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# Partitions of the edge set of a graph into internally disjoint paths

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**Abstract**

A collection $\mathcal{P}$ of nontrivial paths in a graph $G$ is called a *path pile* of $G$, if every edge of $G$ is on exactly one path and no two paths have a common internal vertex. The least number that can be the cardinality of a path pile of $G$ is called the *path piling number* of $G$. It can be shown that $\epsilon \leq \nu + \eta$ where $\epsilon$, $\nu$ and $\eta$ are respectively the size, the order and the path piling number of $G$. In this note we find a criterion to determine when the equality of this relation holds.

For definitions not given and notations not explained in this note, we rely on [5] and occasionally refer the reader to [4]. All graphs considered in this note are finite.

Let $G$ be a graph; its order—the number of vertices—and size—the number of edges—are denoted by $\nu(G)$ and $\epsilon(G)$ respectively. If $P = v_0v_1 \cdots v_n$ is a path in $G$, then $v_1, \ldots, v_{n-1}$ are called the *internal vertices* of $P$ and $v_0, v_n$ are the *external vertices*; sometimes $v_n$ is called the *terminal* of $P$. Let $\mathcal{P}$ be a collection of paths in $G$. A vertex of $G$ is said to be in the *interior* of $\mathcal{P}$, if it is an internal vertex of some path in $\mathcal{P}$; it is said to be on the *exterior* of $\mathcal{P}$, if it is a vertex of some path but not in the interior of $\mathcal{P}$. The set of all exterior vertices of $\mathcal{P}$ is denoted by $\operatorname{ext} \mathcal{P}$. The paths in $\mathcal{P}$ are called *internally disjoint* if no two paths have a common internal vertex.

A set $\mathcal{P}$ of nontrivial paths in a graph $G$ is called a *path pile* of $G$, if every edge is on exactly one path and the paths are internally disjoint. The least number which is the cardinality of a path pile of $G$ is called the *path piling number* of $G$. This number is denoted by $\eta(G)$; it is simply $\eta$ when no ambiguity is possible. (This convention will be adopted for other parameters also.)

A slight generalization of the above notion, called 'graphoidal cover' has been introduced in [1]. (A graphoidal cover includes apart from paths, cycles with 'ordering' of their vertices.) As a particular case of this, a notion called 'acyclic graphoidal cover' has been studied in [7]. This latter one is exactly what we have defined above.

**Remark 1.** Let $\mathcal{P}$ be a path pile of a graph $G$ and $v$ be any vertex of $G$. By counting the paths of $\mathcal{P}$ which contain the edges incident with $v$, we get the following. If $v \in \operatorname{ext} \mathcal{P}$, then $|\mathcal{P}| \geq \deg v$; otherwise, $|\mathcal{P}| \geq \deg v - 1$.

**Remark 2.** Let $\mathcal{P}$ and $G$ be as above. For any $P \in \mathcal{P}$, let $i(P)$ be the number of its internal vertices. Then $\epsilon = \sum_{P \in \mathcal{P}} |E(P)| = \sum [1 + i(P)] = |\mathcal{P}| + \sum i(P) = |\mathcal{P}| + \nu - |\operatorname{ext} \mathcal{P}|$. Therefore $|\mathcal{P}| = \epsilon - \nu + |\operatorname{ext} \mathcal{P}|$. (This relation for graphoidal cover has been observed in [6].) From this, we have the following result.

**Proposition 3.** *For any graph, $\eta \geq \epsilon - \nu$; equality holds if and only if there exists a path pile without exterior vertices.*

Let $\mathfrak{F}$ be the family of all graphs with $\eta = \epsilon - \nu$. The problem of characterizing $\mathfrak{F}$ has been posed in [2]. (This is same as 'determining when a graph contains a set of *internally disjoint paths* with certain properties'. An attempt has been made in [3]; but the criterion employed therein itself involves a class of graphs defined by using the very notion of 'internally disjoint paths'.) In this note we answer this:

**Theorem 4.** *A graph $G$ belongs to $\mathfrak{F}$ if and only if the following holds.*
(**) *For every vertex $\alpha$, no component of $G - \alpha$ is a tree.*

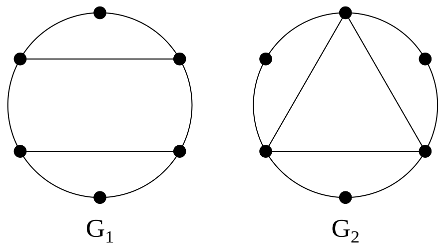
In settling the sufficiency-part of this theorem for graphs with cutvertices the following result serves as a linking tool.

