Letter representations of $m \times n \times p$ proper arrays

Jocelyn Quaintance

Department of Mathematics
West Virginia University
Morgantown, WV 26506
U.S.A.
`jquainta@mix.wvu.edu`

Abstract

Let $m \neq n$. An $m \times n \times p$ *proper array* is a three-dimensional rectangular array composed of directed cubes that obeys certain constraints. Because of these constraints, the $m \times n \times p$ proper arrays may be classified via a schema in which each $m \times n \times p$ proper array is associated with a particular $m \times n$ planar face. By representing each connected component present in the $m \times n$ planar face with a distinct letter, an $m \times n$ array of letters is formed. This $m \times n$ array of letters is the *letter representation* associated with the $m \times n \times p$ proper array. The main result of this paper involves the enumeration of all $m \times n$ letter representations modulo symmetry, where the symmetry is derived from the group $D_2 = C_2 \times C_2$ acting on the set of letter representations. The enumeration is achieved by forming a linear combination of four exponential generating functions, each of which is derived from a particular symmetry operation. This linear combination counts the number of partitions of the set of $m \times n$ letter representations that are inequivalent under D_2.

0   Introduction

0.1   History of the Problem: Two-Dimensional Proper Arrays

This article is an extension of the author's work involving the enumeration of two-dimensional proper arrays [1], [2], [3]. In both the two-dimensional and three-dimensional situations, the basic building block of a proper array is a **directed cube**, where a directed cube is a cube with five faces that containing indentations while the sixth face has a cylindrical plug or **connector**.
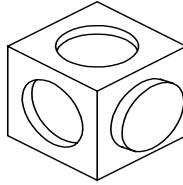
Figure 1: A directed cube. In this illustration, the connector extends from the right face while the other five faces contain circular indentations.

Rectangular arrays can be built by fitting the connector of one directed cube into an indentation of another. If the ensuing rectangular array, which consists m rows and n columnns, obeys certain connectivity constraints, it is said to be an $\boldsymbol{m \times n}$ **proper array** [1], [2], [3]. The $m \times n$ proper array may be classified by its rightmost column of directed cubes. The right edge structure contains all of the path-connected subsets that appear in the $m \times n$ proper array and all of the connectors that extend from this edge.

The right edge structure does not uniquely determine the number $m \times n$ proper arrays since two different $m \times n$ proper arrays may have the same right edge structure. However, the right edge structure is the state for a transition matrix, $\boldsymbol{M_m}$, which for a fixed m and arbitrary n, describes how an $m \times n$ proper array evolves into an $m \times (n+1)$ proper array [1], [2], [12]. Note that $\boldsymbol{M_m}$ is an $r \times r$ matrix, where r is the number, modulo symmetry, of possible rightmost columns associated with $m \times n$ proper arrays [1],[2],[12]. Let $\boldsymbol{S_n}$ be the $1 \times r$ vector $(s_{n,i})$, where $s_{n,i}$ counts the number of $m \times n$ proper arrays, modulo symmetry, having the ith kind of right edge. We find $\boldsymbol{S_n M_m = S_{n+1}}$. For example, in the case of $2 \times n$ proper arrays, there are four possible right edge structures. Hence, $\boldsymbol{M_2}$ is a 4×4 matrix. Let $\boldsymbol{S_1} = [1,\ 2,\ 0,\ 1]$, where the entries of $\boldsymbol{S_1}$ count the number of 2×1 proper arrays, with a particular right edge structure. Note, there are $1 + 2 + 0 + 1 = 4$ distinct 2×1 proper arrays. Then $\boldsymbol{S_1 M_2 = S_2} = [2,\ 8,\ 0,\ 3]$, where the entries of $\boldsymbol{S_2}$ count the number of 2×2 proper arrays with a particular right edge structure. Note, there are $2+8+0+3 = 13$ distinct 2×2 proper arrays. Thus, the right edge structure of an $m \times n$ proper array can determine the total number, modulo symmetry, of $m \times n$ proper arrays [1],[2],[12].

For a fixed n, the author wrote a computer program which constructs $\boldsymbol{M_m}$. In the process of writing this program, the author realized that one of the main research questions associated with the enumeration of $m \times n$ proper is determining, for any given m, the size of r, where r is the number of right edge structures, modulo symmetry, associated with $m \times n$ proper arrays. By having such a formula, the author and various computer scientists [5] would be able to determine the amount of computer time and memory necessary for the construction of $\boldsymbol{M_m}$. This formula consists of a linear combination of four ordinary generating functions, each of which

enumerate a particular subset of right edge configurations [1], [3]. Thus, for the two-dimensional situation, the basis size for $\boldsymbol{M_m}$ is completely determined.

m	1	2	3	4	5	6
$\boldsymbol{M_m}$	1	4	16	72	364	1916

Table 1: The basis size for the transition matrix $\boldsymbol{M_m}$.

0.2 Three-Dimensional Proper Arrays

Next the author decided to enumerate three-dimensional proper arrays. An $\boldsymbol{m \times n \times p}$ **proper array** is an $m \times n \times p$ array of directed cubes that obey certain constraints. These constraints allow the set of $m \times n \times p$ proper arrays to be classified via a schema in which each $m \times n \times p$ proper array is associated with a particular $m \times n$ planar face, called the **preferred face**. The preferred face of the $m \times n \times p$ proper array contains all the path-connected subsets that appear in the $m \times n \times p$ proper array and the number of connectors that extend from the preferred face.

