dyck lattice* $\mathcal{D}_n$*;*
2. *the Bruhat noncrossing partition lattice* $NC(n)$*;*
3. $S_n(312)$ *as a subposet of* S_n *endowed with the strong Bruhat order.*

In particular, $S_n(312)$ *is a distributive lattice with respect to the strong Bruhat order.*

The above results on 312-avoiding permutations give some useful information on the order structure of $S_n(\tau)$, for any $\tau \in S_3$. To this aim, a crucial step is represented by the following general lemma, whose proof can be found in [12].

Lemma 2.1 *Let* $r, c, i : S_n \longrightarrow S_n$ *be the reverse, complement and inverse functions on permutations. Then, with respect to the strong Bruhat order,* i *is an isomorphism, whereas* r, c *are antiisomorphisms.*

As a consequence of this lemma, given $S_n(\tau)$, for some $\tau \in S_k$, endowed with the strong Bruhat order, if we consider the reverse of each element, we get $S_n(\rho)$, with $\rho = r(\tau)$, endowed with the dual order. Analogous considerations can be done for the complement and the inverse functions, whence the following proposition holds.

Proposition 2.1 *For every* $n \in \mathbf{N}$*,* $S_n(312)$ *is order-isomorphic to* $S_n(231)$ *and order-antiisomorphic to* $S_n(132)$ *and* $S_n(213)$*. Therefore all the above posets are distributive lattices. The posets* $S_n(123)$ *and* $S_n(321)$ *are not even lattices, since they do not have minimum and maximum, respectively.*

Clearly, thanks to Lemma 2.1, the posets $S_n(123)$ and $S_n(321)$ are antiisomorphic.

Open problem 1. Describe the poset $S_n(123)$.

Open problem 2. Fixed $k \in \mathbf{N}$, $k > 3$, for which $\tau \in S_k$ is $S_k(\tau)$ a (distributive) lattice under the strong Bruhat order? In case of a positive answer, is it possible to give some alternative combinatorial descriptions of such lattices? We point out that this problem has been solved in [8] for the weak Bruhat order.

3 Motzkin paths

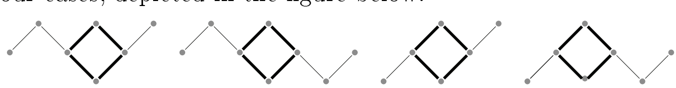
We start by recalling a bijection introduced by Elizalde and Mansour [10] between the set $\mathcal{M}_n$ of Motzkin paths of length n and the set $\mathcal{D}_n^{(3)}$ of Dyck paths of length $2n$ without three consecutive down steps. Every Dyck path $P \in \mathcal{D}_n^{(3)}$ can be uniquely decomposed into factors of the following three types: u, ud, udd. Define a Motzkin path $f(P)$ by translating the above factors according to the following:

$$\begin{aligned} u &\to u \\ ud &\to h \\ udd &\to d\ . \end{aligned}$$

The path $f(P)$ has length n and it is possible to show that the function f is a bijection. Our next proposition shows that f has some more structural properties.

Proposition 3.1 *The function $f : \mathcal{D}_n^{(3)} \longrightarrow \mathcal{M}_n$ is an order-isomorphism.*

Proof. Let $P, Q \in \mathcal{D}_n^{(3)}$ such that $P \preceq Q$. This means that Q is obtained from P by changing a valley into a peak. Call *box* the two steps on which P and Q differ. However, we notice that, unlike it happens for the whole $\mathcal{D}_n$, performing the above operation on paths belonging to $\mathcal{D}_n^{(3)}$ does not necessarily produce a path of the same kind: this occurs precisely when a valley is followed by two or more down steps. When we apply f to P and Q, several different things can happen, according to the type of the steps next to the box. Since the down step of a valley cannot be preceded by two or more down steps, there are only two possibilities for P and Q, namely the box is preceded either by u or by ud. Analogously, the down step of a peak cannot be followed by two or more down steps, whence also in this case we have two different cases, i.e. the box is followed either by u or by du. Therefore we have a total of four cases, depicted in the figure below:



Now apply f to each of the above, to obtain respectively the following four cases on the corresponding Motzkin paths:

As it is clear, each situation yields two Motzkin paths $f(P), f(Q)$ such that $f(P) \preceq f(Q)$, as desired.