**Proposition 5.** *If the exterior of a path pile of a graph $G$ contains cutvertices only, then $G \in \mathfrak{F}$.*

**Proof.** Let $\mathcal{P}$ be one of the path piles which satisfy the hypothesis, such that its exterior is minimal. If $\operatorname{ext} \mathcal{P}$ is nonempty, let $\alpha$ be a cutvertex in $\operatorname{ext} \mathcal{P}$. Then there exist edges $e_1, e_2$ incident with $\alpha$ such that no cycle of $G$ contains both the edges. Obviously the two paths in $\mathcal{P}$ which contain $e_1$ and $e_2$ respectively, have no vertex in common, other than $\alpha$. Hence, from $\mathcal{P}$, replacing these two paths by their join, we can get a path pile whose exterior is $\operatorname{ext} \mathcal{P} - \{\alpha\}$, contradicting the minimization of $\operatorname{ext} \mathcal{P}$. Therefore $\operatorname{ext} \mathcal{P} = \emptyset$ implying $G \in \mathfrak{F}$ by Proposition 3. ☐

The following result serves as a tool for constructing path piles for blocks which satisfy (**); the proof employs the technique of constructing 2-connected graphs recursively from cycles. (cf. [5, Proposition 3.1.2].)

**Proposition 6.** *If $G$ is a 2-connected graph and $\mathcal{Q}$ is a set of nontrivial internally disjoint paths in $G$, then there exists a path pile $\mathcal{P}$ of $G$ containing $\mathcal{Q}$ such that $\operatorname{ext} \mathcal{P} = \operatorname{ext} \mathcal{Q}$.*




Two Graphs in $\mathfrak{F}$.

**Proof.** Let $\mathcal{P}$ be a maximal set of nontrivial internally disjoint paths containing $\mathcal{Q}$ such that $\operatorname{ext}\mathcal{P} = \operatorname{ext}\mathcal{Q}$. Let $H$ be the subgraph formed by all the vertices and all the edges on the paths of $\mathcal{P}$. If $V(H) \neq V(G)$, then by connectedness of $G$, there is an edge $uv$ with $u \in V(H)$ and $v \in V(G) - V(H)$. Since $G$ is 2-connected, $G - u$ contains a path $P$ joining $v$ to a vertex of $H$ such that its interior is disjoint from $V(H)$. Now $\mathcal{P} \cup \{uvP\}$ is a set of internally disjoint paths whose exterior is same as that of $\mathcal{P}$, contradicting the maximization of $\mathcal{P}$. Therefore $V(H) = V(G)$; maximization of $\mathcal{P}$ now implies that $E(H) = E(G)$. Hence $\mathcal{P}$ is a path pile of $G$. □

**Corollary 7.** *If a subgraph of a 2-connected graph $G$ belongs to $\mathfrak{F}$, then $G$ also does so.*

A graph $H$ is said to be a *subdivision* of a graph $G$, if $H$ is obtained from $G$ by subdividing some of the edges, that is, replacing the edges by paths so that the external vertices of any path are the ends of the corresponding edge but any internal vertex is neither in $V(G)$ nor on any other path. Note that $G \in \mathfrak{F} \Rightarrow H \in \mathfrak{F}$.

The key result of this note is proved by making use of subdivisions of $K_4$ and the two graphs in the figure.

