AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **39** (2007), Pages 73–87

Supplementary difference sets constructed from $(q+1)$st cyclotomic classes in GF(q^2)

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Abstract

In this paper, we give Jacobi sums for $(q+1)$st power residue characters on GF(q^2) by a combinatorial method and construct families of supplementary difference sets from $(q+1)$st cyclotomic classes by using these Jacobi sums. Further, we show the theorem on supplementary difference sets on GF(q^2) for $q \equiv 1 \pmod 4$ by M. Xia, T. Xia and J. Seberry can be proved by using these Jacobi sums and we shall prove a similar theorem for $q \equiv 3 \pmod 4$.

1 Introduction

The concept of supplementary difference sets was introduced by J.S. Wallis (now Seberry) [2] in 1972 and many new Hadamard matrices have been constructed from supplementary difference sets [3, 4, 5]. Let $q = ef + 1$ be an odd prime power. A necessary and sufficient condition such that unions of eth cyclotomic classes become supplementary difference sets was given [7]. We need the explicit values of Jacobi sums in the concrete calculations when we apply this theorem to constructions. Jacobi sums are not determined explicitly in general, though Jacobi sums are evaluated for $e \leq 24$ [1]. In this paper, we give the explicit values of Jacobi sums for $(q+1)$st power residue characters on GF(q^2) by combinatorial method and construct families of supplementary difference sets from $(q+1)$st cyclotomic classes by using these Jacobi sums. Let $e = q+1$, $\Omega = \{0, 1, \cdots, e-1\}$ and $A_0, A_1, \cdots, A_{n-1}$ be parts of a partition of Ω such that $|A_i| = \frac{e}{n}$. Let $S_0, S_1, \cdots, S_{e-1}$ be eth cyclotomic classes in GF(q^2). We define the subsets $D_i = \cup_{l \in A_i} S_l$ of GF(q^2) for $i = 0, \cdots, n-1$. Then we show for any partition of Ω, the subsets $D_0, D_1, \cdots, D_{n-1}$ become supplementary difference sets. Furthermore the theorem on supplementary difference sets on GF(q^2) for $q \equiv 1 \pmod 4$ by M. Xia, T. Xia and J. Seberry [6] can be proved by using these Jacobi sums and we shall prove a similar theorem for $q \equiv 3 \pmod 4$.

2 Supplementary difference sets and eth cyclotomic classes

First we give the definition of supplementary difference sets.

Definition 1. *Let G be an additive abelian group of order v and $D_0, \cdots, D_{n-1}$ be subsets of G which contain $k_0, \cdots, k_{n-1}$ elements respectively. For $d \neq 0 \in G$, we define the numbers.*

$$\lambda_i(d) = |\{(r,s) : d = r - s,\ r, s \in D_i\}| \quad \textit{for} \quad i = 0, \cdots, n-1.$$

If $\lambda(d) = \lambda_0(d) + \cdots + \lambda_{n-1}(d)$ has a constant value λ, $D_0, \cdots, D_{n-1}$ are called $n - \{v; k_0, \cdots, k_{n-1}; \lambda\}$ supplementary difference sets. If $k_0 = \cdots = k_{n-1}$, we simplify $D_0, \cdots, D_{n-1}$ to $n - \{v; k_0; \lambda\}$ supplementary difference sets.

The parameters k_i, $(i = 0, \cdots, n-1)$, v, λ satisfy the following relation:

$$\sum_{i=0}^{n-1} k_i(k_i - 1) = \lambda(v - 1).$$

For convenience, we abbreviate supplementary difference set to SDS.

Let $q = ef + 1$ be an odd prime power and $F = \mathrm{GF}(q)$ be the finite field with q elements. Let g be a primitive element of F. The eth cyclotomic classes $S_0, S_1, \cdots, S_{e-1}$ in F are defined as

$$S_i = \{g^{es+i} : s = 0, \cdots, f-1\}.$$

Note that S_0 is the set of eth power residues.

For non-empty subsets $A_0, A_1, \cdots, A_{n-1}$ of $\Omega = \{0, 1, \cdots, e-1\}$, we define the subsets

$$D_i = \bigcup_{l \in A_i} S_l, \quad \text{for } i = 0, \cdots, n-1,$$

of F, that are unions of some eth cyclotomic classes.

A necessary and sufficient condition such that the subsets $D_0, D_1, \cdots, D_{n-1}$ become SDSs was given.

Theorem 1. [7] *Let $A_0, A_1, \cdots, A_{n-1}$ be non-empty subsets of $\Omega = \{0, 1, \cdot, e-1\}$ and $|A_i| = u_i$ for $i = 0, \cdots, n-1$. The subsets $D_i = \bigcup_{l \in A_i} S_l$ of F determined by A_i, $i = 0, \cdots, n-1$, become $n - \{q\ ;\ u_0 f, u_1 f, \cdots, u_{n-1} f\ ;\ \lambda\}$ SDSs if and only if the following equations are satisfied:*

(1) $$\sum_{i=0}^{n-1} u_i(u_i f - 1) \equiv 0 \pmod{e}.$$

(2) $$\sum_{i=0}^{n-1} \sum_{m=0}^{e-1} \pi(\chi^m, \chi^{-t}) \omega_{i,m} \omega_{i,t-m} = 0 \quad \textit{for all } t = 1, \cdots, e-1,$$

where $\omega_{i,m} = \sum_{l \in A_i} \zeta_e^{-lm}$, ζ_e is a primitive eth root of unity, $\pi(\chi^m, \chi^{-t})$ is the Jacobi sum for eth power residue chracters χ^m, χ^{-t}, and χ is a primitive eth power residue charactor. If f is odd, i.e. $-1 \notin S_0$, we have only to verity equation (2) for even t.

3 Jacobi sums

Let χ be a primitive eth power residue character, i.e. $\chi(g) = \zeta_e$ where ζ_e is a primitive eth root of unity and g is a primitve element of F. We let $\chi(0) = 0$. Let χ^0 be the trivial character. Then

$$\pi(\chi^m, \chi^t) = \sum_{\alpha \in F} \chi^m(\alpha)\chi^t(1-\alpha)$$

is called the Jacobi sum for the eth power residue chracters χ^m and χ^t. The following theorem on Jacobi sums is well known.

Theorem 2. *For the eth power residue characters χ^m and χ^t, the following relations on Jacobi sums are satisfied:*

(1) $\pi(\chi^m, \chi^t) = \pi(\chi^t, \chi^m)$.

(2) $\pi(\chi^m, \chi^t) = \chi^m(-1)\pi(\chi^m, \chi^{-m-t})$.

(3) $\pi(\chi^0, \chi^0) = q - 2$.

(4) $\pi(\chi^m, \chi^0) = -1$ *for* $m \neq 0$.