Conversely, an analogous argument shows that, if P, Q are arbitrary Motzkin paths for which $P \preceq Q$, then $f^{-1}(P) \preceq f^{-1}(Q)$, so the proof is complete. ■

The bijection between $\mathcal{D}_n$ and $NC(n)$ recalled in the introduction can be restricted to $\mathcal{D}_n^{(3)}$; the corresponding subset of $NC(n)$ is easily seen to consist of noncrossing partitions whose blocks have cardinality at most 2. Call such partitions *Motzkin noncrossing partitions*. Thanks to the last proposition we can establish the following result.

Theorem 3.1 *The set $MNC(n)$ of Motzkin noncrossing n-partitions can be endowed with a distributive lattice structure, which is isomorphic to the lattice of Motzkin paths of length n. More precisely, given $\pi, \rho \in MCN(n)$, we have that $\pi \preceq \rho$ if ρ is obtained from π by moving the minimum of some block B of π into the block $\tilde{B}$ containing the element $\beta = \max B + 1$ if $\beta = \min \tilde{B}$. In this case, either:*

1. *keep β inside $\tilde{B}$, if $|\tilde{B}| = 1$, or*
2. *add a new block $\overline{B} = \{\beta\}$, if $|\tilde{B}| = 2$.*

Proof. The first part of the theorem is an easy consequence of Proposition 3.1. As far as the covering relation is concerned, the above result for Motzkin noncrossing partitions immediately derives from the analogous one given in [3] for general noncrossing partitions. The only thing to take into account is that a Motzkin noncrossing partitions has blocks of cardinality at most 2, and so, if $|\tilde{B}| = 2$ and $\beta = \max \tilde{B}$, the above mentioned rule cannot be applied since the resulting partition would not belong to $MNC(n)$. ■

Example. Given the partition $2|31|65|74|8 \in MNC(n)$, there are two partitions covering it, which are $2|3|4|65|71|8$ (1 has been moved into a block with two elements) and $2|31|65|7|84$ (4 has been moved into a block with one element). Note that the partitions obtained by moving 2 or 5 are not listed above, since the elements 3 and 7 are not the minima of their blocks.

Remark. Another consequence of Proposition 3.1 is that the rank of a partition of $MNC(n)$ corresponds to the area of the associated Motzkin path, this meaning that two partitions of $MNC(n)$ have the same rank if and only if the associated Motzkin paths have the same area. Also in this case, a formula expressing the area using parameters on partitions (such as cardinality of a block and maximum of a block) can be found as in [3].

Similarly to [3], it is possible to transfer the distributive lattice structure of Motzkin noncrossing partitions to a suitable subset of pattern avoiding permutations via a bar-removing bijection. In [7] it is shown (bijectively) that $S_n(3-21, 31-2)$ is counted by Motzkin numbers. Here we give an essentially equivalent bijection between $MNC(n)$ (and so Motzkin paths) and $S_n(3-21, 31-2)$.

Proposition 3.2 *Removing the vertical bars in Motzkin noncrossing partitions defines a bijection between $MNC(n)$ and the set $S_n(3-21, 31-2)$ of pattern avoiding permutations of $[n]$, for any $n \in \mathbf{N}$.*

Proof. Let π be a permutation of $S_n(3-21, 31-2)$. It is straightforward to see that the associated partition is a noncrossing partition, since π avoids the pattern $31-2$ ([3]). If π contains a block with three or more elements, then the associated permutation would show the forbidden pattern $3-21$, against the hypothesis. So π is a Motzkin noncrossing partition. On the other hand, if $\pi \in MNC(n)$, then the associated permutation avoids the pattern $31-2$. Moreover, if π contains a pattern