After defining the notion of an $m \times n \times p$ proper array, it was an easy exercise to adapt the computer program which constructed $\boldsymbol{M_m}$ into one which constructed $\boldsymbol{M_{m \times n}}$, where $\boldsymbol{M_{m \times n}}$ is the transition matrix that describes how an $m \times n \times p$ proper array evolves into an $m \times n \times (p+1)$ proper array. Note that $\boldsymbol{M_{m \times n}}$ is an $r \times r$ matrix, where r counts, modulo symmetry, the number of $m \times n$ preferred faces associated with $m \times n \times p$ proper arrays. The most important adaptation involved the varying numbers of symmetry maps. Unlike $m \times n$ proper arrays, which have only one symmetry map, $m \times n \times p$ proper arrays have four or eight symmetry maps. More precisely, if $m \neq n$, an $m \times n \times p$ proper array possesses the rectangular symmetry of D_2. However, when $m = n$, an $m \times m \times p$ proper array has D_4 symmetry. Each of these symmetry maps must take the preferred face onto itself.

Once again, we would like to find a formula that predicts r, the basis size of $\boldsymbol{M_{m \times n}}$. The main result of this paper is a first step to constructing such a forumla, when $m \neq n$. In particular, Theorem 2.1 provides a formula, in the form of a linear combination of four exponential generating functions, for calculating, modulo D_2 symmetry, the number of preferred face structures associated with $m \times n \times p$ proper arrays. The techniques used to derive the generating functions are similar in nature to the techniques used by Yoshinaga and Mori [7] and David Branson [8]. The second result provides generating functions which count the number of $m \times n \times p$ proper arrays whose associated $m \times n$ preferred face is fixed by the four symmetry maps of D_2.

0.3 Connections to Percolation Theory

Before continuing, we would like to provide a physical interpretation for an $m \times n \times p$ proper array. If the center of each directed cube is represented as a point of Z^3, with

the connectors providing a network of open paths between the various vertices, then the $m \times n \times p$ proper array is a model of bond percolation [10 Section 1], [11 p. 16]. Furthermore, this bond percolation has the following property, namely, there exists an open cluster that allows water to flow from the back face of the $m \times n \times p$ array to its front face [10 p. 2]. Moreover, since there is a relationship between bond percolation and the Ising Model [10 p. 8], an open question involves the exact nature of the connection between $m \times n \times p$ proper arrays and Ising Models. At this point in time, we have not investigated this connection between the Ising Model and the three-dimensional proper arrays and leave it as an open question for future research.

1 Proper Arrays and Letter Representations

We are now ready to give the definition of an $m \times n \times p$ proper array.

Definition: A (connected) **component** of an array of directed cubes is a (path) connected subset of the array.

Definition: Let A represent an $m \times n \times p$ array of directed cubes. Orient A so that one planar $m \times n$ face has a center at $(0, 1, 0)$ and is perpendicular to the y axis. This $m \times n$ planar face is the **preferred face** of A.

Definition: An $\boldsymbol{m \times n \times p}$ **proper array** is a three-dimensional array of directed cubes that obeys the following two conditions:

1. The array is **5-way flat**. That is, only the preferred face has outward pointing connectors.
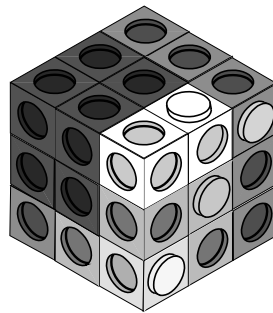

Figure 1.1: This $3 \times 3 \times 3$ array of directed cubes is not 5-way flat since the white component in the upper left corner has a connector that extends from the top face.

2. We define an **island** to be a component that does not reach the preferred face. The array <u>can not</u> have islands.
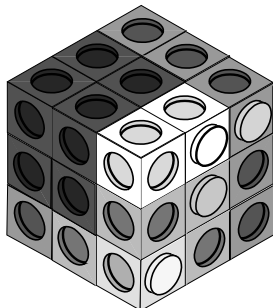
Figure 1.2: The darkest gray (purple) component, which is directly behind the white component, forms an island since it does not reach the preferred face.

An $m \times n \times p$ proper array may be associated with its preferred face, since the preferred face corresponds to the state of a transition matrix. In particular, by letting each distinct component be denoted by a distinct letter, and any outward pointing connectors be depicted as circles around the letters, the geometry of the preferred face is recorded as $m \times n$ array of circled letters. This $m \times n$ array of circled letters is called the **word representation** of the $m \times n \times p$ proper array. It encodes the number of connected components that appear in the $m \times n \times p$ proper array and the connectors that extend from the $m \times n \times p$ proper array. By deleting the circles in the word representation, the **letter representation** of the $m \times n \times p$ proper array is formed.
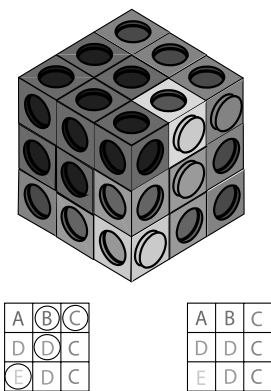

Figure 1.3: An example of a $3 \times 3 \times 3$ proper array. At the bottom left corner, the word representation is given. This word representation records the components as (colored) letters and outward pointing connectors as circles. At the bottom right corner, the letter representation is given.