(5) $\pi(\chi^m, \chi^{-m}) = -\chi^m(-1)$ *for* $m \neq 0$.

Throughout, we assume q is an odd prime power and let $F = \mathrm{GF}(q)$, $F^\times = F - \{0\}$, $K = \mathrm{GF}(q^2)$, $K^\times = K - \{0\}$. We need the following lemma to determine the values of Jacobi sums.

Lemma 1. *Let g be a primitive element of K. For given numbers $1 \leq x, t \leq q$, we have*

$$N = |\{\delta \in F^\times : 1 - g^t\delta \in g^x F^\times\}| = \begin{cases} 1 & \textit{if} \quad x \neq t \quad \textit{and} \quad t \neq 0, \\ 0 & \textit{if} \quad x = t \quad \textit{and} \quad t \neq 0, \\ q-2 & \textit{if} \quad x = t = 0, \\ 0 & \textit{if} \quad x \neq 0 \quad \textit{and} \quad t = 0. \end{cases}$$

Proof. First we assume $t \neq 0$ and $x \neq t$. For δ and $\delta' \in F^\times$, if $1 - g^t\delta \in g^x F^\times$ and $1 - g^t\delta' \in g^x F^\times$, then $g^t(\delta - \delta') \in g^x F^\times$. It follows $\delta = \delta'$, since we have $x = t$ if $\delta \neq \delta'$. If $x = t \neq 0, 1 - g^x\delta \in g^x F^\times$, x should be 0, a contradiction. Hence $N = 0$. The assumption $t = 0$ implies $x = 0$. In this case, for any element $\delta \neq 1 \in F^\times, 1 - \delta \in F^\times$, that is $N = q - 2$. □

Theorem 3. *Put $e = q + 1$. Let χ be a primitive eth power residue character and χ^0 be the trivial character on K. For $0 \leq k,\ k' \leq e - 1$, we have*

$$\pi(\chi^k, \chi^{k'}) = \begin{cases} q^2 - 2 & \textit{if} \quad k = k' = 0, \\ -1 & \textit{if} \quad k + k' = 0, \\ q & \textit{if} \quad k + k' \neq 0,\ k \neq 0 \quad \textit{and} \quad k' \neq 0. \end{cases}$$

Proof. From Theorem 1, we have $\pi(\chi^0,\chi^0)=q^2-2, \pi(\chi^0,\chi^k)=\pi(\chi^k,\chi^0)=-1$ and $\pi(\chi^k,\chi^{-k})=-\chi^k(-1)=-1$ for $k\neq 0$, since $\chi^k(-1)=1$. Assume that $k+k'\neq 0, k\neq 0$ and $k'\neq 0$. From Lemma 1,

$$\begin{aligned}\pi(\chi^k,\chi^{k'}) &= \sum_{\alpha\in K^\times}\chi^k(1-\alpha)\chi^{k'}(\alpha)=\sum_{t=0}^{e-1}\chi^{k'}(g^t)\sum_{\delta\in F^\times}\chi^k(1-g^t\delta)\\ &= (q-2)\chi^{k'}(1)+\sum_{t=1}^{e-1}\chi^{k'}(g^t)(-\chi^{k'}(1)-\chi^k(g^t))=q.\end{aligned}$$

□

Let $L=\mathrm{GF}(q^{2s})$ be an extension of $K=\mathrm{GF}(q^2)$ and $S_{L/K}$ be the relative trace from L to K and $N_{L/K}$ be the relative norm from L to K. Let χ be a multiplicative character on K and λ be an additive character on K. We let for $\beta\in L$,

$$\tilde{\chi}(\beta)=\chi(N_{L/K}\beta),\quad \tilde{\lambda}(\beta)=\lambda(S_{L/K}\beta).$$

Then $\tilde{\chi}$ is a multiplicative character on L and $\tilde{\lambda}$ is an additive character on L, which are called the lift of χ and the lift of λ from K to L respectively. Thus the Gauss sum is defined for the lift of χ and the lift of λ.

The Davenport-Hasse theorem on lifted Gauss sums shows the relationship between the Gauss sums with the characters χ and λ and the Gauss sum with the lifts of χ and λ. The Davenport-Hasse theorem yields the following theorem.

Theorem 4. *If* $k\neq 0$, $k'\neq 0$ $k+k'\neq 0$,

$$\pi(\tilde{\chi^k},\tilde{\chi^{k'}})=(-1)^{s-1}\pi(\chi^k,\chi^{k'})^s=(-1)^{s-1}q^s.$$

Proof. See [1]. □

4 Group ring and characteristic functions

Let χ be a primitive eth power residue character. The characteristic function f_{S_l} of an eth cyclotomic class S_l is given by

$$f_{S_l}(\alpha)=\frac{1}{e}\sum_{k=0}^{e-1}\zeta_e^{-lk}\chi^k(\alpha),\quad \text{for}\quad \alpha\in K.$$

Let $A_0,A_1,\cdots,A_{n-1}$ be non-empty subsets of $\Omega=\{0,1,\cdots,e-1\}$ and $D_i=\bigcup_{l\in A_i}S_l$ be the subsets of K for $i=0,\cdots,n-1$. Then the characteristic function f_{D_i} is given by

$$f_{D_i}(\alpha)=\sum_{l\in A_i}f_{S_l}(\alpha)=\frac{1}{e}\sum_{k=0}^{e-1}\Big(\sum_{l\in A_i}\zeta_e^{-lk}\Big)\chi^k(\alpha)=\frac{1}{e}\sum_{k=1}^{e-1}\omega_{i,k}\chi^k(\alpha),\quad 0\le i\le n-1,$$

for $\alpha \in K$ where $\omega_{i,k} = \sum_{l \in A_i} \zeta_e^{-lk}$.

Let $\boldsymbol{Z}$ be an integer ring. We consider the group ring $\boldsymbol{Z}K$ of the additive group K^+ over $\boldsymbol{Z}$. Let $D_0, D_1, \cdots, D_{n-1}$ be subsets of K. For an element $\mathcal{D}_i = \sum_{\alpha \in D_i} \alpha$ of $\boldsymbol{Z}K$, we set

$$\mathcal{D}_i^{-1} = \sum_{\alpha \in D_i} (-\alpha).$$

We denote the unit of K by $\boldsymbol{o}$ in order to distinguish zero 0 of $\boldsymbol{Z}$. A necessary and sufficient condition that $D_0, D_1, \cdots, D_{n-1}$ become $n - \{q^2; k_0, k_1, \cdots, k_{n-1}; \lambda\}$ SDSs if and only if

$$\sum_{i=0}^{n-1} \mathcal{D}_i \cdot \mathcal{D}_i^{-1} = \sum_{i=0}^{n-1} k_i \boldsymbol{o} + \lambda \sum_{\alpha \in K^\times} \alpha$$

where $k_i = |D_i|$ for $i = 0, \cdots, n-1$. By using the characterisitc function f_{D_i} of a subset D_i, we have

$$\mathcal{D}_i = \sum_{\alpha \in K^\times} f_{D_i}(\alpha)\alpha = \frac{1}{e} \sum_{\alpha \in K^\times} \sum_{k=0}^{e-1} \omega_{i,k} \chi^k(\alpha)\alpha$$

and

$$\mathcal{D}_i^{-1} = \sum_{\alpha \in K^\times} f_{D_i}(\alpha)(-\alpha) = \frac{1}{e} \sum_{\alpha \in K^\times} \sum_{k=0}^{e-1} \omega_{i,k} \chi^k(\alpha)(-\alpha).$$

