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A class of antiautomorphisms of Mendelsohn triple systems with two cycles

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Abstract

A cyclic triple, (a, b, c), is defined to be the set $\{(a, b), (b, c), (c, a)\}$ of ordered pairs. A Mendelsohn triple system of order v, MTS(v), is a pair, (M, β), where M is a set of v points and β is a collection of cyclic triples of pairwise distinct points of M such that any ordered pair of distinct points of M is contained in precisely one cyclic triple of β. An antiautomorphism of a Mendelsohn triple system, (M, β), is a permutation of M which maps β to β^{-1}, where $\beta^{-1} = \{(c, b, a) \mid (a, b, c) \in \beta\}$. In this paper we give necessary and sufficient conditions for the existence of a Mendelsohn triple system of order v admitting an antiautomorphism consisting of two cycles, where one cycle is twice the length of the other.

1 Introduction

A *Steiner triple system of order* v, STS(v), is a pair, (S, β), where S is a set of v points and β is a collection of 3-element subsets of S, called *blocks*, such that any pair of distinct points of S is contained in precisely one block of β. Kirkman [6] has shown that there is an STS(v) if and only if $v \equiv 1$ or $3 \pmod 6$ or $v = 0$.

An *automorphism* of an STS(v) is a permutation of S which maps β to itself. An automorphism, α, on an STS(v), (S, β), is called *cyclic* if the permutation defined by α consists of a single cycle of length v. Peltesohn [8] proved that there exists an STS(v) admitting a cyclic automorphism if and only if $v \equiv 1$ or $3 \pmod 6$ with $v \neq 9$.

We call an automorphism, α, on an STS(v), (S, β), *bicyclic* if the permutation defined by α consists of two cycles. Calahan-Zijlstra and Gardner [1] have shown that there exists an STS(v) admitting a bicyclic automorphism having two cycles of lengths M and N, with $1 < M < N$, if and only if $M \equiv 1$ or $3 \pmod 6$, $M \neq 9$, $M|N$, and $M + N \equiv 1$ or $3 \pmod 6$.

A *cyclic triple*, (a, b, c), is defined to be the set $\{(a, b), (b, c), (c, a)\}$ of ordered pairs. A *Mendelsohn triple system of order* v, MTS(v), is a pair, (M, β), where M is a set of v points and β is a collection of cyclic triples of pairwise distinct points of M, called *triples*, such that any ordered pair of distinct points of M is contained in precisely one element of β. Mendelsohn [7] has shown that necessary and sufficient conditions for the existence of an MTS(v) are that $v \equiv 0$ or 1 (mod 3), with $v \neq 6$.

For an MTS(v), (M, β), we define β^{-1} by $\beta^{-1} = \{(c, b, a) \mid (a, b, c) \in \beta\}$. Then (M, β^{-1}) is an MTS(v) and is called the *converse* of (M, β). An MTS(v) which is isomorphic to its converse is said to be *self-converse*. Kang, Chang, and Yang [5] have shown that a self-converse MTS(v) exists if and only if $v \equiv 0$ or 1 (mod 3) and $v \neq 6$. An *automorphism* of (M, β) is a permutation of M which maps β to itself. An *antiautomorphism* of (M, β) is a permutation of M which maps β to β^{-1}. Clearly, an MTS(v) is self-converse if and only if it admits an antiautomorphism.

An antiautomorphism, α, on an MTS(v), (M, β), is called *cyclic* if the permutation defined by α consists of a single cycle of length v. Carnes [2] proved that there exists an MTS(v) admitting a cyclic antiautomorphism if and only if $v \equiv 0$ or 4 (mod 12).

We call an antiautomorphism, α, on an MTS(v), (M, β), *bicyclic* if the permutation defined by α consists of two cycles of lengths M and N, where $v = M + N$. Carnes, Dye, and Reed [3] have shown that there exists an MTS(v) admitting a bicyclic antiautomorphism having two cycles of equal length if and only if $v \equiv 0$ or 16 (mod 24). Carnes [4] has shown that if an MTS(v) admits a bicyclic antiautomorphism with cycles of length M and N, where $1 < M < N$, then $M|N$, $v = M + N \equiv 0$ or 1 (mod 3), and $M \equiv 0$ or 4 (mod 12) if M is even or $M \equiv 1$ or 3 (mod 6) if M is odd and $M > 2N$.