$3-21$ in the entries π_i, π_k, π_{k+1}, then necessarily $\pi_{k-1} < \pi_k$, otherwise π would have a block with three elements. So the entries π_i, π_{k-1} and π_k are a pattern $3-12$ which induces the presence in π of the forbidden pattern $31-2$ (see [5]). We conclude that $\pi \in S_n(3-21, 31-2)$. ■

To prove that the above bar-removing bijection between $MNC(n)$ and $S_n(3-21, 31-2)$ is also an order-isomorphism, we just notice that such a bijection is obtained by simply restricting the bar-removing isomorphism between $NC(n)$ and $S_n(312)$ considered in [3]. Therefore the following theorem holds.

Theorem 3.2 *Let $(S_n(3-21, 31-2); \leq)$ be the poset obtained by transferring the distributive lattice structure defined in Theorem 3.1 along the bar-removing bijection. This is precisely the subposet induced on $S_n(3-21, 31-2)$ by the strong Bruhat order of the symmetric group S_n. Therefore $S_n(3-21, 31-2)$ is a distributive sublattice of S_n endowed with the strong Bruhat order.*

An immediate consequence of the above theorem is stated in the following, remarkable corollary.

Corollary 3.1 *For any $n \in \mathbf{N}$, the Motzkin lattice $\mathcal{M}_n$ is isomorphic to the lattice $S_n(3-21, 31-2)$ with the strong Bruhat order.*

We conclude this section by generalizing the bijection of Elizalde and Mansour between $\mathcal{D}_n^{(3)}$ and $\mathcal{M}_n$. Denote by $\mathcal{D}_n^{(k)}$ the set of Dyck paths of length $2n$ having at most $k-1$ consecutive down steps and by $\mathcal{C}_n^{[-k+2,1]}$ the set of paths of length n starting from the origin, ending on the x-axis, never falling below the x-axis and using steps of the kind $(1, j)$, for $j \in \{-k+2, -k+1, \ldots, -1, 0, 1\}$ (this notation is borrowed from [11]). Each path in $\mathcal{D}_n^{(k)}$ can be uniquely factorized using factors of type ud^j, for $0 \leq j \leq k-1$. Therefore we can define a bijection analogous to f by mapping the factor ud^{j+1} into the step $(1, -j)$, thus obtaining a path in $\mathcal{C}_n^{[-k+2,1]}$. Call f_k such a bijection (with this notation, clearly $f = f_3$). Using an argument similar to Proposition 3.1, it is possible to show that f_k is an order-isomorphism, for any $k \geq 2$. Moreover, from a general result proved in [11], each set of paths $\mathcal{C}_n^{[-k+2,1]}$ is a distributive lattice with the usual order. As a consequence, our previous results on the order structure of paths, partitions and permutations counted by Motzkin numbers can be extended as follows:

Proposition 3.3 *For any $k \geq 2$, the following distributive lattice structures are isomorphic:*

1. *the set $\mathcal{C}_n^{[-k+2,1]}$ with the usual order on paths;*
2. *the set $kNC(n)$ of noncrossing partitions of an n-set having blocks of cardinality at most $k-1$, endowed with the order inherited by the Bruhat order of $NC(n)$;*
3. *the set of generalized pattern avoiding permutations $S_n(31-2, k-(k-1)(k-2)\cdots 21)$ endowed with the strong Bruhat order.*

When k tends to infinity, we get a bijection f_∞ between Dyck paths of length $2n$ and paths of length n using the unique positive step $(1,1)$ and any kind of negative step $(1,-j)$. This latter class of paths will be called here the class of *Łukasiewicz* paths. Observe that Łukasiewicz paths are usually defined dually (in [2] they correspond to our paths read from right to left), anyway both enumerative results and order properties are not affected by this slight change of notation. The above proposition translates into the fact that the distributive lattices of Łukasiewicz paths are isomorphic to those of Dyck paths, as well as to the Bruhat noncrossing partition lattices and 312-avoiding permutations with the strong Bruhat order.