5 A partition of GF(q^2) and supplementary difference sets

Let q, $A_i\,(0 \le i \le n-1)$, $S_l\,(0 \le l \le e-1)$, $D_i\,(0 \le i \le n-1)$ and χ be as in Theorem 1. The following lemma is used in the proofs of theorems in this and next sections.

Lemma 2. *For* $\omega_{i,k} = \sum_{l \in A_i} \zeta_e^{-lk}$*, we have*

(1) $\displaystyle\sum_{k=0}^{e-1} \omega_{i,k}\omega_{i,t-k} = e\omega_{i,t}$,

(2) $\displaystyle\sum_{i=0}^{e-1} \omega_{i,k}\chi^k(\alpha) = \begin{cases} e & \text{if } \alpha \in D_i, \\ 0 & \text{if } \alpha \notin D_i. \end{cases}$

Proof. (1) $\displaystyle\sum_{k=0}^{e-1} \omega_{i,k}\omega_{i,t-k} = \sum_{k=0}^{e-1} \sum_{a \in A_i} \zeta_e^{-ak} \sum_{b \in A_i} \zeta_e^{-b(t-k)} = \sum_{a \in A_i} \sum_{b \in A_i} \zeta_e^{-bt} \sum_{k=0}^{e-1} \zeta_e^{(b-a)k}$
$\displaystyle= e \sum_{b \in A_i} \zeta_e^{-bt} = e\omega_{i,t}$.

(2) Assume $\alpha \in S_t$.

$$\sum_{k=0}^{e-1} \omega_{i,k}\chi^k(\alpha) = \sum_{k=0}^{e-1}\sum_{a\in A_i} \zeta_e^{-ak}\zeta_e^{kt} = \sum_{a\in A_i}\sum_{k=0}^{e-1} \zeta_e^{k(t-a)} = \begin{cases} e & \text{if } t\in A_i, \\ 0 & \text{if } t\notin A_i. \end{cases}$$

□

Put $e = q+1$. Let $S_0, S_1, \cdots, S_{e-1}$ be eth cyclotimic classes and n be a divisor of e. Let $A_0, A_1, \cdots, A_{n-1}$ be parts of a partition of $\Omega = \{0, 1, \cdots, q\}$ such that $|A_i| = \frac{e}{n}$ for all $i = 0, 1, \cdots, n-1$. Then $D_i = \bigcup_{l\in A_i} S_l$ are parts of a partition of $K^\times$ for $i = 0, 1, \cdots, n-1$. We shall prove the subsets $D_0, D_1, \cdots, D_{n-1}$ become SDSs for any partition of Ω.

Theorem 5. $D_0, D_1, \cdots, D_{n-1}$ *are* $n - \Big\{q^2; \frac{q^2-1}{n}; \frac{q^2-1}{n} - 1\Big\}$ *SDSs.*

Proof. We verify the equations in Theorem 1.

(1) $\displaystyle\sum_{i=0}^{n-1} \frac{q+1}{n}(\frac{q+1}{n}(q-1)-1) \equiv 0 \pmod{q+1}.$

(2) From Lemma 2 and Theorem 3,

$$\begin{aligned}
\sum_{i=0}^{n-1}\sum_{k=0}^{e-1} \pi(\chi^k, \chi^{-t})\omega_{i,k}\omega_{i,t-k} &= \sum_{i=0}^{n-1}\{q \sum_{k=1,\, k\neq t}^{e-1} \omega_{i,k}\omega_{i,t-k} - 2\omega_{i,0}\omega_{i,t}\} \\
&= q\sum_{i=0}^{n-1}\sum_{k=0}^{e-1} \omega_{i,k}\omega_{i,t-k} - 2(q+1)\omega_{i,0}\sum_{i=0}^{n-1}\omega_{i,t} \\
&= e\Big(q - 2\frac{q+1}{n}\Big)\sum_{i=0}^{n-1}\omega_{i,t}.
\end{aligned}$$

Since $D_0, \cdots, D_{n-1}$ are parts of a partition of $K^\times$, $\displaystyle\sum_{i=0}^{n-1}\omega_{i,t} = 0.$ □

Using Theorem 4, we obtain a similar result over an extension $L = \mathrm{GF}(q^{2s})$.

Theorem 6. *Let* $S_0, \cdots, S_{e-1}$ *be eth cyclotomic classes of* L *and* $A_0, \cdots, A_{n-1}$ *be parts of a partition of* $\Omega = \Big\{0, 1, \cdots, e-1\Big\}$ *such that* $|A_i| = \frac{e}{n}$ *for* $i = 0, \cdots, n-1$. *We define the subsets* $D_i = \bigcup_{l\in A_i} S_l$ *of* L *for* $i = 0, \cdots, n-1$. *Then* $D_0, D_1, \cdots, D_{n-1}$ *are* $n - \Big\{q^{2s}; \frac{q^{2s}-1}{n}; \frac{q^{2s}-1}{n} - 1\Big\}$ *SDSs.*

Proof. We verify the equations in Theorem 1.

(1) $\sum_{i=0}^{n-1} \frac{q+1}{n}\Big(\frac{q+1}{n} \cdot \frac{q^{2s}-1}{q+1} - 1\Big) \equiv 0 \pmod{q+1}$.

(2) From Lemma 2 and Theorem 4, we obtain

$$\sum_{i=0}^{n-1}\sum_{k=0}^{e-1} \pi(\tilde{\chi}^t, \tilde{\chi}^{-t})\omega_{i,k}\omega_{i,t-k} = \Big\{(-1)^{s-1}q^s e - 2((-1)^{s-1}q^s+1)\frac{e}{n}\Big\}\sum_{i=0}^{n-1}\omega_{i,t} = 0.$$

□

Furthermore we can construct SDSs on K combining two partitions of Ω.