In this paper we consider the special case where $N = 2M$.

2 Preliminaries

If K is the length of a cycle, $K \in \{M, N\}$, we let the cycles be $(0_i, 1_i, 2_i, \ldots, (K-1)_i)$, $i \in \{0, 1\}$. Let $\Delta = \{0, 1, 2, \ldots, (K-1)\}$. We shall use all additions modulo K in the triples. For $(a_i, b_j, c_k) \in \beta$, i, j, $k \in \{0, 1\}$, let *orbit* $(a_i, b_j, c_k) = \{((a+t)_i, (b+t)_j, (c+t)_k) \mid t \in \Delta, t \text{ even}\} \cup \{((c+t)_k, (b+t)_j, (a+t)_i) \mid t \in \Delta, t \text{ odd}\}$. Clearly the orbits partition β.

We say that a collection of triples, $\bar{\beta}$, is a collection of *base triples* of an MTS(v) under α if the orbits of the triples of $\bar{\beta}$ produce β and exactly one triple of each orbit occurs in $\bar{\beta}$. Also, we say that the *reverse* of the triple (a, b, c) is the triple (c, b, a).

Let (S, β') be an STS(v). Let $\beta = \{(a, b, c), (c, b, a) \mid \{a, b, c\} \in \beta'\}$. Then (S, β) is called the *corresponding* MTS(v), and the identity map on the point set is an antiautomorphism. The previously mentioned results on Steiner triple systems then yield the corresponding Mendelsohn triple systems with the appropriate structures.

3 M even

Lemma 3.1 *If $v = M + N$, $N = 2M$, and $M \equiv 0$ or $4 \pmod{12}$, there exists an MTS(v) which admits a bicyclic antiautomorphism where M and N are the lengths of the cycles.*

Proof: Let $M = 12k$, $N = 24k$.

For $k \geq 1$ the base triples include the following:
$(0_1, 2_1, (12k)_1)$, $(0_1, (12k-1)_1, (12k+1)_1)$,
along with the following and its reverse:
$(0_1, (8k)_1, (16k)_1)$.

For $k = 1$ the base triples include the following:
$(0_1, 5_1, 10_1)$,
along with the following and their reverses:
$(0_0, 0_1, 9_1)$, $(0_0, 1_1, 8_1)$, $(0_0, 2_1, 3_1)$, $(0_0, 4_1, 7_1)$, $(0_0, 5_1, 11_1)$, $(0_0, 6_1, 10_1)$.

For $k = 2$ the base triples include the following:
$(0_1, 11_1, 22_1)$,
along with the following and their reverses:
$(0_1, 3_1, 8_1)$, $(0_1, 18_1, 38_1)$, $(0_0, 0_1, 21_1)$, $(0_0, 1_1, 20_1)$, $(0_0, 2_1, 19_1)$,
$(0_0, 3_1, 18_1)$, $(0_0, 4_1, 17_1)$, $(0_0, 5_1, 11_1)$, $(0_0, 6_1, 13_1)$, $(0_0, 7_1, 16_1)$,
$(0_0, 8_1, 12_1)$, $(0_0, 9_1, 23_1)$, $(0_0, 10_1, 22_1)$, $(0_0, 14_1, 15_1)$.

For $k \geq 3$ the base triples include the following:
$(0_1, (6k+1)_1, (18k-1)_1)$,
along with the following and their reverses:
$(0_0, (2k+t+1)_1, (10k-t-1)_1)$ for $t = 0, 1, \ldots, k-1$,
$(0_0, (3k+t+1)_1, (21k-t-2)_1)$ for $t = 0, 1, \ldots, 3k-3$,
$(0_0, (2+t)_1, (12k-t-2)_1)$ for $t = 0, 1, \ldots, 2k-3$,
$(0_1, (2k+t+1)_1, (4k-t)_1)$ for $t = 0, 1, \ldots, k-2$,
$(0_0, 0_1, 1_1)$, $(0_0, (6k-1)_1, (10k)_1)$, $(0_0, (6k)_1, (12k-1)_1)$, $(0_0, (2k)_1, (21k-1)_1)$,
$(0_1, (3k)_1, (5k)_1)$, $(0_1, (3k+1)_1, (5k-1)_1)$.