**Proposition 8.** *If $G$ is a 2-connected graph satisfying (**) of Theorem 4, then it belongs to $\mathfrak{F}$.*

**Proof.** Since the complete graph $K_4$ and the graphs $G_1$, $G_2$ in the figure belong to $\mathfrak{F}$, by Corollary 7, it is enough to show the existence of a subgraph of $G$ which is isomorphic to a subdivision of one of the former graphs. Let $\alpha$ be a vertex of maximum degree $\Delta$. Let $C$ be one of the cycles in $G - \alpha$ which are at the minimum possible distance from $\alpha$. Let $P_1$ be a shortest path from $\alpha$ to $C$, meeting the latter at $\beta$. Now in $G - \beta$, draw a path $P_2$ from $\alpha$ to $C - \beta$; let $\gamma$ be its terminal. Note that $P_2$ is internally disjoint from $P_1$, for otherwise there would be a cycle $C'$ formed by parts of $P_1$, $P_2$ and $C$ such that $d(\alpha, C') < d(\alpha, C)$, contradicting the choice of $C$. Let $x_1, x_2, \ldots, x_\Delta$ be the neighbours of $\alpha$. Assume that $x_1$ and $x_2$ are on $P_1$ and $P_2$ respectively. For each $x_i$, $3 \leq i \leq k$, let $P_i$ be the path formed by combining $\alpha x_i$ and a path in $G - \alpha$ which joins $x_i$ and $C$. If this path intersects $P_1$ or $P_2$, then a part of this path together with $P_1$, $P_2$ and $C$ form a subgraph which is a subdivision of $G_1$ in the figure; so assume this is internally disjoint from $P_1$ and $P_2$. We also assume that the terminal of $P_i$ is either $\beta$ or $\gamma$ for otherwise $P_1$, $P_2$, $P_i$ and $C$ form a subgraph isomorphic to a subdivision of $K_4$. Further the paths $P_i$, $3 \le i \le \Delta$ can be taken to be internally disjoint for otherwise there would be a subgraph isomorphic to a subdivision of $G_1$. If all these paths have same terminal, say $\beta$, then $\deg \beta \ge \Delta + 1$, a contradiction. Therefore two of them, say $P_3$ and $P_4$, have different terminals. Now $P_1$, $P_2$, $P_3$, $P_4$ and $C$ form a subdivision of $G_2$. □

**The proof of Theorem 4.** First let $G$ have a path pile $\mathcal{P}$ with $\operatorname{ext} \mathfrak{F} = \emptyset$. Suppose the existence of a vertex $\alpha$ such that a component $K$ of $G - \alpha$ is a tree. Let $H = G[V(K) \cup \{\alpha\}]$. Then $\nu(H) = \nu(K) + 1$ and $\epsilon(H) = \deg_H \alpha + \nu(K) - 1$.

Let $\mathcal{Q}$ be the set of all nontrivial restrictions to $H$ of paths in $\mathcal{P}$. Obviously $\mathcal{Q}$ is a path pile of $H$ with $\operatorname{ext} \mathcal{Q} = \emptyset$ or $\{\alpha\}$. By Remark 2, $|\mathcal{Q}| = \epsilon(H) - \nu(H) + |\operatorname{ext} \mathcal{Q}| = \deg_H \alpha - 2 + |\operatorname{ext} \mathcal{Q}|$. Therefore,

$$|\mathcal{Q}| = \begin{cases} \deg_H \alpha - 1 & \text{if } \alpha \in \operatorname{ext} \mathcal{Q}; \\ \deg_H \alpha - 2 & \text{otherwise.} \end{cases}$$

This contradicts Remark 1.

Now let $G$ satisfy $(**)$. For each block $B$ of $G$, let us construct a path pile as follows.

If $V(B)$ has no cutvertex of $G$, i.e., if $B$ is a component of $G$, then by Proposition 8, we can get a path pile $\mathcal{P}_B$ of $B$ with $\operatorname{ext} \mathcal{P}_B = \emptyset$.

If $V(B)$ has exactly one cutvertex of $G$, say $\alpha$, i.e., if $B$ is an endblock (as defined in [4]) and $\alpha$ is the cutvertex in $V(B)$, then as done in the proof of Proposition 8, let a cycle $C$ and paths $P_1$, $P_2$ be found. It is easy to construct a path pile $\mathcal{Q}$ for the subgraph formed by $P_1$, $P_2$ and $C$ such that $\operatorname{ext} \mathcal{Q} = \{\alpha\}$. By Proposition 6, this can be extended to a path pile $\mathcal{P}_B$ of $B$ such that $\operatorname{ext} \mathcal{P}_B = \{\alpha\}$.

Now, suppose that two cutvertices $\alpha, \beta$ are in $V(B)$. If $B$ is a $K_2$, let $\mathcal{P}_B = \{\alpha\beta\}$; otherwise choose two internally disjoint paths $P_1$, $P_2$ joining $\alpha$ and $\beta$. By applying Proposition 6 to $\{P_1, P_2\}$, we can get a path pile $\mathcal{P}_B$ for $B$ with $\operatorname{ext} \mathcal{P}_B = \{\alpha, \beta\}$.

Finally consider the union of all the path piles constructed for the blocks of $G$. If any vertex $\alpha$ is internal to more than one path—$\alpha$ is obviously a cutvertex—replace any such path $P$ by two paths obtained by splitting $P$ at $\alpha$. Repeat this process, till we get a path pile $\mathcal{P}$ of $G$; obviously $\operatorname{ext} \mathcal{P}$ contains only cutvertices. Now Proposition 5 completes the proof. □

**Corollary 9.** *$\mathfrak{F}$ contains the class of all graphs with $\delta \ge 3$.*

A direct proof for this corollary has been obtained in [7].

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