Theorem 7. *Assume* $q \equiv 3 \pmod 4$. *Let* $\Omega = \{0, 1, \cdots, e-1\} = \Omega_0 \cup \Omega_1, \Omega_0 = \{a \in \Omega | a \equiv 0 \pmod 2\}$ *and* $\Omega_1 = \{a \in \Omega | a \equiv 1 \pmod 2\}$. *Let* $S_0, S_1, \cdots, S_{e-1}$ *be eth cyclotomic classes of* $K^\times$, *and* C_0, C_1, C_2 *and* C_3 *be biquadratic cyclotomic classes of* $K^\times$. *Let* A_0 *and* A_2 *be parts of a partition of* Ω_1, *and* A_1 *and* A_3 *parts of a partition of* Ω_0, *and put* $|A_i| = \frac{e}{4}$ *for* $i = 0, 1, 2, 3$. *Then*

$$D_i = \bigcup_{l \in A_i} S_l \cup C_i, \qquad i = 0, 1, 2, 3,$$

are $4 - \Big\{q^2; \frac{q^2-1}{2}; q^2-3\Big\}$ *SDSs.*

Proof. A biquadratic cyclotomic class is a union of eth cyclotomic classes. By setting $B_i = \{a \in \Omega | a \equiv i \pmod 4\}$ and $\mathcal{A}_i = A_i \cup B_i$ for $i = 0, 1, 2, 3$, we can thus write

$$D_i = \bigcup_{l \in \mathcal{A}_i} S_l \quad \text{and} \quad \omega_{i,t} = \sum_{l \in \mathcal{A}_i} \zeta_e^{-lt}, \quad \text{for } i = 0, 1, 2, 3.$$

We verify the equations in Theorem 1. Since $|\mathcal{A}_i| = \frac{e}{4} + \frac{e}{4} = \frac{e}{2}$, $|D_i| = \frac{e}{2}|S_i|$ and $\sum_{i=0}^{3} \frac{e}{2}\Big(\frac{e}{2}|S_0| - 1\Big) \equiv 0 \pmod e$. It is sufficient to verify for every $1 \le t \le e-1$, $e\Big(q - 2 \cdot \frac{e}{2}\Big)\sum_{i=0}^{3}\omega_{i,t} = 0$. The subsets $\bigcup_{l \in \mathcal{A}_i} S_l$ and C_i are parts of partitions of $K^\times$, so that

$$\sum_{i=0}^{3}\omega_{i,t} = \sum_{i=0}^{3}\sum_{l \in \mathcal{A}_i}\zeta_e^{-lt} = \sum_{i=0}^{3}\Big(\sum_{l \in A_i}\zeta_e^{-lt} + \sum_{l \in C_i}\zeta_e^{-lt}\Big) = 2\sum_{l \in \Omega}\zeta_e^{-lt} = 0.$$

□

6 Xia-Xia-Seberry's theorem and the similar theorem for $q \equiv 3 \pmod 4$

M. Xia, T. Xia and J. Seberry constructed $4-\left\{q^2; \frac{1}{2}q(q-2); q(q-2)\right\}$ SDSs from $(q+1)$st cyclotomic classes and biquadratic cyclotomic classes in GF(q^2) for $q \equiv 1 \pmod 4$ [6]. They used $(q;x,y)$-partitions and generalized cyclotomic classes to prove the theorem. Theorem 1 needs Jacobi sums for $2(q+1)$st power residue characters in the concrete calculations to prove Xia-Xia-Seberry's theorem, since the least common multiple of 4 and $q+1$ is $2(q+1)$. Jacobi sums are not determined explicitly in general, although Jacobi sums are evaluated for $e \le 24$ [1].

In what follows, we assume $e = q+1$ and denote $(q+1)$st cylotomic classes by $S_0, S_1, \cdots, S_{e-1}$.

Let $\Omega = \{0, 1, \cdots, e-1\} = \Omega_0 \cup \Omega_1$, $\Omega_0 = \{a \in \Omega \mid a \equiv 0 \pmod 2\}$ and $\Omega_1 = \{a \in \Omega \mid a \equiv 1 \pmod 2\}$. Assume that A_0 is a subset of Ω_1 and A_1 is a subset of Ω_0 such that $|A_0| = |A_1| = (q-1)/4$. Further we let $A_2 = A_0$ and $A_3 = A_1$. Let C_0, C_1, C_2 and C_3 be biquadratic cyclotomic classes.

Lemma 3. *We let* $\omega_{i,k} = \sum_{l \in A_i} \zeta_e^{-lt}$ *and* $v_{i,k'} = \zeta_4^{-ik'}$. *Further we let* χ *be a primitive eth power residue character and* φ *be a primitive biquadratic charactor. Then, for* $\alpha \in GF(q^2)$,

(1) $\sum_{k=0}^{e-1} \omega_{i,k}\chi^k(\alpha) \cdot \sum_{k'=0}^{3} v_{i,k'}\varphi^{k'}(\alpha) = 0.$

(2) *If* k' *is odd, then* $\sum_{i=0}^{3} \omega_{i,k} v_{i,k'} = 0.$

(3) $\sum_{i=0}^{3} \omega_{i,\frac{e}{2}} v_{i,2} = -(q-1).$

(4) $\sum_{i=0}^{3} v_{i,k} v_{i,k'} = \begin{cases} 4 & \text{if } k+k' = 0, \\ 0 & \text{if } k+k' \neq 0. \end{cases}$

Proof. (1) This follows from $\bigcup_{l \in A_i} S_l \cap C_i = \emptyset$.
(2) Since $\omega_{0,k} = \omega_{2,k}$ and $\omega_{1,k} = \omega_{3,k}$,

$$\sum_{t=0}^{3} \omega_{i,k} v_{ik'} = \omega_{0,k}\left(\zeta_4^0 + \zeta_4^{-2k'}\right) + \omega_{1,k}\left(\zeta_4^{-k'} + \zeta_4^{-3k'}\right).$$

This sum is equal to 0 if k' is odd.
(3) Since $\omega_{0,\frac{e}{2}} = \sum_{l \in A_0} \zeta_e^{-\frac{e}{2}l} = -|A_0|$ and $\omega_{1,\frac{e}{2}} = \sum_{l \in A_1} \zeta_e^{-\frac{e}{2}l} = |A_1|$, we have $\sum_{i=0}^{3} \omega_{i,\frac{e}{2}} v_{i,2} = \omega_{0,\frac{e}{2}}\left(\zeta_4^0 + \zeta_4^{2.2}\right) + \omega_{1,\frac{e}{2}}\left(\zeta_4^{-2} + \zeta_4^{-2.3}\right) = -2\Big(|A_0| + |A_1|\Big) = -(q-1).$

(4) $\sum_{i=0}^{3} v_{ik} v_{ik'} = \sum_{i=0}^{3} \zeta_4^{-i(k+k')} = 4$ if $k+k' = 0$ and 0 if $k+k' \neq 0$. □

Theorem 8. [Xia-Xia-Seberry, [6]] *Assume* $q \equiv 1 \pmod 4$. *Then*

$$D_i = \bigcup_{l \in A_i} S_l \cup C_i, \quad \textit{for } i = 0, 1, 2, 3,$$

become $4 - \{q^2; \frac{1}{2}q(q-1); q(q-2)\}$ *SDSs.*

Proof. For every subset D_i, $0 \leq i \leq 3$, $|D_i| = |A_0||S_0| + |C_0| = \frac{1}{2}q(q-1)$, so that we have $\sum_{i=0}^{3} |D_i| = 2q(q-1)$.