Remark 1.1 *Letter Representations are partitions of the preferred face. Hence, two letter representations are regarded as the same if they have the same components but their letters are different. In other words, the letters are just labels for the components and the labels are unimportant. For example, take the letter representation provided in Figure 1.3. If we replace the A with an X, the resulting letter representation is the same as the orginal letter representation.*

The definition of an $m \times n \times p$ proper array is motivated by the desire to ensure that a particular $m \times n$ face of the proper array corresponds to the state of the transition matrix [1], [2], [12]. In order to see how the definition of proper array guarantees this correspondence, we will take a moment to describe how the transition matrix constructs proper arrays. Fix m and n. The matrix multiplication $\boldsymbol{S_{p-1}M_{m \times n}}$ corresponds to attaching an $m \times n$ layer of directed cubes onto a previous constructed $m \times n \times (p-1)$ proper array. The concept of 5-way flat ensures this conncection could only occur on the preferred face. The concept of no islands ensures there are no connected subsets of the $m \times n \times (p-1)$ proper array that do not reach its preferred face. Hence, all the connected subsets of the $m \times n \times (p-1)$ proper array will be affected by the attachment of the new $m \times n$ layer and will percolate throughout the newly constructed $m \times n \times p$ proper array. Since this percolation occurs, we are able to record the state of construction by simply looking at the geometry of the $m \times n$ preferred face of the newly build $m \times n \times p$ proper array. Without the conditions of 5-way flat and no islands, this correspondence between preferred face and construction state would not exist.

2   Enumerating Letter Representations

For the remainder of this paper, we can ignore the three-dimensional context provided by the proper array and work in the two-dimensional setting of the letter representation.

Our goal is to fix m and n, and enumerate, modulo symmetry, all letter representations associated with $m \times n \times p$ proper arrays. If $m \neq n$, the symmetry equivalence is determined by D_2.

Definition: A $\boldsymbol{m \times n}$ **letter array** is an $m \times n$ arrangement of letters which obeys the following condition: Each distinct letter labels an element in the partition of the $m \times n$ array of squares. The comments of Remark 1.1 apply to the set of $m \times n$ letter arrays.

It is an easy exercise to show that $B_{m,n}$, the cardinality of the entire set of $m \times n$ letter arrays, equals to the mnth Bell Number. Define $L_{m,n}$ to be the number of $m \times n$ letter representations modulo D_2 symmetry. Note that $L_{m,n}$ is a lower bound for the basis of the transition matrix $\boldsymbol{M_{m \times n}}$. We calculate $L_{m,n}$ as follows.
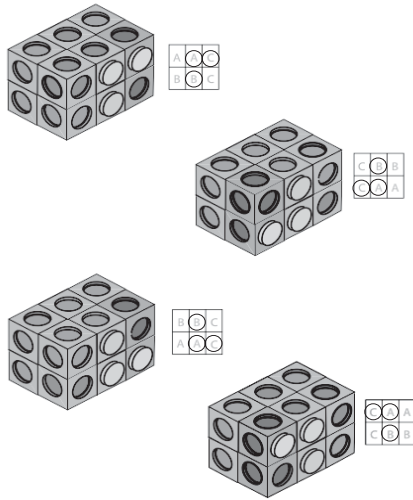
Figure 2.1: The four equivalent versions of a $2 \times 3 \times 2$ array. Let the diagram in the upper left corner represent the orginal array. Then by reading in a clockwise manner, the next diagram depicts a 180° rotation of the orginal array. The bottom right corner depicts a vertical reflection of the orginal array. The remaining diagram depicts a horizontal reflection of the orginal array. These four symmetry images are only counted once in the enumeration procedure.

1. Let $S_{m,n}$ count the $m \times n$ letter arrays that are fixed via horizontal reflection, vertical reflection, and 180° degree rotation.

2. Let $H_{m,n}$ count the $m \times n$ letter arrays fixed via horizontal reflection. Then, $H_{m,n} - S_{m,n}$ counts the $m \times n$ letter arrays that are fixed only by horizontal reflection.

3. Let $V_{m,n}$ count the $m \times n$ letter arrays fixed via vertical reflection. Then, $V_{m,n} - S_{m,n}$ counts the $m \times n$ letter arrays that are fixed only by vertical reflection.

4. Let $R_{m,n}$ count the $m \times n$ letter arrays fixed via 180° rotation. Then, $R_{m,n} - S_{m,n}$ counts the $m \times n$ letter arrays that are fixed only by rotation.

5. Let $C_{m,n} = B_{m,n} - (H_{m,n} - S_{m,n}) - (V_{m,n} - S_{m,n}) - (R_{m,n} - S_{m,n}) - S_{m,n}$. Then, $C_{m,n}$ counts the $m \times n$ letter arrays that are not fixed by any symmetry transformation.