For $k \geq 4$ the base triples include the following and their reverses:
$(0_1, (4k+t+2)_1, (6k-t-2)_1)$ for $t = 0, 1, \ldots, k-4$.

For $k \geq 1$, the remaining triples in the cycle of length M are from a cyclic MTS(M) from [2].

Let $M = 12k + 4$, $N = 24k + 8$.

For $k = 0$ the base triples include the following:
$(0_0, 0_1, 4_1)$, $(0_0, 1_1, 2_1)$, $(0_0, 2_1, 3_1)$, $(0_0, 3_1, 1_1)$, $(0_1, 3_1, 5_1)$.

For $k \geq 1$ the base triples include the following:
$(0_1, 2_1, (12k+4)_1)$, $(0_1, (12k+3)_1, (12k+5)_1)$, $(0_1, (12k+6)_1, (18k+7)_1)$,
along with the following and their reverses:
$(0_0, (t+2)_1, (12k-t+3)_1)$ for $t = 0, 1, \ldots, 3k-1$,

$(0_0,(3k+t+2)_1,(21k-t+6)_1)$ for $t=0,1,\ldots,3k-2$,
$(0_0,0_1,(6k+2)_1)$, $(0_0,1_1,(6k+1)_1)$, $(0_0,(6k+3)_1,(9k+3)_1)$, $(0_1,(5k-1)_1,(5k)_1)$.

For $k\geq 2$ the base triples include the following and their reverses:
$(0_1,(2k+t+1)_1,(4k-t)_1)$ for $t=0,1,\ldots,k-2$,
$(0_1,(3k+1)_1,(5k+1)_1)$.

For $k\geq 3$ the base triples include the following and their reverses:
$(0_1,(4k+t+1)_1,(6k-t-1)_1)$ for $t=0,1,\ldots,k-3$.

For $k\geq 0$, the remaining triples in the cycle of length M are from a cyclic MTS(M) from [2]. □

4 M odd

Lemma 4.1 *If M is odd, there exists an MTS(v) which admits a bicyclic antiautomorphism where $v=M+N$, $N=2M$, M and N being the lengths of the cycles.*

Proof: By the result of Calahan-Zijlstra and Gardner [1], there is an MTS(v) which admits a bicyclic antiautomorphism where $v=M+N$, $N=2M$, M and N being the lengths of the cycles, if $M\equiv 1$ or $3 \pmod 6$, $M\neq 9$. The cases left are $M=9$ and $M\equiv 5 \pmod 6$. We consider the general cases modulo 24.

Let $M=9$, $N=18$.

The base triples are the following:
$(0_0,3_0,4_0)$, $(0_0,0_1,2_0)$, $(0_0,1_1,5_1)$, $(0_0,5_1,13_1)$, $(0_0,9_1,0_1)$, $(0_0,10_1,4_1)$,
$(0_0,16_1,7_1)$, $(0_1,4_1,10_1)$,
along with the following and their reverses:
$(0_0,2_1,3_1)$, $(0_0,6_1,17_1)$, $(0_1,3_1,5_1)$.

Let $M=24k+5$, $N=48k+10$.

For $k\geq 0$, the base triples include the following:
$(0_0,0_1,2_0)$, $(0_0,(12k+1)_0,(12k+1)_1)$, $(0_0,(12k+1)_1,(36k+6)_1)$,
$(0_0,(48k+8)_1,(24k+3)_1)$,
along with the following and their reverses:
$(0_0,2_1,(12k+4)_1)$ and $(0_1,(24k+1)_1,(24k+4)_1)$.

For $k\geq 1$, the base triples include the following and their reverses:
$(0_0,(t+3)_1,(24k-t+2)_1)$ for $t=0,1,\ldots,12k-3$,
$(0_0,1_1,(24k+4)_1)$, $(0_0,(12k+2)_1,(12k+3)_1)$,
$(0_1,(8k+2t+2)_1,(16k-2t+2)_1)$ for $t=0,1,\ldots,2k-1$,
$(0_1,(16k+2t+4)_1,(24k-2t+2)_1)$ for $t=0,1,\ldots,2k-1$.