The characteristic function f_{D_i} is the sum of characteristic function f_{A_i} for the subset $\bigcup_{l \in A_i} S_l$ and the characteristic function f_{C_i} for the biquadratic cyclotomic class C_i:

$$f_{D_i}(\alpha) = f_{A_i}(\alpha) + f_{C_i}(\alpha) = \frac{1}{e}\sum_{k=0}^{e-1} \omega_{i,k}\chi^k(\alpha) + \frac{1}{4}\sum_{k'=0}^{3} v_{i,k'}\varphi^{k'}(\alpha).$$

Therefore

$$\sum_{i=0}^{3} \mathcal{D}_i \cdot \mathcal{D}_i^{-1} = \sum_{i=0}^{3} \sum_{\alpha \in K^\times} (f_{A_i}(\alpha) + f_{C_i}(\alpha))\alpha \cdot \sum_{\beta \in K^\times} (f_{A_i}(\beta) + f_{C_i}(\beta))(-\beta)$$

$$= \sum_{i=0}^{3} \sum_{\alpha \in K^\times} f_{A_i}(\alpha)\alpha \cdot \sum_{\beta \in K^\times} f_{A_i}(\beta)(-\beta) + \sum_{i=0}^{3} \sum_{\alpha \in K^\times} f_{A_i}(\alpha)\alpha \cdot \sum_{\beta \in K^\times} f_{C_i}(\beta)(-\beta)$$

$$+ \sum_{i=0}^{3} \sum_{\alpha \in K^\times} f_{C_i}(\alpha)\alpha \cdot \sum_{\beta \in K^\times} f_{A_i}(\beta)(-\beta) + \sum_{i=0}^{3} \sum_{\alpha \in K^\times} f_{C_i}(\alpha)\alpha \cdot \sum_{\beta \in K^\times} f_{C_i}(\beta)(-\beta).$$

We denote four terms of the above equation by P_1, P_2, P_3 and P_4. We set $m = \omega_{i,0} = |A_i| = (q-1)/4$ for $i = 0, 1, 2, 3$.

(1)
$$\begin{aligned} P_1 &= \sum_{i=0}^{3} \sum_{\alpha \in K^\times} f_{A_i}(\alpha)\alpha \cdot \sum_{\beta \in K^\times} f_{A_i}(\beta)(-\beta) \\ &= \sum_{i=0}^{3} \sum_{\alpha \in K^\times} \frac{1}{e}\sum_{k=0}^{e-1} \omega_{i,k}\chi^k(\alpha) \cdot \sum_{\beta \in K^\times} \frac{1}{e}\sum_{k'=0}^{e-1} \omega_{i,k'}\chi^{k'}(\beta)(\alpha - \beta) \\ &= \frac{1}{e^2}\sum_{i=0}^{3} \sum_{\alpha \in K^\times} \sum_{\beta \in K^\times} \sum_{k=0}^{e-1} \sum_{k'=0}^{e-1} \omega_{i,k}\omega_{i,k'}\chi^k(\alpha)\chi^{k'}(\beta)(\alpha - \beta). \end{aligned}$$

Putting $\gamma = \alpha - \beta$, we obtain

$$\begin{aligned} P_1 \ &= \ \frac{1}{e^2}\sum_{i=0}^{3}\Big\{ \sum_{\alpha \in K^\times} \Big(\sum_{k=0}^{e-1} \omega_{i,k}\chi^k(\alpha)\Big)^2 \boldsymbol{o} \\ &\quad + \sum_{\gamma \in K^\times} \sum_{k=0}^{e-1} \sum_{k'=0}^{e-1} \omega_{i,k}\omega_{i,k'}\chi^{k+k'}(\gamma)\chi^{k'}(-1)\pi(\chi^k, \chi^{k'})\gamma \Big\}. \end{aligned}$$

Substituting the values of Jacobi sums given in Theorem 3 and from Lemma 2,

$$P_1 = \frac{1}{e^2}\sum_{i=0}^{3}\Big\{\sum_{\alpha\in\bigcup_{l\in A_i}S_l} e^2\boldsymbol{o} + \sum_{\gamma\in K^\times}\Big(\omega_{i,0}^2\pi(\chi^0,\chi^0) + \sum_{k=1}^{e-1}\omega_{i,k}\omega_{i,0}\chi^k(\gamma)\pi(\chi^k,\chi^0)$$

$$+\sum_{k'=1}^{e-1}\omega_{i,0}\omega_{i,k'}\chi^{k'}(\gamma)\pi(\chi^0,\chi^{k'}) + \sum_{k=1}^{e-1}\sum_{k'=1}^{e-1}\omega_{i,k}\omega_{i,k'}\chi^{k+k'}(\gamma)\pi(\chi^k,\chi^{k'})\Big)\gamma\Big\}$$

$$= \frac{1}{e^2}\sum_{i=0}^{3}\Big\{|A_i||S_0|e^2\boldsymbol{o} + \sum_{\gamma\in K^\times}\Big(m^2(q^2-2) - 2m(\sum_{k=0}^{e-1}\omega_{i,k}\chi^k(\gamma) - \omega_{i,0})$$

$$+ q(\sum_{k=0}^{e-1}\omega_{i,k}\chi^k(\gamma) - \omega_{i,0})^2 - q(\sum_{k=0}^{e-1}\omega_{i,k}\omega_{i,-k} - \omega_{i,0}^2) - (\sum_{k=0}^{e-1}\omega_{i,k}\omega_{i,-k} - \omega_{i,0}^2)\Big)\gamma\Big\}$$

$$= (q-1)^2\boldsymbol{o} + \frac{1}{e^2}\sum_{i=0}^{3}\sum_{\gamma\in K^\times}\Big(m^2(q+1)^2 - me(q+1) - 2m(q+1)\sum_{k=0}^{e-1}\omega_{i,k}\chi^k(\gamma)$$

$$+ q(\sum_{k=0}^{e-1}\omega_{i,k}\chi^k(\gamma))^2\Big)\gamma$$

$$= (q-1)^2\boldsymbol{o} + 4(m^2-m)\sum_{\gamma\in K^\times}\gamma - \frac{2m}{e}\sum_{i=0}^{3}\sum_{\gamma\in\cup_{l\in A_i}S_l} e\gamma + \frac{q}{e^2}\sum_{i=0}^{3}\sum_{\gamma\in\cup_{l\in A_i}S_l} e^2\gamma$$

$$= (q-1)^2\boldsymbol{o} + \frac{1}{4}(q-1)(q-5)\sum_{\gamma\in K^\times}\gamma + (q+1)\sum_{i=0}^{1}\sum_{\gamma\in\cup_{l\in A_i}S_l}\gamma.$$