Theorem 2.1 *Let $L_{m,n}, C_{m,n}, B_{m,n}, V_{m,n}, H_{m,n}, R_{m,n},$ and $S_{m,n}$ be as previously defined. Then*

$$L_{m,n} = \frac{B_{m,n} + H_{m,n} + V_{m,n} + R_{m,n}}{4}$$

Proof of Theorem 2.1: To calculate the letter representations associated with $m \times n \times p$ proper arrays modulo symmetry, we first determine whether a given $m \times n$ letter array, called A, is fixed via any of the four symmetry transformations. If A is not fixed by any symmetry, it has four equivalent images. However, if A is fixed under a symmetry transformation, it has at most two symmetry images. It follows that

$$L_{m,n} = \frac{C_{m,n}}{4} + \frac{(H_{m,n} - S_{m,n}) + (R_{m,n} - S_{m,n}) + (V_{m,n} - S_{m,n})}{2} + S_{m,n}$$
$$= \frac{B_{m,n} + H_{m,n} + V_{m,n} + R_{m,n}}{4} \qquad \square$$

Remark 2.1 *Theorem 2.1 can be considered to be an immediate consequence of Burnside's Lemma.*

2.1   Numerical Example

In order to understand how the information provided by Theorem 2.1 helps provide a lower bound on the basis size of the transition matrix, look at the following example. Let $m = 3$ and $n = 1$. The goal is to compute the size of the transition matrix associated with the $3 \times 1 \times p$ proper arrays. It can be shown that the transition matrix is a 16×16 matrix [1],[2]. Thus, there are 16 possible 3×1 *word* representations modulo symmetry. Theorem 2.1 simply counts the *letter* representations modulo symmetry. Ignoring symmetry, there are $B_{3,1} = 5$ letter representations. By direct calculation, we find that $H_{3,1} = 3, V_{3,1} = 5$, and $R_{3,1} = 3$. Theorem 2.1 implies that the number of 3×1 letter representations modulo symmetry is $\frac{5+3+5+3}{4} = 4$. Thus, the transition matix associated with the $3 \times 1 \times p$ proper arrays is at least a 4×4 matrix. The goal of our research is to obtain a formula that calculates the actual basis size of the transition matrix. As this example demonstrates, the theorems in this paper provide not actual basis size, but a lower bound.

3   Generating Function for Horizontal/Vertical Symmetry

In order to use Theorem 2.1, we need to find generating functions for $H_{m,n}$, $V_{m,n}$, and $R_{m,n}$, These generating functions are obtained by dividing the $m \times n$ array into small sections determined by the symmetry transformation. For instance, in the case of horizontal reflection, the array is divided into two halves. If two or more transformations are applied, the array is divided into four quarters. In either case, we can arbitrarily fill one of the halves/quarters with any arrangement of letters and then use symmetry to fill the remaining half/quarters of the $m \times n$ array. The key in this technique is to carefully divide the array around the row and/or column that may be fixed under a symmetry transformation. Hence, the generating functions depend on the parity of m and n.

We begin with the generating functions associated with $H_{m,n}$. Due to symmetry, they are also the generating functions associated with $V_{n,m}$. Before going further, we should recall the following representations of the Bell Numbers and the Second Stirling Numbers [4], [7], [8].

Remark 3.1

$$\sum_{n=0}^{\infty} B(n)\frac{x^n}{n!} = e^{e^x - 1}$$

Remark 3.2

$$\sum_{r=0}^{\infty} \left\{ \begin{array}{c} r \\ t \end{array} \right\} \frac{y^r}{r!} = \frac{(e^y - 1)^t}{t!}$$

Using the aforementioned subdivision technique, we find that

$$H_{2m,n} = \sum_{j=1}^{mn} \sum_{s=0}^{[\frac{j}{2}]} \frac{\left\{ \begin{array}{c} mn \\ j \end{array} \right\} 2^{j-3s} j!}{s!(j-2s)!}$$

where

1. j counts the letters in the first m rows. We will refer to the first m rows as the **top half** of the array and the remaining m rows as the **bottom half** of the array.

2. s counts the number of these j letters that are interchanged when reflected to the bottom half of the array.

Figure 3.1: This figure demonstrates the three possiblities that occur to the letters that appear in the top half of the array. A letter can reflect to itself; this is demonstrated by A. A letter can reflect to an entirely new letter; this is demonstrated by B reflecting to X. Otherwise, two letters, such as C and D, can interchange position under reflection.

Theorem 3.1 *The coefficient of t^{mn} in the expansion of $\exp(\frac{1}{2}(e^t + 3)(e^t - 1))$ is* $H_{2m,n}$.

Proof of Theorem 3.1: Let $k = mn$, then

$$H(t) = \sum_{k,j,s=0}^{\infty} \frac{\left\{ \begin{matrix} k \\ j \end{matrix} \right\} t^k 2^{j-3s} j!}{s!(j-2s)!k!} \tag{3.1}$$

To obtain the desired result, use Remark 3.2 and sum over s and j. See the proof of Theorem 3.2 for a more elaborate example.   □

Remark 3.3 *In Equation (3.1), we sum only over a range of values than ensure nonnegative factorials. Instead of explicitly writing the range of summation for each index, we use the shorthand notation of summing each variable from zero to infinity. This convention will be used throughout the paper.*

Next we determine $H_{2m+1,n}$. Finding $H_{2m+1,n}$ is more complicated since we must contend with a middle row that is fixed by the reflection map.

$$H_{2m+1,n} = n! \sum_{i=1}^{mn} \sum_{j=1}^{min(n,i)} \sum_{k=j}^{n} \sum_{s=0}^{\left[\frac{i-j}{s}\right]} \frac{\left\{ \begin{matrix} mn \\ i \end{matrix} \right\} \left\{ \begin{matrix} k \\ j \end{matrix} \right\} B_{n-k} i! 2^{i-j-3s}}{k!(n-k)!s!(i-j-2s)!}$$

where

1. i counts the letters in the top half of the array.
2. j counts the number of those i letters that appear in the middle row.
3. k counts the squares in the middle row occupied by the j letters.
4. s counts the number of the $i - j$ letters that are interchanged when reflected to the bottom half of the array.
5. B_{n-k} counts the $1 \times (n - k)$ letter arrays. These $1 \times (n - k)$ letter arrays only appear in the middle row of array.