For $k=1$, the base triples include the following:
$(0_0,7_0,8_0)$, $(0_0,9_0,12_0)$, $(0_0,10_0,14_0)$, $(0_0,6_0,11_0)$.

For $k \geq 2$, the base triples include the following:
$(0_0, (4k+t+2)_0, (8k-t+1)_0)$ for $t = 0, 1, \ldots, 2k-2$.

For $k = 2$, the base triples include the following:
$(0_0, 19_0, 20_0)$, $(0_0, 22_0, 26_0)$, $(0_0, 18_0, 24_0)$, $(0_0, 13_0, 21_0)$, $(0_0, 14_0, 23_0)$.

For $k \geq 3$, the base triples include the following:
$(0_0, (8k+t+4)_0, (12k-t)_0)$ for $t = 0, 1, \ldots, k-2$,
$(0_0, (6k+1)_0, (10k+2)_0)$, $(0_0, (6k+2)_0, (10k)_0)$, $(0_0, (8k+2)_0, (12k+2)_0)$,
$(0_0, (8k+3)_0, (10k+1)_0)$, $(0_0, (11k)_0, (11k+1)_0)$.

For $k \geq 4$, the base triples include the following:
$(0_0, (9k+t+3)_0, (11k-t-1)_0)$ for $t = 0, 1, \ldots, k-4$.

Let $M = 24k + 11$, $N = 48k + 22$.

For $k \geq 0$, the base triples include the following:
$(0_0, 0_1, 2_0)$, $(0_0, (12k+6)_0, 0_1)$, $(0_0, (12k+5)_1, (36k+16)_1)$,
$(0_0, (48k+20)_1, (24k+9)_1)$,
along with the following and their reverses:
$(0_0, (t+3)_1, (24k-t+8)_1)$ for $t = 0, 1, \ldots, 12k+1$,
$(0_0, 1_1, (24k+10)_1)$, $(0_0, 2_1, (12k+6)_1)$, $(0_1, (24k+6)_1, (24k+7)_1)$,
$(0_1, (24k+8)_1, (24k+10)_1)$.

For $k = 0$, the base triples include the following:
$(0_0, 3_0, 4_0)$.

For $k \geq 1$, the base triples include the following:
$(0_0, (4k+t+4)_0, (8k-t+3)_0)$ for $t = 0, 1, \ldots, 2k-2$,
along with the following and their reverses:
$(0_1, (8k+2t+4)_1, (16k-2t+6)_1)$ for $t = 0, 1, \ldots, 2k-1$,
$(0_1, (16k+2t+8)_1, (24k-2t+4)_1)$ for $t = 0, 1, \ldots, 2k-2$,
$(0_1, (12k+6)_1, (20k+6)_1)$.

For $k = 1$, the base triples include the following:
$(0_0, 14_0, 15_0)$, $(0_0, 9_0, 13_0)$, $(0_0, 7_0, 12_0)$, $(0_0, 10_0, 16_0)$.

For $k \geq 2$, the base triples include the following:
$(0_0, (8k+t+6)_0, (12k-t+4)_0)$ for $t = 0, 1, \ldots, k-2$,
$(0_0, (4k+3)_0, (8k+5)_0)$, $(0_0, (6k+3)_0, (10k+3)_0)$, $(0_0, (6k+4)_0, (10k+5)_0)$,
$(0_0, (8k+4)_0, (10k+4)_0)$, $(0_0, (11k+4)_0, (11k+5)_0)$.

For $k \geq 3$, the base triples include the following:
$(0_0, (9k+t+5)_0, (11k-t+3)_0)$ for $t = 0, 1, \ldots, k-3$.

Let $M = 24k + 17$, $N = 48k + 34$.

For $k \geq 0$, the base triples include the following:
$(0_0, 0_1, 2_0)$, $(0_0, (12k+8)_0, 0_1)$, $(0_0, (12k+9)_1, (36k+26)_1)$,
$(0_0, (48k+32)_1, (24k+15)_1)$,
along with the following and their reverses:

$(0_0,(t+3)_1,(24k-t+14)_1)$ for $t=0,1,\dots,12k+4$,
$(0_0,1_1,(24k+16)_1)$, $(0_0,2_1,(12k+8)_1)$, $(0_1,(12k+8)_1,(12k+10)_1)$,
$(0_1,(20k+12)_1,(20k+16)_1)$, $(0_1,(24k+13)_1,(24k+14)_1)$.