From an enumerative point of view, we observe that for $k=2$ we get the sequence $1,1,1,\ldots$, for $k=3$ we get the Motzkin numbers and for $k=\infty$ we get the Catalan numbers. Therefore the sequences obtained for a generic k interpolate between the Motzkin and the Catalan numbers. A strikingly similar result has been found in [4], where the authors use classes of pattern avoiding permutations different from ours: it would be interesting to relate the two approaches.

4 Schröder paths

In this section we try to find analogous results starting from the lattices of Schröder paths.

A first attempt in this direction consists of reading Schröder paths as special Motzkin paths, namely a Schröder path can be regarded as a Motzkin path in which any set of consecutive horizontal steps has even cardinality. From this point of view, we can consider a suitable restriction of the bijection of Proposition 3.1. As a consequence of this approach, we obtain that Schröder lattices are isomorphic to the lattices of Motzkin noncrossing partitions where any bunch of singletons made of consecutive integers has even cardinality. Unfortunately, we have not been able to determine the set of pattern avoiding permutations associated with the above subset of Motzkin noncrossing partitions via the bar-removing bijection.

Open problem 3. Find a set of patterns T such that $S_n(T)$ corresponds to the set of Schröder paths of length n via a suitable restriction of the bijection between Dyck paths and 312-avoiding permutations recalled in the introduction.

A totally different approach consists of interpreting Schröder paths as Dyck paths with bicoloured peaks. Denote by $\overline{\mathcal{D}}_n$ the set of Dyck paths of length $2n$ whose peaks can be coloured either white or black. There is an obvious bijection between $\overline{\mathcal{D}}_n$ and the set $\mathcal{S}_n$ of Schröder paths of length $2n$ (just map white peaks into simple peaks, black peaks into a pair of consecutive horizontal steps, and leave the remaining steps unchanged; from this bijection, which has been considered in [16], the identity $R_n=\sum_{k=1}^{n}2^kN(n,k)$ immediately follows). Thanks to this simple observation, it is not difficult to find a suitable set of coloured noncrossing partitions in bijection with Schröder paths.

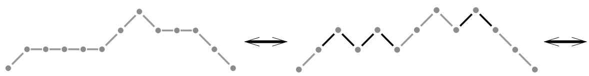


Figure 1: The bijections connecting $\mathcal{S}_n$, $\overline{\mathcal{D}}_n$ and $\overline{NC}(n)$ for $n = 6$

Proposition 4.1 *Denote by $\overline{NC}(n)$ the set of noncrossing partitions of an n-set such that the maximum of the blocks can be either coloured white or black. Then there is a bijection between $\mathcal{S}_n$ and $\overline{NC}(n)$.*

Proof. Given a Schröder path, consider the associated bicoloured Dyck paths and take the noncrossing partition determined by the classical bijection, taking care of colouring each element of the partition with the same colour of the corresponding down step. ■

An example illustrating the bijections connecting $\mathcal{S}_n$, $\overline{\mathcal{D}}_n$ and $\overline{NC}(n)$ for $n = 6$ is given in figure 1.

The elements of $\overline{NC}(n)$ will be called *Schröder noncrossing partitions*. From now on, in a Schröder noncrossing partition we will denote black elements using a horizontal bar, and we will simply call them *coloured* elements.

Similarly to Dyck paths, Schröder paths can be endowed with a natural partial order structure, and the obtained poset is again a distributive lattice [11]. Here we only recall the covering relation: if U is a Schröder path, then a path V covering it ($U \preceq V$) is obtained either by:

- changing a pair du in U into a pair hh in V, or
- changing a pair hh in U into a pair ud in V. Note that, in this second case, the replacement is possible only if the hh in U is followed by an even number of h steps, otherwise the path V would not be a Schröder path.