Figure 3.2: In this case, the middle row is fixed under horizontal reflection. If a letter from the top half of the array occurs in the middle row, it is <u>fixed</u> under reflection. Otherwise, it obeys the three possiblities discussed in the construction of $H_{2m,n}$.

Theorem 3.2 *The coefficient of $y^{mn} x^n$ in the expansion of $\exp(e^{y+x} + \frac{1}{2} e^{2y} - \frac{3}{2})$ is $H_{2m+1,n}$.*

Proof of Theorem 3.2 Let $q = mn$ and define

$$H(x,y) = \sum_{i,j,k,s,n,q=0}^{\infty} \frac{\left\{ \begin{matrix} q \\ i \end{matrix} \right\} \left\{ \begin{matrix} k \\ j \end{matrix} \right\} B_{n-k} i! 2^{i-j-3s} x^n y^q}{k!(n-k)!s!(i-j-2s)!q!}$$

Using Remarks 3.1 and 3.2, we find

$$H(x,y) = \sum_{i,j,s=0}^{\infty} \frac{(e^y-1)^i(e^x-1)^j e^{e^x-1} 2^{i-j-3s}}{j!s!(i-j-2s)!}$$

$$= \sum_{i,s=0}^{\infty} \frac{(e^y-1)^i(e^x+1)^{i-2s} e^{e^x-1} 2^{-s}}{s!(i-2s)!}$$

$$= \sum_{s=0}^{\infty} \frac{(e^y-1)^{2s} e^{e^x-1+(e^y-1)(e^x+1)} 2^{-s}}{s!}$$

$$= \exp(e^{y+x} + \frac{1}{2}e^{2y} - \frac{3}{2}) \qquad \square$$

4   Generating Function for Rotational Symmetry

Our next step is to find a generating function for the rotational symmetry. Observe that $R_{2m,2n} = H_{2m,2n}$, $R_{2m+1,2n} = H_{2n,2m+1}$, and $R_{2m,2n+1} = H_{2m,2n+1}$. It remains to compute $R_{2m+1,2n+1}$. In this situation, the 180° rotation fixes the central square of the array. Our geometric subdivision must avoid this central square and takes the form of an upper L and a lower L.

Figure 4.1: Under 180° rotation, the central square is fixed, and the array is subdivided into two L shapes.

We find that

$$R_{m,n} = \sum_{i=1}^{q} \sum_{s=0}^{[\frac{i}{2}]} \sum_{r=0}^{i-2s} \frac{\left\{ \begin{matrix} q \\ i \end{matrix} \right\} i!(i-2s+1-r)}{s!2^s r!(i-2s-r)!}$$

where

1. $q = n[\frac{m}{2}] + [\frac{n}{2}]$.
2. i counts the letters in the top L.
3. s counts the number of the i letters that are interchanged in the bottom L.
4. r counts the new letters that appear in the bottom L. The central square can contain a letter that does not appear in the rest of the array or is filled by one of the $i - 2s - r$ letters that go to themselves in the bottom L.
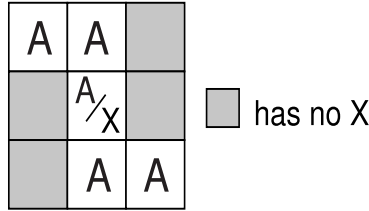

Figure 4.2: The two ways to fill the central square. The central square contains either new letter or a letter from in the upper L that rotates to itself in the bottom L.

Theorem 4.1 *Let m and n be odd integers. Let $q = n[\frac{m}{2}] + [\frac{n}{2}]$. The coefficient of t^q in the expansion of $\exp(2e^t - 2 + t + \frac{1}{2}(e^t - 1)^2)$ is $R_{m,n}$.*

Proof of Theorem 4.1:
Let

$$R(t) = \sum_{q,i,s,r=0}^{\infty} \frac{\left\{ \begin{array}{c} q \\ i \end{array} \right\} t^q i! (i - 2s - r + 1)}{q! 2^s s! r! (i - 2s - r)!}$$

By using the techniques of Theorem 3.1, we obtain the desired result.   □

Remark 4.1 *We should note that the theorems in Sections 3 and 4 could be proven using the context of involutions acting on a finite set. In particular, suppose the involution is acting on a finite set in manner which provides t matching pairs and u fixed points. Then, by the argument used to derive $H_{2m+1,n}$, the number of partitions of the set that are inequivalent under the involution is $\exp(e^{y+x} + \frac{1}{2}e^{2y} - \frac{3}{2})$, where the powers of x correspond to u and the powers of y correspond to t. Thus, $H_{2m,n}$ is the case when $t = mn$ and $u = 0$, $H_{2m+1,n}$ is the case when $t = mn$ and $u = n$, and $R_{m,n}$ (with m, n odd) is the case when $t = q = n[\frac{m}{2}] + [\frac{n}{2}]$ and $u = 1$.*

5 Generating Function for Fully Symmetrical Letter Representation

We define an $m \times n$ letter representation to be **fully symmetric** if and only if it is fixed via horizontal <u>and</u> vertical reflection. Let $S_{m,n}$ count the number of fully symmetrical letter representations. Although $S_{m,n}$ only appears in the proof of Theorem