(2)
$$P_2 = \sum_{i=0}^{3}\sum_{\alpha\in K^\times} f_{A_i}(\alpha)\alpha\cdot\sum_{\beta\in K^\times} f_{C_i}(\beta)(-\beta)$$

$$= \sum_{i=0}^{3}\frac{1}{e}\sum_{\alpha\in K^\times}\sum_{k=0}^{e-1}\omega_{i,k}\chi^k(\alpha)\alpha\cdot\frac{1}{4}\sum_{\beta\in K^\times}\sum_{k'=0}^{3} v_{i,k'}\varphi^{k'}(\beta)(-\beta)$$

$$= \frac{1}{4e}\sum_{i=0}^{3}\sum_{\alpha\in K^\times}\sum_{\beta\in K^\times}\sum_{k=0}^{e-1}\sum_{k'=0}^{3}\omega_{i,k}v_{i,k'}\chi^k(\alpha)\varphi^{k'}(\beta)(\alpha-\beta).$$

Putting $\gamma = \alpha - \beta$, we get

$$P_2 = \frac{1}{4e}\sum_{i=0}^{3}\Big\{\sum_{\alpha\in K^\times}\Big(\sum_{k=0}^{e-1}\omega_{i,k}\chi^k(\alpha)\Big)\Big(\sum_{k'=0}^{3}v_{i,k'}\varphi^{k'}(\alpha)\Big)\boldsymbol{o}$$

$$+ \sum_{\gamma\in K^\times}\sum_{k=0}^{e-1}\sum_{k'=0}^{3}\omega_{i,k}v_{i,k'}\chi^k(\gamma)\varphi^{k'}(\gamma)\varphi^{k'}(-1)\pi(\chi^k,\varphi^{k'})\gamma\Big\}.$$

From Lemma 3 and Theorem 3,

$$\begin{aligned}
P_2 &= \frac{1}{4e}\sum_{\gamma\in K^\times}\Big\{(4m(q^2-2)-\sum_{i=0}^{3}\Big(\sum_{k=0}^{e-1}\omega_{i,k}\chi^k(\gamma)-\omega_{i,0}\Big)\\
&\quad -m\sum_{i=0}^{3}\Big(\sum_{k'=0}^{3}v_{i,k'}\varphi^{k'}(\gamma)-v_{i,0}\Big)\\
&\quad +\sum_{k=1,k\neq\frac{e}{2}}^{e-1}(\sum_{i=0}^{3}\omega_{i,k}v_{i,2})\chi^k(\gamma)\varphi^2(\gamma)\pi(\chi^k,\varphi^2)+\sum_{i=0}^{3}\omega_{i,\frac{e}{2}}v_{i,2}\pi(\chi^{\frac{e}{2}},\varphi^2)\Big\}\gamma\\
&= \frac{1}{4e}\sum_{\gamma\in K^\times}\Big\{4mq^2-\sum_{i=0}^{3}\sum_{k=0}^{e-1}\omega_{i,k}\chi^k(\gamma)-m\sum_{i=0}^{3}\sum_{k'=0}^{3}v_{i,k'}\varphi^{k'}(\gamma)\\
&\quad +q\Big(\sum_{k=1}^{e-1}\Big(\sum_{i=0}^{3}\omega_{i,k}v_{i,2}\Big)\chi^k(\gamma)\varphi^2(\gamma)-\sum_{i=0}^{3}\omega_{i,\frac{e}{2}}v_{i,2}\Big)-\sum_{i=0}^{3}\omega_{i,\frac{e}{2}}v_{i,2}\Big\}\gamma.
\end{aligned}$$

Since $\displaystyle\sum_{i=0}^{3}v_{i,k'}=4$ if $k'=0$ and 0 if $k'\neq 0$, we have

$$\begin{aligned}
P_2 &= \frac{1}{4e}\sum_{\gamma\in K^\times}\Big((4m+1)(q^2-1)-\sum_{i=0}^{3}\sum_{k=0}^{e-1}\omega_{i,k}\chi^k(\gamma)+q\sum_{i=0}^{3}\sum_{k=0}^{e-1}\omega_{i,k}v_{i,2}\chi^k(\gamma)\varphi^2(\gamma)\Big)\gamma\\
&= \frac{q(q-1)}{4}\sum_{\gamma\in K^\times}\gamma-\frac{q+1}{2}\sum_{i=0}^{1}\sum_{\gamma\in\cup_{l\in A_i}S_l}\gamma.
\end{aligned}$$

(3) From $\chi^k(-1)=\varphi^k(-1)=1$,

$$P_3=\sum_{i=0}^{3}\sum_{\gamma\in K^\times}f_{C_i}(\alpha)\alpha\cdot\sum_{\beta\in K^\times}f_{A_i}(\beta)(-\beta)=P_2.$$

(4) Similarly we have, from Lemma 3 and Theorem 3,

$$\begin{aligned}
P_4 &= \sum_{i=0}^{3}\sum_{\alpha\in K^\times}f_{C_i}(\alpha)\alpha\cdot\sum_{\beta\in K^\times}f_{C_i}(\beta)(-\beta)\\
&= \frac{1}{4^2}\Big\{\sum_{i=0}^{3}\sum_{\alpha\in K^\times}\Big(\sum_{k=0}^{3}v_{i,k}\varphi^k(\alpha)\Big)^2\boldsymbol{o}\\
&\qquad +\sum_{\gamma\in K^\times}\sum_{k=0}^{3}\sum_{k'=0}^{3}\Big(\sum_{i=0}^{3}v_{i,k}v_{i,k'}\Big)\varphi^k(\gamma)\varphi^{k'}(\gamma)\pi(\varphi^k,\varphi^{k'})\gamma\Big\}\\
&= \frac{1}{4^2}\Big\{\sum_{i=0}^{3}\sum_{\alpha\in C_i}4^2\boldsymbol{o}+4\sum_{\gamma\in K^\times}\sum_{k=0}^{3}\pi(\varphi^k,\varphi^{-k})\gamma\Big\}\\
&= (q^2-1)\boldsymbol{o}+\frac{1}{4}\Big(q^2-5\Big)\sum_{\gamma\in K^\times}\gamma.
\end{aligned}$$

Altogether, we get

$$\sum_{k=0}^{3} \mathcal{D}_i \cdot \mathcal{D}_i^{-1} = \sum_{i=1}^{4} P_i = 2q(q-1)\boldsymbol{o} + q(q-2) \sum_{\gamma \in K^\times} \gamma.$$

Hence D_1, D_2, D_3, D_4 are $4 - \{q^2; \frac{1}{2}q(q-1); q(q-2)\}$ SDSs. □

We have a similar theorem for $q \equiv 3 \pmod 4$.