The natural order on Schröder paths of length $2n$ can be transferred to $\overline{NC}(n)$ by means of the bijection of Proposition 4.1. We have the following theorem:

Theorem 4.1 *(Characterization of coverings) Given two coloured noncrossing partitions $\pi, \rho \in \overline{NC}(n)$, we have $\pi \preceq \rho$ if and only if ρ is obtained from π by either*

1. *unbarring a coloured element of π, or*
2. *moving the minimum of some block B of π into the block $\tilde{B}$ containing the element $\beta = \max B + 1$ only when β is not coloured; moreover:*
   (a) if $\beta = \max \tilde{B}$, then keep β inside $\tilde{B}$ and bar it;
   (b) if $\beta \neq \max \tilde{B}$, then add the coloured block $\overline{B} = \{\overline{\beta}\}$.

Proof (sketch). We can proceed as we did in Theorem 4.1 of [3] for the covering relation on $NC(n)$, so we omit a detailed proof. However, it is worth noticing that the bijection between $\mathcal{S}_n$ and $\overline{\mathcal{D}}_n$ implies that if $P, Q \in \overline{\mathcal{D}}_n$ are such that $P \preceq Q$, then Q is obtained from P by either changing a black peak into a white peak or replacing a valley with a black peak (observe that this last operation on valleys can be only performed when the steps are both white). ■

Example. Given the partition $\bar{5}43|62|871|\bar{9} \in \overline{NC}(n)$, there are precisely four partitions covering it, which are $543|62|871|\bar{9}$ ($\bar{5}$ has been unbarred), $\bar{5}4|\bar{6}32|871|\bar{9}$ (3 has been moved and 6 was the maximum of its block), $\bar{5}43|6|\bar{7}|821|\bar{9}$ (2 has been moved and 7 was not the maximum of its block) and $\bar{5}43|62|871|9$ ($\bar{9}$ has been unbarred). Note that the partition obtained by moving 1 into the block containing $\bar{9}$ (i. e. the maximum of its block plus 1) is not listed above, since $\bar{9}$ is coloured.

The area $A(P)$ of a Schröder path P can be derived from the Dyck path P' obtained by replacing each double horizontal step with a coloured peak. If C is the number of coloured peaks of P', then it is easily seen that $A(P) = A(P') - C$. Now, the rank of the associated Schröder noncrossing partition π can be expressed by recalling the formula in [3] for the rank of a noncrossing partition. Denoting by $\pi' \in NC$ the (noncoloured) noncrossing partition associated with π, we have

$$A(\pi') = \sum_{i=1}^{k} \left(|B_i| \left(2b_i - 2\sum_{j=1}^{i-1} |B_j| - |B_i| \right) \right) ,$$

whence the rank of π is given by:

$$r(\pi) = A(\pi') - c(\pi) ,$$

where $c(\pi)$ is the number of coloured elements of π.

Following the lines of [3], we now look for a suitable set of coloured pattern avoiding permutations in bijection with both Schröder paths and Schröder noncrossing partitions. The study of the enumerative properties of coloured pattern avoiding permutations has been pursued by several authors, see for example [13]. The next result has been independently proved by Egge [9] using algebraic arguments; here we propose a bijective proof, as well as a presumably new order structure connecting a certain class of permutations with Schröder paths and Schröder noncrossing partitions.

Theorem 4.2 *Removing the vertical bars in Schröder noncrossing partitions defines a bijection between $\overline{NC}(n)$ and the set $\overline{S}_n(2\bar{1}, \bar{2}\bar{1}, 312, \bar{3}12)$, for any $n \in \mathbf{N}$.*

Proof. Let π be a partition of $\overline{NC}(n)$. We show that π avoids the four patterns $312, \bar{2}\bar{1}, 2\bar{1}, \bar{3}12$.

If π' denotes the associated permutation via the bar-removing bijection, then it is known [3] that π' is a 312-avoiding permutation, since π is a noncrossing partition (just recall the standard representation of partitions given in the introduction).

Suppose that π' contains $\bar{2}\bar{1}$. Since in π only the maxima of the blocks can be coloured, it means that π contains two maxima in decreasing order, which is not possible due to our standard notation.

If π' contains $2\bar{1}$ in its elements a and $\bar{b}$, with $a > \bar{b}$, then, regarded as elements of π, they belong to two different blocks and $\bar{b}$ is the maximum of its block. Then, considering $\bar{b}$ and the maximum of the block containing a, two maxima in decreasing order would appear in π, against the hypothesis.