2.1, we can apply the techniques of Sections 3 and 4 to determine the generating functions for $S_{m,n}$. Since the fully symmetrical letter representations are invariant under two symmetry transformations, the underlying $m \times n$ array is divided into four quadrants. The upper left quadrant is arbitrarily filled and the remaining three quadrants are completed by applying the two symmetry transformations to the upper left quadrant. There are two types of letters in the upper left quadrant. The first type of letter is called a singleton letter. A **singleton letter** is a letter whose image, in the remaining three quadrants, is never another letter that appears in the upper left quadrant. There are five possible ways a letter can be a singleton. These five ways are illustrated in Figure 5.1
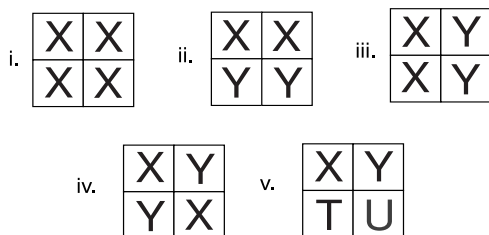

Figure 5.1: The five ways a singleton letter can be transformed under symmetry.

The second type of letter present in the upper left quadrant can be considered to be part of a double pair. A letter is part of **double pair** when its image in one of the other three quadrants is another letter orginally present in the upper left quadrant. Figure 5.2 illustrates the six ways double pairs transform in a fully symmetrical manner.

We begin by assuming the row and column dimensions are even. The geometric construction depends on locating the singleton letters and the double pairs present in the upper left quadrant. In particular,

$$S_{2m,2n} = \sum_{j=1}^{mn} \sum_{s=0}^{[\frac{j}{2}]} \frac{\left\{ \begin{array}{c} mn \\ j \end{array} \right\} 6^s 5^{j-2s} j!}{2^s s!(j-2s)!}$$

where

A	B	B	A		A	B	A	B
		i.					ii.	
B	A	A	B		C	D	C	D

A	B	A	B		A	B	D	C
		iii.					iv.	
A	B	A	B		B	A	C	D

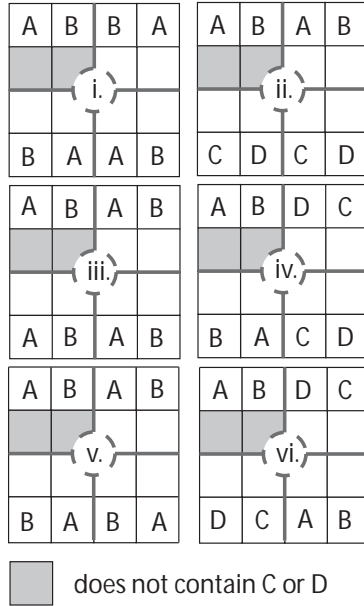| A	B	A	B		A	B	D	C
		v.					vi.	
B	A	B	A		D	C	A	B

does not contain C or D

Figure 5.2: The six ways two letters, each appearing in the upper left quadrant, form double pairs.

1. j counts the letters in the upper left quadrant.
2. s counts the number of these j letters that are interchanged under symmetry operations.

The proof of Theorem 5.1 is similiar to that of Theorem 3.1 and will be omitted.

Theorem 5.1 *The coefficient of t^{mn} in the expansion of $\exp(5(e^t - 1)e^{3(e^t+1)^2})$ is $S_{2m,2n}$.*

Next we calculate $S_{2m,2n+1}$. By symmetry, the calculation for $S_{2m,2n+1}$ determines $S_{2n+1,2m}$. In this particular case, we decompose the $2m \times (2n+1)$ array into four quadrants and a middle column and fill the upper left quadrant with an arbitrary letter configuration. Using symmetry, we complete the remaining three quadrants as illustrated by Figures 5.1 and 5.2. However, we need to determine whether a singleton letter or letter from a double pair may occur in the middle column. In particular, the two singleton types illustrated by Diagrams i and ii in Figure 5.1 and the double pair illustrated by Diagram i of Figure 5.2 may occur in the middle column. The middle column is completed as follows. The first m squares of the middle column, called the **top half of the middle column**, are filled by letters

that appeared in the quadrants or by new letters. Then, using horizontal reflection, the corresponding positions in the lower half of the middle column are filled. Hence,

$$S_{2m,2n+1} = \sum_{i=1}^{mn} \sum_{F=0}^{[\frac{i}{2}]} \sum_{M=0}^{[\frac{i-2F}{2}]} \sum_{f=0}^{i-2F-2M} \sum_{p=0}^{i-2F-2M-f} \sum_{l=0}^{m} \sum_{k=min(1,m-l)}^{m-l} \sum_{t=0}^{[\frac{k}{2}]}$$

$$\frac{\left\{ \begin{array}{c} mn \\ i \end{array} \right\} i!\, 5^F 3^f 2^{-F-M+k-3t} m!\, (2i - 2f - p - 4F - 2M)^l \left\{ \begin{array}{c} m-l \\ k \end{array} \right\} k!}{F!\, M!\, f!\, p!\, (i - 2F - 2M - f - p)!\, l!\, (m - l)!\, (k - 2t)!\, t!}$$

where

1. i counts the letters in the upper left quadrant.
2. F counts the double pairs of the upper left hand quadrant that cannot occur in the middle column.
3. M counts the double pairs of the upper left quadrant that can occur in the middle column.
4. f counts the singleton letters of the upper left quadrant that cannot occur in the middle column.
5. p counts the letters in the upper left quadrant that are the singleton letters illustrated by Diagram i of Figure 5.1
6. l counts the squares in the top half of the middle column that are filled by letters that appeared in the quadrants
7. k counts the letters in the top half of the middle column that do not appear in the quadrants.
8. t counts the interchanges that occur as the k letters are reflected to the bottom half of the middle column.