Let $q+1 \equiv 0 \pmod{2^t}$ for $t \geq 2$ such that $(q+1)/2^t$ is odd. Let $\Omega = \{0, 1, \cdots, e-1\} = \Omega_0 \cup \Omega_1$ be as in Theorem 8. Let A_i, $i \equiv 0 \pmod 2$, $0 \leq i \leq 2^t - 1$, be parts of a partition of Ω_1 and A_i, $i \equiv 1 \pmod 2$, $0 \leq i \leq 2^t - 1$, be parts of a partition of Ω_0. We assume $|A_i| = e/2^t = (q+1)/2^t$ and put $A_{i+2^t} = A_i$ for $i = 0, 1, \cdots, 2^t - 1$. Denote 2^{t+1}th cyclotomic classes by $C_0, C_1, \cdots, C_{2^{t+1}-1}$.

Lemma 4. *Let χ be a primitive eth power residue character and φ be a primitive 2^{t+1}th power residue character on $GF(q^2)$. We put $\omega_{i,k} = \sum_{l \in A_i} \zeta_e^{-lk}$ and $v_{i,k} = \zeta_{2^{t+1}}^{-ik}$. Then, for an element $\alpha \in GF(q^2)$,*

(1) $\displaystyle\sum_{i=0}^{2^{t+1}-1} \omega_{i,k} = \begin{cases} 2e & \textit{if } k = 0, \\ 0 & \textit{if } k \neq 0. \end{cases}$

(2) $\displaystyle\sum_{k=0}^{e-1} \omega_{i,k}\chi^k(\alpha) \cdot \sum_{k'=0}^{2^{t+1}-1} v_{i,k'}\varphi^{k'}(\alpha) = 0.$

(3) *If k' is odd ,* $\displaystyle\sum_{i=0}^{2^{t+1}-1} \omega_{i,k} v_{i,k'} = 0.$

(4) $\displaystyle\sum_{k=0}^{2^t-1} \omega_{i,-\frac{e}{2^t}k} v_{i,2k} = 0.$

(5) $\displaystyle\sum_{i=0}^{2^{t+1}-1} v_{i,2k} = \begin{cases} 2^{t+1} & \textit{if } k = 0, \\ 0 & \textit{if } k \neq 0. \end{cases}$

(6) $\displaystyle\sum_{i=0}^{2^{t+1}-1} v_{i,k} v_{i,k'} = \begin{cases} 2^{t+1} & \textit{if } k + k' = 0, \\ 0 & \textit{if } k + k' \neq 0. \end{cases}$

Proof. (1) $\displaystyle\sum_{i=0}^{2^{t+1}-1} \omega_{i,k} = \sum_{i=0}^{2^{t+1}-1} \sum_{l \in A_i} \zeta_e^{-lk} = 2\sum_{l \in \Omega} \zeta_e^{-lk} = 2e$ if $k = 0$ and 0 if $k \neq 0$.

(2) This follows from $\displaystyle\bigcup_{l \in A_i} S_l \cap C_i = \emptyset$.

(3) From $\omega_{i+2^t,k} = \omega_{i,k}$, we have

$$\sum_{i=0}^{2^{t+1}-1} \omega_{i,k} v_{i,k'} = \sum_{i=0}^{2^t-1} \omega_{i,k}\left(\zeta_{2^{t+1}}^{-ik'} + \zeta_{2^{t+1}}^{-(i+2^t)k'}\right) = 0.$$

(4) $\sum_{k=0}^{2^t-1} \omega_{i,-\frac{e}{2^t}k} v_{i,2k} = \sum_{k=0}^{2^t-1} \sum_{a\in A_i} \zeta_e^{\frac{e}{2^t}ka} \zeta_{2^{t+1}}^{-2ki} = \sum_{a\in A_i}\sum_{k=0}^{2^t-1} \zeta_{2^t}^{k(a-i)} = 0$, since $\bigcup_{l\in A_i} S_l \cap C_i = \emptyset$.

(5) $\sum_{i=0}^{2^{t+1}-1} v_{i,2k} = \sum_{i=0}^{2^{t+1}-1} \zeta_{2^t}^{-ki} = 2^{t+1}$ if $k=0$, and 0 if $k \neq 0$.

(6) $\sum_{i=0}^{2^{t+1}-1} v_{i,k} v_{i,k'} = \sum_{i=0}^{2^{t+1}-1} \zeta_{2^{t+1}}^{-i(k+k')} = 0$ if $k+k' \neq 0$ and 2^{t+1} if $k+k'=0$. □

Theorem 9. *We define the subsets* $D_i = \bigcup_{l\in A_i} S_l \cup C_i$ *of* K *for* $i = 0, 1, \cdots, 2^{t+1}-1$. *Then* $D_0, D_1, \cdots D_{2^{t+1}-1}$ *are* $2^{t+1} - \left\{q^2; \frac{3(q^2-1)}{2^{t+1}}; \frac{1}{2^{t+1}}\left(9q^2 - 3(2^{t+1}+3)\right)\right\}$ *SDSs.*

Proof. For every subset D_i, $|D_i| = |A_0||S_0| + |C_0| = \frac{3(q^2-1)}{2^{t+1}}$, so that $\sum_{i=0}^{2^{t+1}-1} |D_i| = 3(q^2-1)$. The characteristic function f_{D_i} is a sum of the characteristic function f_{A_i} of $\bigcup_{l\in A_i} S_l$ and the characteristic function f_{C_i} . We calculate the sum $\sum_{i=0}^{2^{t+1}-1} \mathcal{D}_i \cdot \mathcal{D}_i^{-1}$ by separating four terms similar to the proof of Theorem 8. Put $n = 2^{t+1}$ and $m = \omega_{i,0} = |A_i| = \frac{q+1}{2^{t+1}}$.

$$\begin{aligned}
(1)\ P_1 &= \sum_{i=0}^{n-1} \sum_{\alpha\in K^\times} f_{A_i}(\alpha)\alpha \cdot \sum_{\beta\in K^\times} f_{A_i}(\beta)(-\beta) \\
&= \frac{1}{e^2}\sum_{i=0}^{n-1}\Big\{ \sum_{\alpha\in K^\times}\Big(\sum_{k=0}^{e-1}\omega_{ik}\chi^k(\alpha)\Big)^2 \boldsymbol{o} \\
&+ \sum_{\gamma\in K^\times}\Big((m^2-m)e^2 - 2me\sum_{k=0}^{e-1}\omega_{ik}\chi^k(\gamma) + q\Big(\sum_{k=0}^{e-1}\omega_{ik}\chi^k(\gamma)\Big)^2\Big)\gamma\Big\}.
\end{aligned}$$