Let us suppose that π' contains a $\bar{3}12$ pattern in the elements $\bar{a}$, b and c, with $\bar{a} > c > b$. Then, in π, b and c lie in two different blocks. Suppose that $\bar{a}$ is the maximum of the block containing b. Let d be the maximum of the block containing c. Clearly $d > a$, since maxima are in increasing order. The elements $\bar{a}, b, c, d$ constitute a crossing being $\bar{a}$ in the same block of b, d in the same block of c and $b < c < a < d$. This is not possible since $\pi \in \overline{NC}(n)$. If $\bar{a}$ is not the maximum of the block of b, the same argument of the previous point can be repeated considering the maximum g of the block containing b. So π' is also a $\bar{3}12$-pattern avoiding permutation, whence $\pi' \in \overline{S}_n(2\bar{1}, \bar{2}\bar{1}, 312, \bar{3}12)$.

Vice versa, given $\pi' \in \overline{S}_n(2\bar{1}, \bar{2}\bar{1}, 312, \bar{3}12)$, consider the partition π obtained by inserting a vertical bar before each left-to-right maximum other than the first one. In this way, the maxima of the blocks of π are precisely the left-to-right maxima of π'. Moreover, the fact that π' avoids the two patterns $\bar{2}\bar{1}, 2\bar{1}$ implies that the only elements of π which can be coloured are the maxima of its blocks. Finally, the avoidance of the two patterns $312, \bar{3}12$ forces the partition π to be both in standard notation and noncrossing. ■

Remark. The above set of coloured pattern avoiding permutations clearly coincides with $\Theta_n(\bar{2}\bar{1}, 2\bar{1})$, where Θ_n is the set of coloured permutations of length n avoiding any coloured version of the pattern 312 (and so $|\Theta_n| = 2^n C_n$).

Using the above bar-removing bijection we can now transfer the order structure of Schröder paths to the set $\overline{S}_n(2\bar{1}, \bar{2}\bar{1}, 312, \bar{3}12)$. What we obtain is clearly a distributive lattice; its covering relation is described in the next proposition, whose proof is omitted.

Proposition 4.2 *Given $\pi, \rho \in \overline{S}_n(2\bar{1}, \bar{2}\bar{1}, 312, \bar{3}12)$, it is $\pi \prec \rho$ if and only if ρ is obtained from π by either:*

1. *unbarring an element of π, or*

2. *interchanging the element a immediately preceding a left-to-right maximum of π with $\beta + 1$, where β is the left-to-right maximum before a, and colouring $\beta + 1$; this last operation can be performed exclusively when a and $\beta + 1$ are both unbarred.*

Example. The reader can reconsider the example presented at the end of Theorem 4.1: just delete the vertical bars and read the covering rules according to the last proposition.

Remark. We recall that it is possible to define a notion of Bruhat order on coloured permutations, as it is reported, for instance, in [6]. Unfortunately, the restriction of this Bruhat order to $\overline{S}_n(2\bar{1}, \bar{2}\bar{1}, 312, \bar{3}12)$ does not match our posets.

Open problem 4. Concerning the above remark, the Bruhat order on $\overline{S}_n$ is defined in [6] as the Bruhat order on the set of permutations with ground set $\{1, \dots n, \overline{1}, \dots \overline{n}\}$, where the elements are linearly ordered as they are listed above (i.e., $1 < \cdots < n < \overline{1} < \cdots < \overline{n}$). Is it possible to find a suitable linear order on $\{1, \dots n, \overline{1}, \dots \overline{n}\}$ such that the resulting Bruhat order on $\overline{S}_n$ coincides with our partial order?

Let $\pi \in \overline{S}$; we denote by $inv(\pi)$ the set of the inversions of π and $nb(\pi)$ the number of the unbarred entries of π. Then the following proposition holds:

Proposition 4.3 *The rank of an element* $\pi \in \overline{S}_n(2\bar{1}, \bar{2}\bar{1}, 312, \bar{3}12)$ *is given by*

$$r(\pi) = 2|inv(\pi)| + nb(\pi) \ . \tag{1}$$

Proof. We proceed by induction.