Theorem 5.2 *The coefficient of $y^m x^{mn}$ in the expansion of $\exp(\frac{1}{2}[e^x(-4 + 2e^y) + e^{2x}(e^{2y} + 5) - 4])$ is $S_{2m,2n+1}$.*

To prove Theorem 5.2, we make use of the following version of Taylor's Theorem.

Remark 5.1

$$e^{a\frac{d^n}{dw^n}} e^{bw} = e^{ab^n} e^{bw}$$

Proof of Theorem 5.2 Let $q = mn$. Then,

$$S(x,y) = \sum_{\substack{i,F,M,f,p,\\m,l,k,t,q=0}}^{\infty} \frac{\left\{\begin{matrix} q \\ i \end{matrix}\right\}\left\{\begin{matrix} m-l \\ k \end{matrix}\right\} i!\,k!(2i-2f-p-4F-2M)^l 5^F 3^f 2^{-F-M+k-3t} x^q y^m}{F!M!f!p!(i-2F-2M-f-p)!\,l!\,(m-l)!\,t!\,(k-2t)!q!}$$

$$= \sum_{\substack{i,F,M,f,\\p,l,k,t=0}}^{\infty} \frac{(e^x-1)^i(e^y-1)^k 5^F 3^f 2^{-F-M+k-3t}(2i-2f-p-4F-2M)^l y^l}{F!M!f!p!(i-2F-2M-f-p)!\,l!\,(k-2t)!t!}$$

$$= \sum_{\substack{i,F,M,f,\\p,l,k,t=0}}^{\infty} \frac{(e^x-1)^i(e^y-1)^k 5^F 3^f 2^{-F-M-t}(2i-2f-p-4F-2M)^l y^l [dz]^{2t} z^k}{F!M!f!p!(i-2F-2M-f-p)!\,l!\,k!\,t!}\Bigg|_{z=2}$$

$$= \sum_{\substack{i,F,M,f,\\p,l,k=0}}^{\infty} \frac{(e^x-1)^i(e^y-1)^k 5^F 3^f 2^{-F-M}(2i-2f-p-4F-2M)^l y^l e^{\frac{(dz)^2}{2}} z^k}{F!M!f!p!(i-2F-2M-f-p)!\,l!\,k!}\Bigg|_{z=2}$$

Due to Remark 5.1, we obtain

$$S(x,y) = \sum_{i,F,M,f,p,l=0}^{\infty} \frac{(e^x-1)^i 5^F 3^f 2^{-F-M}(2i-2f-p-4F-2M)^l y^l e^{\frac{1}{2}(e^{2y}+2e^y-3)}}{F!M!f!p!(i-2F-2M-f-p)!\,l!}$$

$$= \sum_{i,F,M,f,p=0}^{\infty} \frac{(e^x-1)^i 5^F 3^f 2^{-F-M} e^{y(2i-2f-p-4F-2M)} e^{\frac{1}{2}(e^{2y}+2e^y-3)}}{F!M!f!p!(i-2F-2M-f-p)!}$$

Let $M = u - F$. The above line becomes

$$S(x,y) = \sum_{i,F,u,f,p=0}^{\infty} \frac{[(e^x-1)e^{2y}]^i [3e^{-2y}]^f [5e^{-2y}]^F [e^{-y}]^p [2e^{2y}]^{-u} e^{\frac{1}{2}(e^{2y}+2e^y-3)}}{F!(u-F)!f!p!(i-2u-f-p)!}$$

$$= \sum_{i,F,u=0}^{\infty} \frac{[(e^x-1)e^{2y}]^i [5e^{-2y}]^F [1+3e^{-2y}+e^{-y}]^{i-2u} [2e^{2y}]^{-u} e^{\frac{1}{2}(e^{2y}+2e^y-3)}}{F!(u-F)!(i-2u)!}$$

$$= \sum_{i,u=0}^{\infty} \frac{[(e^x-1)e^{2y}]^i [1+5e^{-2y}]^u [1+3e^{-2y}+e^{-y}]^{i-2u} [2e^{2y}]^{-u} e^{\frac{1}{2}(e^{2y}+2e^y-3)}}{u!(i-2u)!}$$

$$= \exp\left(\frac{1}{2}[e^x(-4+2e^y)+(e^{2y}+5)e^{2x}-4]\right) \qquad \square$$

The final sum of this section counts $S_{(2m+1)\times(2n+1)}$. In this case, the geometric decomposition involves four quadrants, a central square, and a middle cross, where a **middle cross** is the union of the middle column and middle row minus the central square. Thus, a middle cross consists of the top and bottom halves of the middle column along with the left and right halves of the middle row. In this situation, **the top half of the middle column** is the m squares above the central square, and **the left half of the middle row** is the n squares to the left of the central square.