For $k=0$, the base triples include the following:
$(0_0,5_0,6_0)$, $(0_0,4_0,7_0)$.

For $k\geq 1$, the base triples include the following:
$(0_0,(4k+t+4)_0,(8k-t+3)_0)$ for $t=0,1,\dots,2k-2$,
along with the following and their reverses:
$(0_1,(8k+2t+6)_1,(16k-2t+10)_1)$ for $t=0,1,\dots,2k-1$,
$(0_1,(16k+4t+12)_1,(24k-4t+10)_1)$ for $t=0,1,\dots,k-1$,
$(0_1,(16k+4t+14)_1,(24k-4t+16)_1)$ for $t=0,1,\dots,k-1$.

For $k=1$, the base triples include the following:
$(0_0,16_0,17_0)$, $(0_0,10_0,14_0)$, $(0_0,13_0,18_0)$, $(0_0,9_0,15_0)$, $(0_0,12_0,19_0)$.

For $k\geq 2$, the base triples include the following:
$(0_0,(8k+t+5)_0,(12k-t+7)_0)$ for $t=0,1,\dots,k$,
$(0_0,(6k+3)_0,(10k+6)_0)$, $(0_0,(6k+4)_0,(10k+5)_0)$, $(0_0,(8k+4)_0,(10k+4)_0)$,
$(0_0,(11k+5)_0,(11k+6)_0)$.

For $k\geq 3$, the base triples include the following:
$(0_0,(9k+t+6)_0,(11k-t+4)_0)$ for $t=0,1,\dots,k-3$.

Let $M=24k+23$, $N=48k+46$.

For $k\geq 0$, the base triples include the following:
$(0_0,0_1,2_0)$, $(0_0,(12k+10)_0,0_1)$, $(0_0,(12k+13)_1,(36k+36)_1)$,
$(0_0,(48k+44)_1,(24k+21)_1)$,
along with the following and their reverses:
$(0_0,(t+3)_1,(24k-t+20)_1)$ for $t=0,1,\dots,12k+6$,
$(0_0,1_1,(24k+22)_1)$, $(0_0,2_1,(12k+12)_1)$, $(0_0,(12k+10)_1,(12k+11)_1)$,
$(0_1,(8k+2t+8)_1,(16k-2t+12)_1)$ for $t=0,1,\dots,2k$,
$(0_1,(16k+2t+14)_1,(24k-2t+20)_1)$ for $t=0,1,\dots,2k+1$,
$(0_1,(24k+19)_1,(24k+22)_1)$.

For $k=0$, the base triples include the following:
$(0_0,6_0,7_0)$, $(0_0,8_0,11_0)$, $(0_0,5_0,9_0)$.

For $k\geq 1$, the base triples include the following:
$(0_0,(4k+t+6)_0,(8k-t+5)_0)$ for $t=0,1,\dots,2k-2$,
$(0_0,(4k+5)_0,(8k+7)_0)$, $(0_0,(6k+5)_0,(10k+9)_0)$, $(0_0,(6k+6)_0,(10k+7)_0)$,
$(0_0,(8k+6)_0,(10k+8)_0)$, $(0_0,(8k+8)_0,(12k+11)_0)$, $(0_0,(11k+9)_0,(11k+10)_0)$.

For $k\geq 2$, the base triples include the following:
$(0_0,(8k+t+9)_0,(12k-t+9)_0)$ for $t=0,1,\dots,k-2$,
$(0_0,(9k+t+8)_0,(11k-t+8)_0)$ for $t=0,1,\dots,k-2$. □

5 Conclusion

By the lemmas in the previous two sections, we have the following theorem.

Theorem 5.1 *There exists an MTS(v) which admits a bicyclic antiautomorphism where* $v = M + N$, $N = 2M$, *M and N being the lengths of the cycles, if and only if M is odd or* $M \equiv 0$ *or* $4 \pmod{12}$.

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