If $r(\pi) = 0$, then $\pi = \bar{1}\bar{2} \dots \bar{n}$ and $inv(\pi) = \emptyset$, $nb(\pi) = 0$, whence formula (1) is true.

Suppose that $r(\pi) = 2|inv(\pi)| + nb(\pi)$ for $\pi \in \overline{S}_n(2\bar{1}, \bar{2}\bar{1}, 312, \bar{3}12)$ such that $r(\pi) = s$. Let ρ be a permutation of $\overline{S}_n(2\bar{1}, \bar{2}\bar{1}, 312, \bar{3}12)$ such that $\pi \preceq \rho$, then $r(\rho) = s + 1$. We have to show that $r(\rho) = 2|inv(\rho)| + nb(\rho)$. There are two possibilities for ρ: ρ is obtained from π either by unbarring an element or by interchanging the elements of a pattern 12 of π obeying condition 2 of Proposition 4.2 to obtain a pattern $\bar{2}1$ in ρ (in this way ρ has precisely one more inversion than π). In the first case $inv(\rho) = inv(\pi)$ and $nb(\rho) = nb(\pi) + 1$. Then,

$$r(\rho) = r(\pi) + 1 = 2|inv(\pi)| + nb(\pi) + 1 = 2|inv(\rho)| + nb(\rho) \ .$$

In the second case $|inv(\rho)| = |inv(\pi)| + 1$ and $nb(\rho) = nb(\pi) - 1$. Then,

$$r(\rho) = 2|inv(\pi)| + nb(\pi) + 1 = 2(|inv(\rho)| - 1) + nb(\rho) + 1 + 1 = 2|inv(\rho)| + nb(\rho) \ .$$

In both cases, formula (1) holds. ■

5 Hints for further work

In this last section we propose some ideas to get a better insight into the properties of the above considered order structures.

Given a Dyck path P of length $2n$, it is very natural to consider the Dyck path $m(P)$ obtained by reading P from right to left. So, for example, if $P = uuuuudduddudd$, then $m(P) = uuuduuduudddd$. The function m maps $\mathcal{D}_n$ into itself, and it is clearly an involution which preserves the area, therefore it is a rank-preserving involution. More precisely, m is an order-isomorphism of $\mathcal{D}_n$. Therefore, if we transfer m to $NC(n)$ and $S_n(312)$, we obtain an order-isomorphism (still to be denoted m) of both the Bruhat noncrossing partition lattice of order n and the set of 312-avoiding permutations of length n with the strong Bruhat order. The next proposition allows to determine $m(\pi)$ for any $\pi \in NC(n)$. The translation of this result on $S_n(312)$ is straightforward.

Proposition 5.1 *Let* $\pi = B_1|B_2|\cdots|B_k \in NC(n)$. *Then* $m(\pi) = C_1|C_2|\cdots|C_k \in NC(n)$ *where* $|C_i| = \max B_{k-i+1} - \max B_{k-i}$ *and* $\max C_i = \sum_{j=k-i+1}^{k} |B_j|$.

Proof. First of all we observe that a noncrossing partition is uniquely determined by the cardinalities and the maxima of its blocks.

Let P be the Dyck path associated with π. By definition, the partition $m(\pi)$ is obtained by numbering the down steps of P in decreasing order, then labelling each of its up steps with the number of the down step it is matched with and taking as blocks the sets of labels of consecutive sequences of up steps. Now suppose that $m(\pi) = C_1|C_2|\cdots|C_k$ is written in standard notation, as usual. Since the difference between the maxima of two consecutive blocks B and B' of π represents the number of consecutive up steps of P between the two sequences of down steps corresponding to B and B', it is clear that $|C_i| = \max B_{k-i+1} - \max B_{k-i}$. Moreover, the maximum c of a block of $m(\pi)$ coincides with the number of down steps of P following the up step corresponding to c, and so $\max C_i = \sum_{j=k-i+1}^{k} |B_j|$. ■