For this situation, we will not begin by analyzing the components in the upper left quadrant. Instead, we will work first with the middle cross and then extend outward

to the quadrants. In particular, we begin by filling in the top half of the middle column in an arbitrary manner. Next, using the horizontal reflective symmetry, we determine bottom half of the middle column. A letter from the top half, when reflected to the bottom, can either go to itself, go to an entirely new letter, or become interchanged with another letter present in the top half. In all three cases, these top half letters (and possibly new bottom half letters) may occur in the quadrants. This situation is illustrated by Figure 5.3.
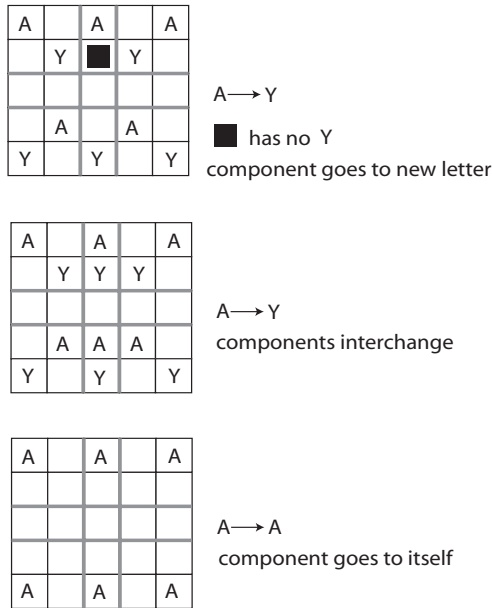

$A \longrightarrow Y$

■ has no Y component goes to new letter

$A \longrightarrow Y$

components interchange

$A \longrightarrow A$

component goes to itself

Figure 5.3: The three ways A, a letter from the top half of the middle column, is transformed, via horizontal reflection, to the bottom half of the middle column. In all three cases, note that A and its image have a unique way of appearing in the quadrants.

The next step is to determine the middle row. This is done by first filling the left half of the middle row with letters occuring in the middle column. The only such middle column letters that may be used are those letters which are fixed under horizontal reflection. To fill in the remaining spaces of the middle row minus the central square, we repeat the procedure used to determine the middle column. The only difference is that we use a vertical reflection. Once again, these middle row letters may appear in the quadrants. To visualize this situation, simply rotate the diagrams in Figure 5.3 by 90 degrees. Next, we are in a position to determine the quadrants. This is done by first placing the cross letters throughout the quadrants. Then, for the remaining spaces of the quadrants, we repeat the geometric analysis of singletons and double pairs used for determining $S_{2m,2n}$. The last step is to fill the central square with an

entirely new letter, a singleton from the quadrant that is fixed under symmetry, or a letter from the middle cross that is also fixed under symmetry.

Following the steps outlined above, we arrive at a summation of 22 variables. The derivation of the exponential generating function utilizes the techniques of Theorem 5.2. Details are available, upon request, from the author.

Theorem 5.3 *The coeffiecient of $y^m x^n l^{mn}$ in the expansion of*
$\exp(l + x + y) \exp(2e^{2l} + \frac{1}{2}e^{2l+2x} + \frac{1}{2}e^{2l+2y} + e^{l+x+y} - 2e^l - 2)$ *is $S_{2m+1,2n+1}$.*

6 Further Research

By using a particular decomposition of the Bell Numbers [4],[7],[8] and applying various symmetry transformations to $m \times n$ rectangular arrays, we are able to calculate the number of $m \times n$ letter representations modulo D_2 symmetry. The number of $m \times n$ letter representations only provides a lower bound for the basis size of the transition matrix. In order to find an upper bound for the cardinality of the basis, it is necessary to extrapolate the results of this paper to the case of $m \times n$ arrays of circled letters (i.e. word representations of Section 1). Currently, we are formulating the generating functions used in the enumeration of $m \times n$ arrays of circled letters.

Another promising avenue of research involves square arrays, where $m = n$. Since a square array has D_4 symmetry, the formula to enumerate $m \times m$ letter representations modulo symmetry will be more complicated than the formula provided by Theorem 2.1. We are in the process of calculating the generating functions necessary in the enumeration of $m \times m$ letter representations. The ultimate goal will be to adapt the geometric constructions used in the enumeration of $m \times m$ letter representations to the case of $m \times m$ word representations. Once this is accomplished, an upper bound for the basis size of the associated transition matrix will be completely determined for all integers m and n.

Finally, a different type of research would involve exploring the connections between $m \times n \times p$ proper arrays and percolation theory. At the present time, we have not explored the connection in any depth but realize that the stochastic and probabilisitic techniques of percolation theory could, when applied to the representation of an $m \times n \times p$ proper array as a bond percolation on Z^3 with an open cluster at the origin (see Section 0.3), give rise to a whole new category of results.

Acknowledgments

Appendix A: Numerical Data

The following table provides, for small integer values of m and n, numerical values for $H_{m,n}, V_{m,n}, R_{m,n}$, and $S_{m,n}$. All the values came from the generating functions given by Theorems 3.1, 3.2, 4.1, 5.1, 5.2, and 5.3 and were verified by a short Maple program the author created.

$m \times n$	$H_{m,n}$	$V_{m,n}$	$R_{m,n}$	$S_{m,n}$
2×3	31	12	31	13
2×4	164	164	164	36
2×5	999	999	999	107
3×2	12	31	31	13
3×4	339	6841	6841	469
3×5	2210	51790	127643	3835

Table 2: Numerical Data for certain $m \times n$ letter representations

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