From Lemma 4,

$$P_1 = \frac{1}{e^2}n|A_0|\,|S_0|e^2\boldsymbol{o} + (m^2-m)n\sum_{\gamma\in K^\times}\gamma - \frac{2m}{e}\sum_{\gamma\in K^\times}\sum_{k=0}^{e-1}\Big(\sum_{i=0}^{n-1}\omega_{ik}\Big)\chi^k(\gamma)\gamma$$

$$+\frac{q}{e^2}\sum_{i=0}^{n-1}\sum_{\gamma\in K^\times}\Big(\sum_{k=0}^{e-1}\omega_{ik}\chi^k(\gamma)\Big)^2\gamma$$

$$= 2mn\frac{q^2-1}{q+1}\boldsymbol{o} + n(m^2-m)\sum_{\gamma\in K^\times}\gamma - \frac{2m}{e^2}2e^2\sum_{\gamma\in K^\times}\gamma + \frac{q}{e^2}2e^2\sum_{\gamma\in K^\times}\gamma$$

$$= 2(q^2-1)\boldsymbol{o} + \frac{1}{2^{t-1}}\Big(q^2-1-2^t\Big)\sum_{\gamma\in K^\times}\gamma.$$

(2) By Lemma 4 and Theorem 3,

$$P_2 = \sum_{i=0}^{n-1}\sum_{\alpha\in K^\times} f_{A_i}(\alpha)\alpha\cdot\sum_{\beta\in K^\times} f_{C_i}(\beta)(-\beta)$$

$$= \frac{1}{ne}\sum_{\gamma\in K^\times}\sum_{k=0}^{e-1}\sum_{k'=0}^{2^t-1}\sum_{i=0}^{n-1}\omega_{i,k}v_{i,2k'}\chi^k(\gamma)\varphi^{2k'}(\gamma)\pi(\chi^k,\varphi^{2k'})\gamma$$

$$= \frac{1}{ne}\sum_{\gamma\in K^\times}\Big\{\sum_{i=0}^{n-1}\omega_{i,0}v_{i,0}\pi(\chi^0,\chi^0) + \sum_{k=1}^{e-1}\Big(\sum_{i=0}^{n-1}\omega_{i,k}\Big)\chi^k(\gamma)\pi(\chi^k,\varphi^0)$$

$$+m\sum_{k'=1}^{2^t-1}\Big(\sum_{i=0}^{n-1}v_{i,2k'}\Big)\varphi^{2k'}(\gamma)\pi(\chi^0,\varphi^{2k'})$$

$$+\sum_{i=0}^{n-1}\sum_{k=1}^{e-1}\sum_{k'=1,k\neq-\frac{e}{2^t}k'}^{2^t-1}\omega_{i,k}v_{i,2k'}\chi^k(\gamma)\varphi^{2k'}(\gamma)\pi(\chi^k,\varphi^{2k'})$$

$$+\sum_{i=0}^{n-1}\sum_{k'=1}^{2^t-1}\omega_{i,-\frac{e}{2^t}k'}v_{i,2k'}\pi\big(\chi^{-\frac{e}{2^t}k'}\varphi^{2k'}\big)\Big\}\gamma$$

$$= \frac{1}{ne}\sum_{\gamma\in K^\times}\Big\{2^{t+1}m(q^2-2) + q\sum_{i=0}^{n-1}\Big(\sum_{k=1}^{e-1}\sum_{k'=1}^{2^t-1}\omega_{i,k}v_{i,2k'}\chi^k(\gamma)\varphi^{2k'}(\gamma)$$

$$-\sum_{k'=1}^{2^t-1}\omega_{i,-\frac{e}{2^t}k'}v_{i,2k'}\Big) - \sum_{i=0}^{n-1}\sum_{k'=1}^{2^t-1}\omega_{i,-\frac{e}{2^t}k'}v_{i,2k'}\Big\}\gamma$$

$$= \frac{1}{ne}\sum_{\gamma\in K^\times}\Big(2^{t+1}m(q^2-2+q+1)-2qe\Big)\gamma$$

$$= \frac{q^2-1}{2^t}\sum_{\gamma\in K^\times}\gamma.$$

(3) Similar to the proof of Theorem 8, we obtain $P_3 = P_2$.

(4) By Lemma 4 and Theorem 3,

$$\begin{aligned} P_4 &= \sum_{i=0}^{n-1} \sum_{\alpha\in K^\times} f_{C_i}(\alpha)\alpha \cdot \sum_{\beta\in K^\times} f_{C_i}(\beta)(-\beta) \\ &= \frac{1}{n^2}\Big\{\sum_{i=0}^{n-1} \sum_{\alpha\in K^\times} \Big(\sum_{k=0}^{n-1} v_{i,k}\varphi^k(\alpha)\Big)^2 \boldsymbol{o} \\ &\quad + \sum_{\gamma\in K^\times} \sum_{k=0}^{n-1}\sum_{k'=0}^{n-1}\Big(\sum_{i=0}^{n-1} v_{i,k}v_{i,k'}\Big)\varphi^{k+k'}(\gamma)\varphi^{k'}(-1)\pi(\varphi^k,\varphi^{k'})\gamma\Big\} \\ &= (q^2-1)\boldsymbol{o} + \frac{2^{t+1}}{n^2}\sum_{\gamma\in K^\times}\sum_{k=0}^{n-1}\pi(\varphi^k,\varphi^{-k})\varphi^k(-1)\gamma \\ &= (q^2-1)\boldsymbol{o} + \frac{1}{n}\Big(q^2-1-2^{t+1}\Big)\sum_{\gamma\in K^\times}\gamma. \end{aligned}$$

Altogether, we obtain

$$\sum_{i=0}^{n-1}\mathcal{D}_i\cdot\mathcal{D}_i^{-1} = 3(q^2-1)\boldsymbol{o} + \frac{1}{2^{t+1}}\Big(9q^2-3(2^{t+1}+3)\Big)\sum_{\gamma\in K^\times}\gamma.$$

Thus $D_0, D_1, \cdots, D_{2^{t+1}-1}$ are $2^{t+1} - \Big\{q^2; \frac{3(q^2-1)}{2^{t+1}}; \frac{1}{2^{t+1}}\Big(9q^2-3(2^{t+1}+3)\Big)\Big\}$ SDSs. □

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(Received 14 July 2006; revised 30 Oct 2006)