It is clear that an analogous involution can be defined also for Motzkin and Schröder paths. As far as Motzkin paths are concerned, there are two possible approaches. First, given a Motzkin path $P \in \mathcal{M}_n$, one can read it from right to left, so obtaining another Motzkin path of $\mathcal{M}_n$. On the other hand, one can restrict m to the set $\mathcal{D}_n^{(3)}$ of Dyck paths of length $2n$ having at most two consecutive down steps. In this way, the image of m is the set ${}^{(3)}\mathcal{D}_n$ of Dyck paths without three consecutive up steps. Anyway, both in the Motzkin and Schröder case, it seems not too difficult to find results on partitions and permutations analogous to the last proposition.

A much more difficult task consists of interpreting the bar-removing bijection in an alternative way. More precisely, given a noncrossing partition π written in standard notation, we associate with it the permutation obtained by reading each block of π as a cycle. For instance, the partition 543|62|871|9 is mapped into the permutation (543)(62)(871)(9). It is evident that the permutations obtained in this way have a special cycle structure [14]; it would be interesting to see if such a structure can be expressed in terms of (possibly generalized) pattern avoidance. Moreover, transferring to this set of permutations the order structure of Dyck paths leads to a new partial order on permutations, whose properties are probably worth being investigated.

We point out that the above map from noncrossing partitions to permutations written in cycle form has already been considered in [14], where the author describes the partial order obtained in S_n by transferring the *refinement order* of NC_n.

References

[1] E. Babson and E. Steingrímsson, Generalized permutation patterns and a classification of the Mahonian statistics, *Sém. Lothar. Combin.* 44 (2000), Art. B44b, 18 pp.

[2] C. Banderier and P. Flajolet, Basic analytic combinatorics of directed lattice paths, *Theoret. Comput. Sci.* 281 (2002), 37–80.

[3] E. Barcucci, A. Bernini, L. Ferrari and M. Poneti, A distributive lattice structure connecting Dyck paths, noncrossing partitions and 312-avoiding permutations, *Order* 22 (2005), 311–328.

[4] E. Barcucci, A. Del Lungo, E. Pergola and R. Pinzani, From Motzkin to Catalan permutations, *Discrete Math.* 217 (2000), 33–49.

[5] A. Bernini, L. Ferrari and R. Pinzani, Enumerating permutations avoiding three Babson-Steingrímsson patterns, *Ann. Combin.* 9 (2005), 137–162.

[6] A. Björner and F. Brenti, *Combinatorics of Coxeter groups*, Graduate Text in Mathematics 231, Springer, New York, 2005.

[7] A. Claesson, Generalized pattern avoidance, *European J. Combin.* 22 (2001), 961–971.

[8] B. Drake, The weak order on pattern-avoiding permutations, Proceedings of FPSAC 2005.

[9] E. S. Egge, Restricted colored permutations and Chebyshev polynomials, *(submitted).*

[10] S. Elizalde and T. Mansour, Restricted Motzkin permutations, Motzkin paths, continued fractions and Chebyshev polynomials, *Discrete Math.* 305 (2005), 170–189.

[11] L. Ferrari and R. Pinzani, Lattices of lattice paths, *J. Statist. Plann. Inference* 135 (2005), 77–92.

[12] F. Incitti, The Bruhat order on the involutions of the hyperoctahedral group, *European J. Combin.* 24 (2003), 825–848.

[13] T. Mansour, Pattern avoidance in coloured permutations, *Sem. Lothar. Combin.* 46 (2001), B46g, 12 pp.

[14] J. MacCammond, Noncrossing partitions in surprising locations, *Amer. Math. Monthly* 113 (2006), 598–610.

[15] R. Simion, Noncrossing partitions, *Discrete Math.* 217 (2000), 367–409.

[16] R. A. Sulanke, The Narayana distribution, *J. Statist. Plann. Inference* 101 (2002), 311–326.

(Received 25 Oct 2006)